EQUIVARIANT CYCLIC HOMOLOGY FOR QUANTUM GROUPS

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Abstract. We define equivariant periodic cyclic homology for bornological quantum groups. Generalizing corresponding results from the group case, we show that the theory is homotopy invariant, stable and satisfies excision in both variables. Along the way we prove Radford's formula for the antipode of a bornological quantum group. Moreover we discuss anti-Yetter-Drinfeld modules and establish an analogue of the Takesaki-Takai duality theorem in the setting of bornological quantum groups.

1. Introduction

Equivariant cyclic homology can be viewed as a noncommutative generalization of equivariant de Rham cohomology. For actions of finite groups or compact Lie groups, different aspects of the theory have been studied by various authors [4], [5], [6], [17], [18]. In order to treat noncompact groups as well, a general framework for equivariant cyclic homology following the Cuntz-Quillen formalism [8], [9], [10] has been introduced in [25]. For instance, in the setting of discrete groups or totally disconnected groups this yields a new approach to classical constructions in algebraic topology [26]. However, in contrast to the previous work mentioned above, a crucial feature of the construction in [25] is the fact that the basic ingredient in the theory is not a complex in the usual sense of homological algebra. In particular, the theory does not fit into the traditional scheme of defining cyclic homology using cyclic modules or mixed complexes.

In this note we define equivariant periodic cyclic homology for quantum groups. This generalizes the constructions in the group case developped in [25]. Again we work in the setting of bornological vector spaces. Correspondingly, the appropriate notion of a quantum group in this context is the concept of a bornological quantum group introduced in [27]. This class of quantum groups includes all locally compact groups and their duals as well as all algebraic quantum groups in the sense of van Daele [24]. As in the theory of van Daele, an important ingredient in the definition of a bornological quantum group is the Haar measure. It is crucial for the duality theory and also explicitly used at several points in the construction of the homology theory presented in this paper. However, with some modifications our definition of equivariant cyclic homology could also be adapted to a completely algebraic setting using Hopf algebras with invertible antipodes instead.

From a conceptual point of view, equivariant cyclic homology should be viewed as a homological analogon to equivariant KK-theory [15], [16]. The latter has been extended by Baaj and Skandalis to coactions of Hopf-C^*-algebras [3]. However, in our situation it is more convenient to work with actions instead of coactions.

An important ingredient in equivariant cyclic homology is the concept of a covariant module [25]. In the present paper we will follow the terminology introduced in [12] and call these objects anti-Yetter-Drinfeld modules instead. In order to construct

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the natural symmetry operator on these modules in the general quantum group setting we prove a formula relating the fourth power of the antipode with the modular functions of a bornological quantum group and its dual. In the context of finite dimensional Hopf algebras this formula is a classical result due to Radford [23].
Although anti-Yetter-Drinfeld modules occur naturally in the constructions one should point out that our theory does not fit into the framework of Hopf-cyclic cohomology [13]. Still, there are relations to previous constructions for Hopf algebras by Akbarpour and Khalkhali [1], [2] as well as Neshveyev and Tuset [22]. Remark in particular that cosemisimple Hopf algebras or finite dimensional Hopf algebras can be viewed as bornological quantum groups. However, basic examples show that the homology groups defined in [1], [2], [22] only reflect a small part of the information contained in the theory described below.
Let us now describe how the paper is organized. In section 2 we recall the definition of a bornological quantum group. We explain some basic features of the theory including the definition of the dual quantum group and the Pontrjagin duality theorem. This is continued in section 3 where we discuss essential modules and comodules over bornological quantum groups as well as actions on algebras and their associated crossed products. We prove an analogue of the Takesaki-Takai duality theorem in this setting. Section 4 contains the discussion of Radford's formula relating the antipode with the modular functions of a quantum group and its dual. In section 5 we study anti-Yetter-Drinfeld modules over bornological quantum groups and introduce the notion of a paracomplex. Section 6 contains a discussion of equivariant differential forms in the quantum group setting. After these preparations we define equivariant periodic cyclic homology in section 7. In section 8 we show that our theory is homotopy invariant, stable and satisfies excision in both variables. Finally, section 9 contains a brief comparison of our theory with the previous approaches mentioned above.
Throughout the paper we work over the complex numbers. For simplicity we have avoided the use of pro-categories in connection with the Cuntz-Quillen formalism to a large extent.

2. Bornological quantum groups

The notion of a bornological quantum group was introduced in [27]. We will work with this concept in our approach to equivariant cyclic homology. For information on bornological vector spaces and more details we refer to [14], [20], [27]. All bornological vector spaces are assumed to be convex and complete.
A bornological algebra H is called essential if the multiplication map induces an isomorphism $H \hat{\otimes}_H H \cong H$. The multiplier algebra $M(H)$ of a bornological algebra H consists of all two-sided multipliers of H, the latter being defined by the usual algebraic conditions. There exists a canonical bounded homomorphism $\iota : H \to M(H)$. A bounded linear functional $\phi : H \to \mathbb{C}$ on a bornological algebra is called faithful if $\phi(xy) = 0$ for all $y \in H$ implies $x = 0$ and $\phi(xy) = 0$ for all x implies $y = 0$. If there exists such a functional the map $\iota : H \to M(H)$ is injective. In this case one may view H as a subset of the multiplier algebra $M(H)$.
In the sequel H will be an essential bornological algebra with a faithful bounded linear functional. For technical reasons we assume moreover that the underlying bornological vector space of H satisfies the approximation property.
A module M over H is called essential if the module action induces an isomorphism $H \hat{\otimes}_H M \cong M$. Moreover an algebra homomorphism $f : H \to M(K)$ is essential if f turns K into an essential left and right module over H. Assume that $\Delta : H \to M(H \hat{\otimes} H)$ is an essential homomorphism. The map Δ is called a comultiplication if it is coassociative, that is, if $(\Delta \hat{\otimes} \mathrm{id})\Delta = (\mathrm{id} \hat{\otimes} \Delta)\Delta$ holds. Moreover the Galois

maps $\gamma_l, \gamma_r, \rho_l, \rho_r : H \hat{\otimes} H \to M(H \hat{\otimes} H)$ for Δ are defined by

$$\gamma_l(x \otimes y) = \Delta(x)(y \otimes 1), \qquad \gamma_r(x \otimes y) = \Delta(x)(1 \otimes y)$$
$$\rho_l(f \otimes g) = (x \otimes 1)\Delta(y), \qquad \rho_r(x \otimes y) = (1 \otimes x)\Delta(y).$$

Let $\Delta : H \to M(H \hat{\otimes} H)$ be a comultiplication such that all Galois maps associated to Δ define bounded linear maps from $H \hat{\otimes} H$ into itself. If ω is a bounded linear functional on H we define for every $x \in H$ a multiplier $(\mathrm{id} \hat{\otimes} \omega)\Delta(x) \in M(H)$ by

$$(\mathrm{id} \hat{\otimes} \omega)\Delta(x) \cdot y = (\mathrm{id} \hat{\otimes} \omega)\gamma_l(x \otimes y)$$
$$y \cdot (\mathrm{id} \hat{\otimes} \omega)\Delta(x) = (\mathrm{id} \hat{\otimes} \omega)\rho_l(y \otimes x).$$

In a similar way one defines $(\omega \hat{\otimes} \mathrm{id})\Delta(x) \in M(H)$. A bounded linear functional $\phi : H \to \mathbb{C}$ is called left invariant if

$$(\mathrm{id} \hat{\otimes} \phi)\Delta(x) = \phi(x)1$$

for all $x \in H$. Analogously one defines right invariant functionals.
Let us now recall the definition of a bornological quantum group.

Definition 2.1. *A bornological quantum group is an essential bornological algebra H satisfying the approximation property with a comultiplication $\Delta : H \to M(H \hat{\otimes} H)$ such that all Galois maps associated to Δ are isomorphisms together with a faithful left invariant functional $\phi : H \to \mathbb{C}$.*

The definition of a bornological quantum group is equivalent to the definition of an algebraic quantum group in the sense of van Daele [24] provided the underlying bornological vector space carries the fine bornology. The functional ϕ is unique up to a scalar and referred to as the left Haar functional of H.

Theorem 2.2. *Let H be a bornological quantum group. Then there exists an essential algebra homomorphism $\epsilon : H \to \mathbb{C}$ and a linear isomorphism $S : H \to H$ which is both an algebra antihomomorphism and a coalgebra antihomomorphism such that*

$$(\epsilon \hat{\otimes} \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \hat{\otimes} \epsilon)\Delta$$

and

$$\mu(S \hat{\otimes} \mathrm{id})\gamma_r = \epsilon \hat{\otimes} \mathrm{id}, \qquad \mu(\mathrm{id} \hat{\otimes} S)\rho_l = \mathrm{id} \hat{\otimes} \epsilon.$$

Moreover the maps ϵ and S are uniquely determined.

Using the antipode S one obtains that every bornological quantum group is equipped with a faithful right invariant functional ψ as well. Again, such a functional is uniquely determined up to a scalar. There are injective bounded linear maps $\mathcal{F}_l, \mathcal{F}_r, \mathcal{G}_l, \mathcal{G}_r : H \to H' = \mathrm{Hom}(H, \mathbb{C})$ defined by the formulas

$$\mathcal{F}_l(x)(h) = \phi(hx), \qquad \mathcal{F}_r(x)(h) = \phi(xh)$$
$$\mathcal{G}_l(x)(h) = \psi(hx), \qquad \mathcal{G}_r(x)(h) = \psi(xh).$$

The images of these maps coincide and determine a vector space $\hat{H}$. Moreover, there exists a unique bornology on $\hat{H}$ such that these maps are bornological isomorphisms. The bornological vector space $\hat{H}$ is equipped with a multiplication which is induced from the comultiplication of H. In this way $\hat{H}$ becomes an essential bornological algebra and the multiplication of H determines a comultiplication on $\hat{H}$.

Theorem 2.3. *Let H be a bornological quantum group. Then $\hat{H}$ with the structure maps described above is again a bornological quantum group. The dual quantum group of $\hat{H}$ is canonically isomorphic to H.*

Explicitly, the duality isomorphism $P : H \to \hat{\hat{H}}$ is given by $P = \hat{\mathcal{G}}_l \mathcal{F}_l S$ or equivalently $P = \hat{\mathcal{F}}_r \mathcal{G}_r S$. Here we write $\hat{\mathcal{G}}_l$ and $\hat{\mathcal{F}}_r$ for the maps defined above associated to the dual Haar functionals on $\hat{H}$. The second statement of the previous theorem should be viewed as an analogue of the Pontrjagin duality theorem.
In [27] all calculations were written down explicitly in terms of the Galois maps and their inverses. However, in this way many arguments tend to become lengthy and not particularly transparent. To avoid this we shall use the Sweedler notation in the sequel. That is, we write

$$\Delta(x) = x_{(1)} \otimes x_{(2)}$$

for the coproduct of an element x, and accordingly for higher coproducts. Of course this has to be handled with care since expressions like the previous one only have a formal meaning. Firstly, the element $\Delta(x)$ is a multiplier and not contained in an actual tensor product. Secondly, we work with completed tensor products which means that even a generic element in $H \hat{\otimes} H$ cannot be written as a finite sum of elementary tensors as in the algebraic case.

3. Actions, coactions and crossed products

In this section we review the definition of essential comodules over a bornological quantum group and their relation to essential modules over the dual. Moreover we consider actions on algebras and their associated crossed products and prove an analogue of the Takesaki-Takai duality theorem.
Let H be a bornological quantum group. Recall from section 2 that a module V over H is called essential if the module action induces an isomorphism $H \hat{\otimes}_H V \cong V$. A bounded linear map $f : V \to W$ between essential H-modules is called H-linear or H-equivariant if it commutes with the module actions. We denote the category of essential H-modules and equivariant linear maps by $H\text{-}\mathsf{Mod}$. Using the comultiplication of H one obtains a natural H-module structure on the tensor product of two H-modules and $H\text{-}\mathsf{Mod}$ becomes a monoidal category in this way.
We will frequently use the regular actions associated to a bornological quantum group H. For $t \in H$ and $f \in \hat{H}$ one defines

$$t \rightharpoonup f = f_{(1)}\, f_{(2)}(t), \qquad f \leftharpoonup t = f_{(1)}(t) f_{(2)}$$

and this yields essential left and right H-module structures on $\hat{H}$, respectively.
Dually to the concept of an essential module one has the notion of an essential comodule. Let H be a bornological quantum group and let V be a bornological vector space. A coaction of H on V is a right H-linear bornological isomorphism $\eta : V \hat{\otimes} H \to V \hat{\otimes} H$ such that the relation

$$(\mathrm{id} \otimes \gamma_r)\eta_{12}(\mathrm{id} \otimes \gamma_r^{-1}) = \eta_{12}\eta_{13}$$

holds.

Definition 3.1. *Let H be a bornological quantum group. An essential H-comodule is a bornological vector space V together with a coaction $\eta : V \hat{\otimes} H \to V \hat{\otimes} H$.*

A bounded linear map $f : V \to W$ between essential comodules is called H-colinear if it is compatible with the coactions in the obvious sense. We write $\mathsf{Comod}\text{-}H$ for the category of essential comodules over H with H-colinear maps as morphisms. The category $\mathsf{Comod}\text{-}H$ is a monoidal category as well.
If the quantum group H is unital, a coaction is the same thing as a bounded linear map $\eta : V \to V \hat{\otimes} H$ such that $(\eta \hat{\otimes} \mathrm{id})\eta = (\mathrm{id} \hat{\otimes} \Delta)\eta$ and $(\mathrm{id} \hat{\otimes} \epsilon)\eta = \mathrm{id}$.
Modules and comodules over bornological quantum groups are related in the same way as modules and comodules over finite dimensional Hopf algebras.

Theorem 3.2. *Let H be a bornological quantum group. Every essential left H-module is an essential right $\hat{H}$-comodule in a natural way and vice versa. This yields inverse isomorphisms between the category of essential H-modules and the category of essential $\hat{H}$-comodules. These isomorphisms are compatible with tensor products.*

Since it is more convenient to work with essential modules instead of comodules we will usually prefer to consider modules in the sequel.
An essential H-module is called projective if it has the lifting property with respect to surjections of essential H-modules with bounded linear splitting. It is shown in [27] that a bornological quantum group H is projective as a left module over itself. This can be generalized as follows.

Lemma 3.3. *Let H be a bornological quantum group and let V be any essential H-module. Then the essential H-modules $H \hat{\otimes} V$ and $V \hat{\otimes} H$ are projective.*

Proof. Let V_τ be the space V equipped with the trivial H-action induced by the counit. We have a natural H-linear isomorphism $\alpha_l : H \hat{\otimes} V \to H \hat{\otimes} V_\tau$ given by $\alpha_l(x \otimes v) = x_{(1)} \otimes S(x_{(2)}) \cdot v$. Similarly, the map $\alpha_r : V \hat{\otimes} H \to V_\tau \hat{\otimes} H$ given by $\alpha_r(v \otimes x) = S^{-1}(x_{(1)}) \cdot v \otimes x_{(2)}$ is an H-linear isomorphism. Since H is projective this yields the claim. □
Using category language an H-algebra is by definition an algebra in the category H-Mod. We formulate this more explicitly in the following definition.

Definition 3.4. *Let H be a bornological quantum group. An H-algebra is a bornological algebra A which is at the same time an essential H-module such that the multiplication map $A \hat{\otimes} A \to A$ is H-linear.*

If A is an H-algebra we will also speak of an action of H on A. Remark that we do not assume that an algebra has an identity element. The unitarization A^+ of an H-algebra A becomes an H-algebra by considering the trivial action on the extra copy $\mathbb{C}$.
According to theorem 3.2 we can equivalently describe an H-algebra as a bornological algebra A which is at the same time an essential $\hat{H}$-comodule such that the multiplication is $\hat{H}$-colinear.
Under additional assumptions there is another possibility to describe this structure which resembles the definition of a coaction in the setting of C^*-algebras. Let us call an essential bornological algebra A regular if it is equipped with a faithful bounded linear functional and satisfies the approximation property. If A is regular it follows from [27] that the natural bounded linear map $A \hat{\otimes} H \to M(A \hat{\otimes} H)$ is injective.

Definition 3.5. *Let H be a bornological quantum group. An algebra coaction of H on a regular bornological algebra A is an essential algebra homomorphism $\alpha : A \to M(A \hat{\otimes} H)$ such that the coassociativity condition*

$$(\alpha \hat{\otimes} \mathrm{id})\alpha = (\mathrm{id} \hat{\otimes} \Delta)\alpha$$

holds and the maps α_l and α_r from $A \hat{\otimes} H$ to $M(A \hat{\otimes} H)$ given by

$$\alpha_l(a \otimes x) = (1 \otimes x)\alpha(a), \qquad \alpha_r(a \otimes x) = \alpha(a)(1 \otimes x)$$

induce bornological automorphisms of $A \hat{\otimes} H$.

It can be shown that an algebra coaction $\alpha : A \to M(A \hat{\otimes} H)$ on a regular bornological algebra A satisfies $(\mathrm{id} \hat{\otimes} \epsilon)\alpha = \mathrm{id}$. In particular, the map α is always injective.

Proposition 3.6. *Let H be a bornological quantum group and let A be a regular bornological algebra. Then every algebra coaction of $\hat{H}$ on A corresponds to a unique H-algebra structure on A and vice versa.*

Proof. Assume that α is an algebra coaction of $\hat{H}$ on A and define $\eta = \alpha_r$. By definition η is a right $\hat{H}$-linear automorphism of $A \hat{\otimes} \hat{H}$. Moreover we have for $a \in A$ and $f, g \in \hat{H}$ the relation

$$\begin{aligned}(\mathrm{id} \hat{\otimes} \gamma_r)\eta^{12}(\mathrm{id} \hat{\otimes} \gamma_r^{-1})(a \otimes f \otimes g) &= (\mathrm{id} \hat{\otimes} \gamma_r)((\alpha(a) \otimes 1)(1 \otimes \gamma_r^{-1}(f \otimes g))) \\ &= (\mathrm{id} \hat{\otimes} \Delta)(\alpha(a))(1 \otimes \gamma_r \gamma_r^{-1}(f \otimes g)) \\ &= (\alpha \hat{\otimes} \mathrm{id})(\alpha(a))(1 \otimes f \otimes g) \\ &= \eta^{12}\eta^{13}(a \otimes f \otimes g)\end{aligned}$$

in $M(A \hat{\otimes} \hat{H} \hat{\otimes} \hat{H})$. Using that A is regular we deduce that

$$(\mathrm{id} \hat{\otimes} \gamma_r)\eta^{12}(\mathrm{id} \hat{\otimes} \gamma_r^{-1})(a \otimes f \otimes g) = \eta^{12}\eta^{13}(a \otimes f \otimes g)$$

in $A \hat{\otimes} \hat{H} \hat{\otimes} \hat{H}$ and hence η defines a right $\hat{H}$-comodule structure on A. In addition we have

$$\begin{aligned}(\mu \hat{\otimes} \mathrm{id})\eta^{13}\eta^{23}(a \otimes b \otimes f) &= (\mu \hat{\otimes} \mathrm{id})\eta^{13}(a \otimes \alpha(b)(1 \otimes f)) \\ &= \alpha(a)\alpha(b)(1 \otimes f) = \alpha(ab)(1 \otimes f) = \eta(\mu \hat{\otimes} \mathrm{id})(a \otimes b \otimes f)\end{aligned}$$

and it follows that A becomes an H-algebra using this coaction.
Conversely, assume that A is an H-algebra implemented by the coaction $\eta : A \hat{\otimes} \hat{H} \to A \hat{\otimes} \hat{H}$. Define bornological automorphisms η_l and η_r of $A \hat{\otimes} \hat{H}$ by

$$\eta_l = (\mathrm{id} \hat{\otimes} S^{-1})\eta^{-1}(\mathrm{id} \hat{\otimes} S), \qquad \eta_r = \eta.$$

The map η_l is left $\hat{H}$-linear for the action of $\hat{H}$ on the second tensor factor and η_r is right $\hat{H}$-linear. Since η is a compatible with the multiplication we have

$$\eta_r(\mu \hat{\otimes} \mathrm{id}) = (\mu \hat{\otimes} \mathrm{id})\eta_r^{13}\eta_r^{23}$$

and

$$\eta_l(\mu \hat{\otimes} \mathrm{id}) = (\mu \hat{\otimes} \mathrm{id})\eta_l^{23}\eta_l^{13}.$$

In addition one has the equation

$$(\mathrm{id} \hat{\otimes} \mu)\eta_l^{12} = (\mathrm{id} \hat{\otimes} \mu)\eta_r^{13}$$

relating η_l and η_r. These properties of the maps η_l and η_r imply that

$$\alpha(a)(b \otimes f) = \eta_r(a \otimes f)(b \otimes 1), \qquad (b \otimes f)\alpha(a) = (b \otimes 1)\eta_l(a \otimes f)$$

defines an algebra homomorphism α from A to $M(A \hat{\otimes} \hat{H})$. As in the proof of proposition 7.3 in [27] one shows that α is essential. Observe that we may identify the natural map $A \hat{\otimes}_A (A \hat{\otimes} \hat{H}) \to A \hat{\otimes} \hat{H}$ induced by α with $\eta_r^{13} : A \hat{\otimes}_A (A \hat{\otimes} \hat{H} \hat{\otimes}_{\hat{H}} \hat{H}) \to (A \hat{\otimes}_A A) \hat{\otimes} (\hat{H} \hat{\otimes}_{\hat{H}} \hat{H})$ since A is essential.
The maps α_l and α_r associated to the homomorphism α can be identified with η_l and η_r, respectively. Finally, the coaction identity $(\mathrm{id} \hat{\otimes} \gamma_r)\eta^{12}(\mathrm{id} \hat{\otimes} \gamma_r^{-1}) = \eta^{12}\eta^{13}$ implies $(\alpha \hat{\otimes} \mathrm{id})\alpha = (\mathrm{id} \hat{\otimes} \Delta)\alpha$. Hence α defines an algebra coaction of $\hat{H}$ on A.
It follows immediately from the constructions that the two procedures described above are inverse to each other. □

To every H-algebra A one may form the associated crossed product $A \rtimes H$. The underlying bornological vector space of $A \rtimes H$ is $A \hat{\otimes} H$ and the multiplication is defined by the chain of maps

$$A \hat{\otimes} H \hat{\otimes} A \hat{\otimes} H \xrightarrow{\gamma_r^{24}} A \hat{\otimes} H \hat{\otimes} A \hat{\otimes} H \xrightarrow{\mathrm{id} \hat{\otimes} \lambda \hat{\otimes} \mathrm{id}} A \hat{\otimes} A \hat{\otimes} H \xrightarrow{\mu \hat{\otimes} \mathrm{id}} A \hat{\otimes} H$$

where λ denotes the action of H on A. Explicitly, the multiplication in $A \rtimes H$ is given by the formula

$$(a \rtimes x)(b \rtimes y) = a x_{(1)} \cdot b \otimes x_{(2)} y$$

for $a, b \in A$ and $x, y \in H$. On the crossed product $A \rtimes H$ one has the dual action of $\hat{H}$ defined by

$$f \cdot (a \rtimes x) = a \rtimes (f \rightharpoonup x)$$

for all $f \in \hat{H}$. In this way $A \rtimes H$ becomes an $\hat{H}$-algebra. Consequently one may form the double crossed product $A \rtimes H \rtimes \hat{H}$. In the remaing part of this section we discuss the Takesaki-Takai duality isomorphism which clarifies the structure of this algebra.

First we describe a general construction which will also be needed later in connection with stability of equivariant cyclic homology. Assume that V is an essential H-module and that A is an H-algebra. Moreover let $b : V \times V \to \mathbb{C}$ be an equivariant bounded linear map. We define an H-algebra $l(b; A)$ by equipping the space $V \hat{\otimes} A \hat{\otimes} V$ with the multiplication

$$(v_1 \otimes a_1 \otimes w_1)(v_2 \otimes a_2 \otimes w_2) = b(w_1, v_2)\, v_1 \otimes a_1 a_2 \otimes w_2$$

and the diagonal H-action.

As a particular case of this construction consider the space $V = \hat{H}$ with the regular action of H given by $(t \rightharpoonup f)(x) = f(xt)$ and the pairing

$$\beta(f, g) = \hat{\psi}(fg).$$

We write $\mathcal{K}_H$ for the algebra $l(\beta; \mathbb{C})$ and $A \hat{\otimes} \mathcal{K}_H$ for $l(\beta; A)$. Remark that the action on $A \hat{\otimes} \mathcal{K}_H$ is not the diagonal action in general. We denote an element $f \otimes a \otimes g$ in this algebra by $|f\rangle \otimes a \otimes \langle g|$ in the sequel. Using the isomorphism $\hat{\mathcal{F}}_r S^{-1} : \hat{H} \to H$ we identify the above pairing with a pairing $H \times H \to \mathbb{C}$. The corresponding action of H on itself is given by left multiplication and using the normalization $\phi = S(\psi)$ we obtain the formula

$$\beta(x, y) = \beta(S\mathcal{G}_r S(x), S\mathcal{G}_r S(y)) = \beta(\mathcal{F}_l(x), \mathcal{F}_l(y)) = \phi(S^{-1}(y)x) = \psi(S(x)y)$$

for the above pairing expressed in terms of H.

Let H be a bornological quantum group and let A be an H-algebra. We define a bounded linear map $\gamma_A : A \rtimes H \rtimes \hat{H} \to A \hat{\otimes} \mathcal{K}_H$ by

$$\gamma_A(a \rtimes x \rtimes \mathcal{F}_l(y)) = |y_{(1)} S(x_{(2)})\rangle \otimes y_{(2)} S(x_{(1)}) \cdot a \otimes \langle y_{(3)}|$$

and it is easily verified that γ_A is an equivariant bornological isomorphism. In addition, a straightforward computation shows that γ_A is an algebra homomorphism. Consequently we obtain the following analogue of the Takesaki-Takai duality theorem.

Proposition 3.7. *Let H be a bornological quantum group and let A be an H-algebra. Then the map $\gamma_A : A \rtimes H \rtimes \hat{H} \to A \hat{\otimes} \mathcal{K}_H$ is an equivariant algebra isomorphism.*

For algebraic quantum groups a discussion of Takesaki-Takai duality is contained in [11]. More information on similar duality results in the context of Hopf algebras can be found in [21].

If $H = \mathcal{D}(G)$ is the smooth convolution algebra of a locally compact group G then an H-algebra is the same thing as a G-algebra. As a special case of proposition 3.7 one obtains that for every G-algebra A the double crossed product $A \rtimes H \rtimes \hat{H}$ is isomorphic to the G-algebra $A \hat{\otimes} \mathcal{K}_G$ used in [25].

4. Radford's formula

In this section we prove a formula for the fourth power of the antipode in terms of the modular elements of a bornological quantum group and its dual. This formula was obtained by Radford in the setting of finite dimensional Hopf algebras [23].

Let H be a bornological quantum group. If ϕ is a left Haar functional on H there exists a unique multiplier $\delta \in M(H)$ such that

$$(\phi \hat{\otimes} \mathrm{id})\Delta(x) = \phi(x)\delta$$

for all $x \in H$. The multiplier δ is called the modular element of H and measures the failure of ϕ from being right invariant. It is shown in [27] that δ is invertible with inverse $S(\delta) = S^{-1}(\delta) = \delta^{-1}$ and that one has $\Delta(\delta) = \delta \otimes \delta$ as well as $\epsilon(\delta) = 1$. In terms of the dual quantum group the modular element δ defines a character, that is, an essential homomorphism from $\hat{H}$ to $\mathbb{C}$. Similarly, there exists a unique modular element $\hat{\delta} \in M(\hat{H})$ for the dual quantum group which satisfies

$$(\hat{\phi} \hat{\otimes} \mathrm{id})\hat{\Delta}(f) = \hat{\phi}(f)\hat{\delta}$$

for all $f \in \hat{H}$.

The Haar functionals of a bornological quantum group are uniquely determined up to a scalar multiple. In many situations it is convenient to fix a normalization at some point. However, in the discussion below it is not necessary to keep track of the scaling of the Haar functionals. If ω and η are linear functionals we shall write $\omega \equiv \eta$ if there exists a nonzero scalar λ such that $\omega = \lambda\eta$. We use the same notation for elements in a bornological quantum group or linear maps that differ by some nonzero scalar multiple. Moreover we shall identify H with its double dual using Pontrjagin duality.

To begin with observe that the bounded linear functional $\delta \rightharpoonup \phi$ on H defined by

$$(\delta \rightharpoonup \phi)(x) = \phi(x\delta)$$

is faithful and satisfies

$$((\delta \rightharpoonup \phi)\hat{\otimes} \mathrm{id})\Delta(x) = (\phi \hat{\otimes} \mathrm{id})(\Delta(x\delta)(1 \otimes \delta^{-1})) = \phi(x\delta)\delta\delta^{-1} = (\delta \rightharpoonup \phi)(x).$$

It follows that $\delta \rightharpoonup \phi$ is a right Haar functional on H. In a similar way we obtain a right Haar functional $\phi \leftharpoonup \delta$ on H. Hence

$$\delta \rightharpoonup \phi \equiv \psi \equiv \phi \leftharpoonup \delta$$

by uniqueness of the right Haar functional which yields in particular the relations

$$\mathcal{F}_l(x\delta) \equiv \mathcal{G}_l(x), \qquad \mathcal{F}_r(\delta x) \equiv \mathcal{G}_r(x)$$

for the Fourier transform.

According to Pontrjagin duality we have $x = \hat{\mathcal{G}}_l\mathcal{F}_l S(x)$ for all $x \in H$. Since $\hat{\mathcal{G}}_l(f) \equiv \hat{\mathcal{F}}_l(f\hat{\delta})$ for every $f \in \hat{H}$ as well as

$$\begin{aligned}(\mathcal{F}_l(S(x))\hat{\delta})(h) &= \phi(h_{(1)}S(x))\hat{\delta}(h_{(2)}) = \phi(hS(x_{(2)}))\hat{\delta}(\delta S^2(x_{(1)})) \\ &\equiv \mathcal{F}_l(S(x_{(2)}))(h)\hat{\delta}(x_{(1)}) = \mathcal{F}_l(S(x \leftharpoonup \hat{\delta}))(h)\end{aligned}$$

we obtain $x \equiv \hat{\mathcal{F}}_l\mathcal{F}_l S(x \leftharpoonup \hat{\delta})$ or equivalently

$$S^{-1}(\hat{\delta} \rightharpoonup x) \equiv \hat{\mathcal{F}}_l\mathcal{F}_l(x). \tag{4.1}$$

Using equation (4.1) and the formula $x \equiv S^{-1}\hat{\mathcal{F}}_l\mathcal{F}_r(x)$ obtained from Pontrjagin duality we compute

$$\hat{\mathcal{F}}_l\mathcal{F}_l(\hat{\delta}^{-1} \rightharpoonup S^2(x)) \equiv S^{-1}S^2(x) = S(x) \equiv \hat{\mathcal{F}}_l\mathcal{F}_r(x)$$

which implies

$$\mathcal{F}_l(S^2(x)) \equiv \mathcal{F}_r(\hat{\delta} \rightharpoonup x) \tag{4.2}$$

since $\hat{\mathcal{F}}_l$ is an isomorphism. Similarly, we have $\hat{\mathcal{F}}_l S\mathcal{G}_l \equiv \mathrm{id}$ and using $\hat{\mathcal{F}}_l(f) \equiv \hat{\mathcal{G}}_l(f\hat{\delta}^{-1})$ together with

$$\begin{aligned}(S\mathcal{G}_l(x)\hat{\delta}^{-1})(h) &= \psi(S(h_{(1)})x)\hat{\delta}^{-1}(h_{(2)}) \\ &\equiv \psi(S(h)x_{(2)})\hat{\delta}^{-1}(x_{(1)}) = S\mathcal{G}_l(x \leftharpoonup \hat{\delta}^{-1})(h)\end{aligned}$$

we obtain $\hat{\mathcal{G}}_l S\mathcal{G}_l(x) \equiv x \leftharpoonup \hat{\delta}$. According to the relation $\mathcal{F}_l(x\delta) \equiv \mathcal{G}_l(x)$ this may be rewritten as $S^{-1}\hat{\mathcal{F}}_r\mathcal{F}_l(x\delta) \equiv x \leftharpoonup \hat{\delta}$ which in turn yields

$$\hat{\mathcal{F}}_r\mathcal{F}_l(x) \equiv S((x\delta^{-1}) \leftharpoonup \hat{\delta}). \tag{4.3}$$

Due to Pontrjagin duality we have $x = \hat{\mathcal{F}}_r\mathcal{G}_r S(x)$ for all $x \in H$ and using $\mathcal{G}_r(x) \equiv \mathcal{F}_r(\delta x)$ we obtain

$$\hat{\mathcal{F}}_r\mathcal{F}_r(S(x)) \equiv \hat{\mathcal{F}}_r\mathcal{G}_r(\delta^{-1}S(x)) = x\delta. \tag{4.4}$$

According to equation (4.3) and equation (4.4) we have

$$\begin{aligned}\hat{\mathcal{F}}_r\mathcal{F}_l(S^{-2}(\delta^{-1}(x \leftharpoonup \hat{\delta}^{-1})\delta)) &\equiv S((S^{-2}(\delta^{-1}(x \leftharpoonup \hat{\delta}^{-1})\delta)\delta^{-1}) \leftharpoonup \hat{\delta}) \\ &\equiv S(S^{-2}(\delta^{-1}x)) = S^{-1}(\delta^{-1}x) \equiv \hat{\mathcal{F}}_r\mathcal{F}_r(x)\end{aligned}$$

and since $\hat{\mathcal{F}}_r$ is an isomorphism this implies

$$\mathcal{F}_r(S^2(x)) \equiv \mathcal{F}_l(\delta^{-1}(x \leftharpoonup \hat{\delta}^{-1})\delta). \tag{4.5}$$

Assembling these relations we obtain the following result.

Proposition 4.1. *Let H be a bornological quantum group and let δ and $\hat{\delta}$ be the modular elements of H and $\hat{H}$, respectively. Then*

$$S^4(x) = \delta^{-1}(\hat{\delta} \rightharpoonup x \leftharpoonup \hat{\delta}^{-1})\delta$$

for all $x \in H$.

Proof. Using equation (4.2) and equation (4.5) we compute

$$\mathcal{F}_l(S^4(x)) \equiv \mathcal{F}_r(\hat{\delta} \rightharpoonup S^2(x)) = \mathcal{F}_r(S^2(\hat{\delta} \rightharpoonup x)) \equiv \mathcal{F}_l(\delta^{-1}(\hat{\delta} \rightharpoonup x \leftharpoonup \hat{\delta}^{-1})\delta)$$

which implies

$$S^4(x) \equiv \delta^{-1}(\hat{\delta} \rightharpoonup x \leftharpoonup \hat{\delta}^{-1})\delta$$

for all $x \in H$ since $\mathcal{F}_l$ is an isomorphism. The claim follows from the observation that both sides of the previous equation define algebra automorphisms of H. □

5. Anti-Yetter-Drinfeld modules

In this section we introduce the notion of an anti-Yetter-Drinfeld module over a bornological quantum group. Moreover we discuss the concept of a paracomplex. We begin with the definition of an anti-Yetter-Drinfeld module. In the context of Hopf algebras this notion was introduced in [12].

Definition 5.1. *Let H be a bornological quantum group. An H-anti-Yetter-Drinfeld module is an essential left H-module M which is also an essential left $\hat{H}$-module such that*

$$t \cdot (f \cdot m) = (S^2(t_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(t_{(3)})) \cdot (t_{(2)} \cdot m).$$

for all $t \in H, f \in \hat{H}$ and $m \in M$. A homomorphism $\xi : M \to N$ between anti-Yetter-Drinfeld modules is a bounded linear map which is both H-linear and $\hat{H}$-linear.

We will not always mention explicitly the underlying bornological quantum group when dealing with anti-Yetter-Drinfeld modules. Moreover we shall use the abbreviations AYD-module and AYD-map for anti-Yetter-Drinfeld modules and their homomorphisms.
According to theorem 3.2 a left $\hat{H}$-module structure corresponds to a right H-comodule structure. Hence an AYD-module can be described equivalently as a bornological vector space M equipped with an essential H-module structure and an H-comodule structure satisfying a certain compatibility condition. Formally, this compatibility condition can be written down as

$$(t \cdot m)_{(0)} \otimes (t \cdot m)_{(1)} = t_{(2)} \cdot m_{(0)} \otimes t_{(3)} m_{(1)} S(t_{(1)})$$

for all $t \in H$ and $m \in M$.
We want to show that AYD-modules can be interpreted as essential modules over a certain bornological algebra. Following the notation in [12] this algebra will be denoted by $\mathsf{A}(H)$. As a bornological vector space we have $\mathsf{A}(H) = \hat{H} \hat{\otimes} H$ and the multiplication is defined by the formula

$$(f \otimes x) \cdot (g \otimes y) = f(S^2(x_{(1)}) \rightharpoonup g \leftharpoonup S^{-1}(x_{(3)})) \otimes x_{(2)} y.$$

There exists an algebra homomorphism $\iota_H : H \to M(\mathsf{A}(H))$ given by

$$\iota_H(x) \cdot (g \otimes y) = S^2(x_{(1)}) \rightharpoonup g \leftharpoonup S^{-1}(x_{(3)}) \otimes x_{(2)} y$$

and

$$(g \otimes y) \cdot \iota_H(x) = g \otimes yx.$$

It is easily seen that ι_H is injective. Similarly, there is an injective algebra homomorphism $\iota_{\hat{H}} : \hat{H} \to M(\mathsf{A}(H))$ given by

$$\iota_{\hat{H}}(f) \cdot (g \otimes y) = fg \otimes y$$

as well as

$$(g \otimes y) \cdot \iota_{\hat{H}}(f) = g(S^2(y_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(y_{(3)})) \otimes y_{(2)}$$

and we have the following result.

Proposition 5.2. *For every bornological quantum group H the bornological algebra $\mathsf{A}(H)$ is essential.*

Proof. The homomorphism $\iota_{\hat{H}}$ induces on $\mathsf{A}(H)$ the structure of an essential left $\hat{H}$-module. Similarly, the space $\mathsf{A}(H)$ becomes an essential left H-module using the homomorphism ι_H. In fact, if we write $\hat{H}_\tau$ for the space $\hat{H}$ equipped with the trivial H-action the map $c : \mathsf{A}(H) \to \hat{H}_\tau \hat{\otimes} H$ given by

$$c(f \otimes x) = S(x_{(1)}) \rightharpoonup f \leftharpoonup x_{(3)} \otimes x_{(2)}$$

is an H-linear isomorphism. Actually, the actions of $\hat{H}$ and H on $\mathsf{A}(H)$ are defined in such a way that $\mathsf{A}(H)$ becomes an AYD-module. Using the canonical isomorphism $H \hat{\otimes}_H \mathsf{A}(H) \cong \mathsf{A}(H)$ one obtains an essential $\hat{H}$-module structure on $H \hat{\otimes}_H \mathsf{A}(H)$ given explicitly by the formula

$$f \cdot (x \otimes g \otimes y) = x_{(2)} \otimes (S(x_{(1)}) \rightharpoonup f \leftharpoonup x_{(3)}) g \otimes y.$$

Correspondingly, we obtain a natural isomorphism $\hat{H} \hat{\otimes}_{\hat{H}} (H \hat{\otimes}_H \mathsf{A}(H)) \cong \mathsf{A}(H)$. It is straighforward to verify that the identity map $\hat{H} \hat{\otimes} (H \hat{\otimes} \mathsf{A}(H)) \cong \hat{H} \hat{\otimes} H \hat{\otimes} \hat{H} \hat{\otimes} H \to \mathsf{A}(H) \hat{\otimes} \mathsf{A}(H)$ induces an isomorphism $\hat{H} \hat{\otimes}_{\hat{H}} (H \hat{\otimes}_H \mathsf{A}(H)) \to \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} \mathsf{A}(H)$. This yields the claim. □

We shall now characterize AYD-modules as essential modules over $\mathsf{A}(H)$.

Proposition 5.3. *Let H be a bornological quantum group. Then the category of AYD-modules over H is isomorphic to the category of essential left $\mathsf{A}(H)$-modules.*

Proof. Let $M \cong \mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)} M$ be an essential $\mathsf{A}(H)$-module. Then we obtain a left H-module structure and a left $\hat{H}$-module structure on M using the canonical homomorphisms $\iota_H : H \to M(\mathsf{A}(H))$ and $\iota_{\hat{H}} : \hat{H} \to M(\mathsf{A}(H))$. Since the action of H on $\mathsf{A}(H)$ is essential we have natural isomorphisms

$$H\hat{\otimes}_H M \cong H\hat{\otimes}_H \mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)} M \cong \mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)} M \cong M$$

and hence M is an essential H-module. Similarly we have

$$\hat{H}\hat{\otimes}_{\hat{H}} M \cong \hat{H}\hat{\otimes}_{\hat{H}} \mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)}\hat{\otimes} M \cong \mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)} M \cong M$$

since $\mathsf{A}(H)$ is an essential $\hat{H}$-module. These module actions yield the structure of an AYD-module on M.
Conversely, assume that M is an H-AYD-module. Then we obtain an $\mathsf{A}(H)$-module structure on M by setting

$$(f \otimes t) \cdot m = f \cdot (t \cdot m)$$

for $f \in \hat{H}$ and $t \in H$. Since M is an essential H-module we have a natural isomorphism $H\hat{\otimes}_H M \cong M$. As in the proof of proposition 5.2 we obtain an induced essential $\hat{H}$-module structure on $H\hat{\otimes}_H M$ and canonical isomorphisms $\mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)} M \cong \hat{H}\hat{\otimes}_{\hat{H}}(H\hat{\otimes}_H M) \cong M$. It follows that M is an essential $\mathsf{A}(H)$-module.
The previous constructions are compatible with morphisms and it is easy to check that they are inverse to each other. This yields the assertion. □
There is a canonical operator T on every AYD-module which plays a crucial role in equivariant cyclic homology. In order to define this operator it is convenient to pass from $\hat{H}$ to H in the first tensor factor of $\mathsf{A}(H)$. More precisely, consider the bornological isomorphism $\lambda : \mathsf{A}(H) \to H\hat{\otimes}H$ given by

$$\lambda(f \otimes y) = \hat{\mathcal{F}}_l(f) \otimes y \leftharpoonup \hat{\delta}^{-1}$$

where $\hat{\delta} \in M(\hat{H})$ is the modular function of $\hat{H}$. The inverse map is given by

$$\lambda^{-1}(x \otimes y) = S\mathcal{G}_l(x) \otimes y \leftharpoonup \hat{\delta}.$$

It is straightforward to check that the left H-action on $\mathsf{A}(H)$ corresponds to

$$t \cdot (x \otimes y) = t_{(3)} x S(t_{(1)}) \otimes t_{(2)} y$$

and the left $\hat{H}$-action becomes

$$f \cdot (x \otimes y) = (f \rightharpoonup x) \otimes y$$

under this isomorphism. The right H-action is identified with

$$(x \otimes y) \cdot t = x \otimes y(t \leftharpoonup \hat{\delta}^{-1})$$

and the right $\hat{H}$-action corresponds to

$$(x \otimes y) \cdot g = x_{(2)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(S^{-2}(x_{(1)})\delta) \otimes \hat{\delta}(y_{(1)}) y_{(3)} \leftharpoonup \hat{\delta}^{-1}$$

where δ is the modular function of H. Using this description of $\mathsf{A}(H)$ we obtain the following result.

Proposition 5.4. *The bounded linear map $T : \mathsf{A}(H) \to \mathsf{A}(H)$ defined by*

$$T(x \otimes y) = x_{(2)} \otimes S^{-1}(x_{(1)}) y$$

is an isomorphism of $\mathsf{A}(H)$-bimodules.

Proof. It is evident that T is a bornological isomorphism with inverse given by $T^{-1}(x \otimes y) = x_{(2)} \otimes x_{(1)}y$. We compute

$$\begin{aligned} T(t \cdot (x \otimes y)) &= T(t_{(3)}xS(t_{(1)}) \otimes t_{(2)}y) \\ &= t_{(5)}x_{(2)}S(t_{(1)}) \otimes S^{-1}(t_{(4)}x_{(1)}S(t_{(2)}))t_{(3)}y \\ &= t_{(3)}x_{(2)}S(t_{(1)}) \otimes t_{(2)}S^{-1}(x_{(1)})y \\ &= t \cdot T(x \otimes y) \end{aligned}$$

and it is clear that T is left $\hat{H}$-linear. Consequently T is a left $\mathsf{A}(H)$-linear map. Similarly, we have

$$T((x \otimes y) \cdot t) = T(x \otimes y(t \leftharpoonup \hat{\delta}^{-1})) = x_{(2)} \otimes S^{-1}(x_{(1)})y(t \leftharpoonup \hat{\delta}^{-1}) = T(x \otimes y) \cdot t$$

and hence T is right H-linear. In order to prove that T is right $\hat{H}$-linear we compute

$$\begin{aligned} &T^{-1}((x \otimes y) \cdot g) \\ &= T^{-1}(x_{(2)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(S^{-2}(x_{(1)})\delta) \otimes \hat{\delta}(y_{(1)})y_{(3)} \leftharpoonup \hat{\delta}^{-1}) \\ &= x_{(3)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(S^{-2}(x_{(1)})\delta) \otimes \hat{\delta}(y_{(1)})x_{(2)}(y_{(3)} \leftharpoonup \hat{\delta}^{-1}) \\ &= x_{(3)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(S^{-2}(\hat{\delta} \rightharpoonup x_{(1)})\delta) \otimes \hat{\delta}(y_{(1)})(x_{(2)}y_{(3)}) \leftharpoonup \hat{\delta}^{-1} \end{aligned}$$

which according to proposition 4.1 is equal to

$$\begin{aligned} &= x_{(3)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(\delta S^2(x_{(1)} \leftharpoonup \hat{\delta})) \otimes \hat{\delta}(y_{(1)})(x_{(2)}y_{(3)}) \leftharpoonup \hat{\delta}^{-1} \\ &= x_{(6)}(S^2(x_{(2)})S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)})S^{-1}(x_{(4)}))(S^{-2}(x_{(5)})\delta) \\ &\qquad \otimes \hat{\delta}(x_{(1)}y_{(1)})(x_{(3)}y_{(3)}) \leftharpoonup \hat{\delta}^{-1} \\ &= (x_{(2)} \otimes x_{(1)}y) \cdot g \\ &= T^{-1}(x \otimes y) \cdot g. \end{aligned}$$

We conclude that T is a right $\mathsf{A}(H)$-linear map as well. □

If M is an arbitrary AYD-module we define $T : M \to M$ by

$$T(F \otimes m) = T(F) \otimes m$$

for $F \otimes m \in \mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)}M$. Due to proposition 5.3 and lemma 5.4 this definition makes sense.

Proposition 5.5. *The operator T defines a natural isomorphism $T : \mathrm{id} \to \mathrm{id}$ of the identity functor on the category of* AYD*-modules.*

Proof. It is clear from the construction that $T : M \to M$ is an isomorphism for all M. If $\xi : M \to N$ is an AYD-map the equation $T\xi = \xi T$ follows easily after identifying ξ with the map $\mathrm{id}\hat{\otimes}\xi : \mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)}M \to \mathsf{A}(H)\hat{\otimes}_{\mathsf{A}(H)}N$. This yields the assertion. □

If the bornological quantum group H is unital one may construct the operator T on an AYD-module M directly from the coaction $M \to M\hat{\otimes}H$ corresponding to the action of $\hat{H}$. More precisely, one has the formula

$$T(m) = S^{-1}(m_{(1)}) \cdot m_{(0)}$$

for every $m \in M$.

Using the terminology of [25] it follows from proposition 5.5 that the category of AYD-modules is a para-additive category in a natural way. This leads in particular to the concept of a paracomplex of AYD-modules.

Definition 5.6. *A paracomplex $C = C_0 \oplus C_1$ consists of* AYD*-modules C_0 and C_1 and* AYD*-maps $\partial_0 : C_0 \to C_1$ and $\partial_1 : C_1 \to C_0$ such that*

$$\partial^2 = \mathrm{id} - T$$

where the differential $\partial : C \to C_1 \oplus C_0 \cong C$ is the composition of $\partial_0 \oplus \partial_1$ with the canonical flip map.

The morphism ∂ in a paracomplex is called a differential although it usually does not satisfy the relation $\partial^2 = 0$. As for ordinary complexes one defines chain maps between paracomplexes and homotopy equivalences. We always assume that such maps are compatible with the AYD-module structures. Let us point out that it does not make sense to speak about the homology of a paracomplex in general.
The paracomplexes we will work with arise from paramixed complexes in the following sense.

Definition 5.7. *A paramixed complex M is a sequence of AYD-modules M_n together with AYD-maps b of degree -1 and B of degree $+1$ satisfying $b^2 = 0$, $B^2 = 0$ and*

$$[b, B] = bB + Bb = \mathrm{id} - T.$$

If T is equal to the identity operator on M this reduces of course to the definition of a mixed complex.

6. Equivariant differential forms

In this section we define equivariant differential forms and the equivariant X-complex. Moreover we discuss the properties of the periodic tensor algebra of an H-algebra. These are the main ingredients in the construction of equivariant cyclic homology.
Let H be a bornological quantum group. If A is an H-algebra we obtain a left action of H on the space $H \hat{\otimes} \Omega^n(A)$ by

$$t \cdot (x \otimes \omega) = t_{(3)} x S(t_{(1)}) \otimes t_{(2)} \cdot \omega$$

for $t, x \in H$ and $\omega \in \Omega^n(A)$. Here $\Omega^n(A) = A^+ \hat{\otimes} A^{\hat{\otimes} n}$ for $n > 0$ is the space of noncommutative n-forms over A with the diagonal H-action. For $n = 0$ one defines $\Omega^0(A) = A$. There is a left action of the dual quantum group $\hat{H}$ on $H \hat{\otimes} \Omega^n(A)$ given by

$$f \cdot (x \otimes \omega) = (f \rightharpoonup x) \otimes \omega = f(x_{(2)}) x_{(1)} \otimes \omega.$$

By definition, the equivariant n-forms $\Omega^n_H(A)$ are the space $H \hat{\otimes} \Omega^n(A)$ together with the H-action and the $\hat{H}$-action described above. We compute

$$\begin{aligned} t \cdot (f \cdot (x \otimes \omega)) &= t \cdot (f(x_{(2)}) x_{(1)} \otimes \omega) \\ &= f(x_{(2)}) t_{(3)} x_{(1)} S(t_{(1)}) \otimes t_{(2)} \cdot \omega \\ &= (S^2(t_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(t_{(5)})) \cdot (t_{(4)} x S(t_{(2)}) \otimes t_{(3)} \cdot \omega) \\ &= (S^2(t_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(t_{(3)})) \cdot (t_{(2)} \cdot (x \otimes \omega)) \end{aligned}$$

and deduce that $\Omega^n_H(A)$ is an H-AYD-module. We let $\Omega_H(A)$ be the direct sum of the spaces $\Omega^n_H(A)$.
Now we define operators d and b_H on $\Omega_H(A)$ by

$$d(x \otimes \omega) = x \otimes d\omega$$

and

$$b_H(x \otimes \omega da) = (-1)^{|\omega|}(x \otimes \omega a - x_{(2)} \otimes (S^{-1}(x_{(1)}) \cdot a)\omega).$$

The map b_H should be thought of as a twisted version of the usual Hochschild operator. We compute

$$\begin{aligned} b_H^2(x\otimes \omega dadb) &= (-1)^{|\omega|+1} b_H(x\otimes \omega dab - x_{(2)}\otimes (S^{-1}(x_{(1)})\cdot b)\omega da)\\ &= (-1)^{|\omega|+1} b_H(x\otimes \omega d(ab) - x\otimes \omega adb - x_{(2)}\otimes (S^{-1}(x_{(1)})\cdot b)\omega da)\\ &= -(x\otimes \omega ab - x_{(2)}\otimes S^{-1}(x_{(1)})\cdot (ab)\omega - x\otimes \omega ab + x_{(2)}\otimes (S^{-1}(x_{(1)})\cdot b)\omega a\\ &\qquad - x_{(2)}\otimes (S^{-1}(x_{(1)})\cdot b)\omega a + x_{(2)}\otimes S^{-1}(x_{(1)})\cdot (ab)\omega) = 0 \end{aligned}$$

which shows that b_H^2 is a differential as in the nonequivariant situation. Let us discuss the compatibility of d and b_H with the AYD-module structure. It is easy to check that d is an AYD-map and that the operator b_H is $\hat{H}$-linear. Moreover we compute

$$\begin{aligned} b_H(t\cdot (x\otimes \omega da)) &= (-1)^{|\omega|}(t_{(4)}xS(t_{(1)})\otimes (t_{(2)}\cdot \omega)(t_{(3)}\cdot a)\\ &\qquad - t_{(6)}x_{(2)}S(t_{(1)})\otimes (S^{-1}(t_{(5)}x_{(1)}S(t_{(2)}))t_{(4)}\cdot a)(t_{(3)}\cdot \omega))\\ &= (-1)^{|\omega|}(t_{(3)}xS(t_{(1)})\otimes t_{(2)}\cdot (\omega a) - t_{(4)}x_{(2)}S(t_{(1)})\otimes (t_{(2)}S^{-1}(x_{(1)})\cdot a)(t_{(3)}\cdot \omega))\\ &= t\cdot b_H(x\otimes \omega da) \end{aligned}$$

and deduce that b_H is an AYD-map as well.

Similar to the non-equivariant case we use d and b_H to define an equivariant Karoubi operator κ_H and an equivariant Connes operator B_H by

$$\kappa_H = 1 - (b_H d + d b_H)$$

and

$$B_H = \sum_{j=0}^{n} \kappa_H^j d$$

on $\Omega_H^n(A)$. Let us record the following explicit formulas. For $n>0$ we have

$$\kappa_H(x\otimes \omega da) = (-1)^{n-1} x_{(2)}\otimes (S^{-1}(x_{(1)})\cdot da)\omega$$

on $\Omega_H^n(A)$ and in addition $\kappa_H(x\otimes a) = x_{(2)}\otimes S^{-1}(x_{(1)})\cdot a$ on $\Omega_H^0(A)$. For the Connes operator we compute

$$B_H(x\otimes a_0da_1\cdots da_n) = \sum_{i=0}^{n} (-1)^{ni} x_{(2)}\otimes S^{-1}(x_{(1)})\cdot (da_{n+1-i}\cdots da_n).da_0\cdots da_{n-i}$$

Furthermore, the operator T is given by

$$T(x\otimes \omega) = x_{(2)}\otimes S^{-1}(x_{(1)})\cdot \omega$$

on equivariant differential forms. Observe that all operators constructed so far are AYD-maps and thus commute with T according to proposition 5.5.

Lemma 6.1. *On $\Omega_H^n(A)$ the following relations hold:*

a) $\kappa_H^{n+1} d = Td$
b) $\kappa_H^n = T + b_H\kappa_H^n d$
c) $\kappa_H^n b_H = b_H T$
d) $\kappa_H^{n+1} = (\mathrm{id} - db_H)T$
e) $(\kappa_H^{n+1} - T)(\kappa_H^n - T) = 0$
f) $B_H b_H + b_H B_H = \mathrm{id} - T$

Proof. a) follows from the explicit formula for κ_H. b) We compute

$$\begin{aligned} \kappa_H^n(x\otimes a_0da_1\cdots da_n) &= x_{(2)}\otimes S^{-1}(x_{(1)})\cdot (da_1\cdots da_n)a_0\\ &= x_{(2)}\otimes S^{-1}(x_{(1)})\cdot (a_0da_1\cdots da_n) + (-1)^n b_H(x_{(2)}\otimes S^{-1}(x_{(1)})\cdot (da_1\cdots da_n)da_0)\\ &= x_{(2)}\otimes S^{-1}(x_{(1)})\cdot (a_0da_1\cdots da_n) + b_H\kappa_H^n d(x\otimes a_0da_1\cdots da_n) \end{aligned}$$

which yields the claim. c) follows by applying b_H to both sides of b). d) Apply κ_H to b) and use a). e) is a consequence of b) and d). f) We compute

$$B_H b_H + b_H B_H = \sum_{j=0}^{n-1} \kappa_H^j d b_H + \sum_{j=0}^{n} b_H \kappa_H^j d = \sum_{j=0}^{n-1} \kappa_H^j (d b_H + b_H d) + \kappa_H^n b_H d$$
$$= \mathrm{id} - \kappa_H^n (1 - b_H d) = \mathrm{id} - \kappa_H^n (\kappa_H + d b_H) = \mathrm{id} - T + d b_H T - T d b_H = \mathrm{id} - T$$

where we use d) and b). □

From the definition of B_H and the fact that $d^2 = 0$ we obtain $B_H^2 = 0$. Let us summarize this discussion as follows.

Proposition 6.2. *Let A be an H-algebra. The space $\Omega_H(A)$ of equivariant differential forms is a paramixed complex in the category of* AYD*-modules.*

We remark that the definition of $\Omega_H(A)$ for $H = \mathcal{D}(G)$ differs slightly from the definition of $\Omega_G(A)$ in [25] if the locally compact group G is not unimodular. However, this does not affect the definition of the equivariant homology groups.

In the sequel we will drop the subscripts when working with the operators on $\Omega_H(A)$ introduced above. For instance, we shall simply write b instead of b_H and B instead of B_H.

The n-th level of the Hodge tower associated to $\Omega_H(A)$ is defined by

$$\theta^n \Omega_H(A) = \bigoplus_{j=0}^{n-1} \Omega_H^j(A) \oplus \Omega_H^n(A)/b(\Omega_H^{n+1}(A)).$$

Using the grading into even and odd forms we see that $\theta^n \Omega_H(A)$ together with the boundary operator $B + b$ becomes a paracomplex. By definition, the Hodge tower $\theta\Omega_H(A)$ of A is the projective system $(\theta^n \Omega_H(A))_{n \in \mathbb{N}}$ obtained in this way.

From a conceptual point of view it is convenient to work with pro-categories in the sequel. The pro-category $\mathrm{pro}(\mathcal{C})$ over a category $\mathcal{C}$ consists of projective systems in $\mathcal{C}$. A pro-H-algebra is simply an algebra in the category $\mathrm{pro}(H\text{-}\mathsf{Mod})$. For instance, every H-algebra becomes a pro-H-algebra by viewing it as a constant projective system. More information on the use of pro-categories in connection with cyclic homology can be found in [10], [25].

Definition 6.3. *Let A be a pro-H-algebra. The equivariant X-complex $X_H(A)$ of A is the paracomplex $\theta^1 \Omega_H(A)$. Explicitly, we have*

$$X_H(A)\colon\ \Omega_H^0(A) \underset{b}{\overset{d}{\rightleftarrows}} \Omega_H^1(A)/b(\Omega_H^2(A)).$$

We are interested in particular in the equivariant X-complex of the periodic tensor algebra $\mathcal{T}A$ of an H-algebra A. The periodic tensor algebra $\mathcal{T}A$ is the even part of $\theta\Omega(A)$ equipped with the Fedosov product given by

$$\omega \circ \eta = \omega\eta - (-1)^{|\omega|} d\omega d\eta$$

for homogenous forms ω and η. The natural projection $\theta\Omega(A) \to A$ restricts to an equivariant homomorphism $\tau_A : \mathcal{T}A \to A$ and we obtain an extension

$$\mathcal{J}A \rightarrowtail \mathcal{T}A \xrightarrow{\tau_A} A$$

of pro-H-algebras.

The main properties of the pro-algebras $\mathcal{T}A$ and $\mathcal{J}A$ are explained in [20], [25]. Let us recall some terminology. We write $\mu^n : N^{\hat{\otimes} n} \to N$ for the iterated multiplication in a pro-H-algebra N. Then N is called locally nilpotent if for every equivariant pro-linear map $f : N \to C$ with constant range C there exists $n \in \mathbb{N}$ such that

$f\mu^n = 0$. It is straightforward to check that the pro-H-algebra $\mathcal{J}A$ is locally nilpotent.

An equivariant pro-linear map $l : A \to B$ between pro-H-algebras is called a lonilcur if its curvature $\omega_l : A\hat{\otimes}A \to B$ defined by $\omega_l(a,b) = l(ab) - l(a)l(b)$ is locally nilpotent, that is, if for every equivariant pro-linear map $f : B \to C$ with constant range C there exists $n \in \mathbb{N}$ such that $f\mu_B^n \omega_l^{\hat{\otimes} n} = 0$. The term lonilcur is an abbreviation for "equivariant pro-linear map with *lo*cally *nil*potent *cur*vature". Since $\mathcal{J}A$ is locally nilpotent the natural map $\sigma_A : A \to \mathcal{T}A$ is a lonilcur.

Proposition 6.4. *Let A be an H-algebra. The pro-H-algebra $\mathcal{T}A$ and the lonilcur $\sigma_A : A \to \mathcal{T}A$ satisfy the following universal property. If $l : A \to B$ is a lonilcur into a pro-H-algebra B there exists a unique equivariant homomorphism $[[l]] : \mathcal{T}A \to B$ such that $[[l]]\sigma_A = l$.*

An important ingredient in the Cuntz-Quillen approach to cyclic homology [8], [9], [10] is the concept of a quasifree pro-algebra. The same is true in the equivariant theory.

Definition 6.5. *A pro-H-algebra R is called H-equivariantly quasifree if there exists an equivariant splitting homomorphism $R \to \mathcal{T}R$ for the projection τ_R.*

We state some equivalent descriptions of equivariantly quasifree pro-H-algebras.

Theorem 6.6. *Let H be a bornological quantum group and let R be a pro-H-algebra. The following conditions are equivalent:*

a) R is H-equivariantly quasifree.

b) There exists an equivariant pro-linear map $\nabla : \Omega^1(R) \to \Omega^2(R)$ satisfying

$$\nabla(a\omega) = a\nabla(\omega), \qquad \nabla(\omega a) = \nabla(\omega)a - \omega da$$

for all $a \in R$ and $\omega \in \Omega^1(R)$.

c) There exists a projective resolution $0 \to P_1 \to P_0 \to R^+$ of the R-bimodule R^+ of length 1 in $\mathrm{pro}(H\text{-}\mathsf{Mod})$.

A map $\nabla : \Omega^1(R) \to \Omega^2(R)$ satisfying condition b) in theorem 6.6 is also called an equivariant graded connection on $\Omega^1(R)$.

We have the following basic examples of quasifree pro-H-algebras.

Proposition 6.7. *Let A be any H-algebra. The periodic tensor algebra $\mathcal{T}A$ is H-equivariantly quasifree.*

An important result in theory of Cuntz and Quillen relates the X-complex of the periodic tensor algebra $\mathcal{T}A$ to the standard complex of A constructed using noncommutative differential forms. The comparison between the equivariant X-complex and equivariant differential forms is carried out in the same way as in the group case [25].

Proposition 6.8. *There is a natural isomorphism $X_H(\mathcal{T}A) \cong \theta\Omega_H(A)$ such that the differentials of the equivariant X-complex correspond to*

$$\partial_1 = b - (\mathrm{id} + \kappa)d \qquad \text{on } \theta\Omega_H^{odd}(A)$$

$$\partial_0 = -\sum_{j=0}^{n-1} \kappa^{2j} b + B \qquad \text{on } \Omega_H^{2n}(A).$$

Theorem 6.9. *Let H be a bornological quantum group and let A be an H-algebra. Then the paracomplexes $\theta\Omega_H(A)$ and $X_H(\mathcal{T}A)$ are homotopy equivalent.*

For the proof of theorem 6.9 it suffices to observe that the corresponding arguments in [25] are based on the relations obtained in proposition 6.1.

7. Equivariant periodic cyclic homology

In this section we define equivariant periodic cyclic homology for bornological quantum groups.

Definition 7.1. *Let H be a bornological quantum group and let A and B be H-algebras. The equivariant periodic cyclic homology of A and B is*

$$HP^H_*(A,B) = H_*(\mathrm{Hom}_{\mathsf{A}(H)}(X_H(\mathcal{T}(A \rtimes H \rtimes \hat{H})), X_H(\mathcal{T}(B \rtimes H \rtimes \hat{H})))).$$

We write $\mathrm{Hom}_{\mathsf{A}(H)}$ for the space of AYD-maps and consider the usual differential for a Hom-complex in this definition. Using proposition 5.5 it is straightforward to check that this yields indeed a complex. Remark that both entries in the above Hom-complex are only paracomplexes.
Let us consider the special case that $H = \mathcal{D}(G)$ is the smooth group algebra of a locally compact group G. In this situation the definition of HP^H_* reduces to the definition of HP^G_* given in [25]. This is easily seen using the Takesaki-Takai isomorphism obtained in proposition 3.7 and the results from [27].
As in the group case HP^H_* is a bifunctor, contravariant in the first variable and covariant in the second variable. We define $HP^H_*(A) = HP^H_*(\mathbb{C}, A)$ to be the equivariant periodic cyclic homology of A and $HP^*_H(A) = HP^H_*(A, \mathbb{C})$ to be equivariant periodic cyclic cohomology. There is a natural associative product

$$HP^H_i(A,B) \times HP^H_j(B,C) \to HP^H_{i+j}(A,C), \qquad (x,y) \mapsto x \cdot y$$

induced by the composition of maps. Every equivariant homomorphism $f : A \to B$ defines an element in $HP^H_0(A,B)$ denoted by $[f]$. The element $[\mathrm{id}] \in HP^H_0(A,A)$ is denoted 1 or 1_A. An element $x \in HP^H_*(A,B)$ is called invertible if there exists an element $y \in HP^H_*(B,A)$ such that $x \cdot y = 1_A$ and $y \cdot x = 1_B$. An invertible element of degree zero is called an HP^H-equivalence. Such an element induces isomorphisms $HP^H_*(A,D) \cong HP^H_*(B,D)$ and $HP^H_*(D,A) \cong HP^H_*(D,B)$ for all H-algebras D.

8. Homotopy invariance, stability and excision

In this section we show that equivariant periodic cyclic homology is homotopy invariant, stable and satisfies excision in both variables. Since the arguments carry over from the group case with minor modifications most of the proofs will only be sketched. More details can be found in [25].
We begin with homotopy invariance. Let B be a pro-H-algebra and consider the Fréchet algebra $C^\infty[0,1]$ of smooth functions on the interval $[0,1]$. We denote by $B[0,1]$ the pro-H-algebra $B \hat{\otimes} C^\infty[0,1]$ where the action on $C^\infty[0,1]$ is trivial. A (smooth) equivariant homotopy is an equivariant homomorphism $\Phi : A \to B[0,1]$ of H-algebras. Evaluation at the point $t \in [0,1]$ yields an equivariant homomorphism $\Phi_t : A \to B$. Two equivariant homomorphisms from A to B are called equivariantly homotopic if they can be connected by an equivariant homotopy.

Theorem 8.1 (Homotopy invariance)**.** *Let A and B be H-algebras and let $\Phi : A \to B[0,1]$ be a smooth equivariant homotopy. Then the elements $[\Phi_0]$ and $[\Phi_1]$ in $HP^H_0(A,B)$ are equal. Hence the functor HP^H_* is homotopy invariant in both variables with respect to smooth equivariant homotopies.*

Recall that $\theta^2\Omega_H(A)$ is the paracomplex $\Omega^0_H(A) \oplus \Omega^1_H(A) \oplus \Omega^2_H(A)/b(\Omega^3_H(A))$ with the usual differential $B + b$ and the grading into even and odd forms for any pro-H-algebra A. There is a natural chain map $\xi^2 : \theta^2\Omega_H(A) \to X_H(A)$.

Proposition 8.2. *Let A be an equivariantly quasifree pro-H-algebra. Then the map $\xi^2 : \theta^2\Omega_H(A) \to X_H(A)$ is a homotopy equivalence.*

A homotopy inverse is constructed using an equivariant connection for $\Omega^1(A)$. Now let $\Phi : A \to B[0,1]$ be an equivariant homotopy. The derivative of Φ is an equivariant linear map $\Phi' : A \to B[0,1]$. If we view $B[0,1]$ as a bimodule over itself the map Φ' is a derivation with respect to Φ in the sense that $\Phi'(ab) = \Phi'(a)\Phi(b) + \Phi(a)\Phi'(b)$ for $a, b \in A$. We define an AYD-map $\eta : \Omega^n_H(A) \to \Omega^{n-1}_H(B)$ for $n > 0$ by

$$\eta(x \otimes a_0 da_1 \dots da_n) = \int_0^1 x \otimes \Phi_t(a_0)\Phi'_t(a_1) d\Phi_t(a_2) \cdots d\Phi_t(a_n) dt$$

and an explicit calculation yields the following result.

Lemma 8.3. *We have $X_H(\Phi_1)\xi^2 - X_H(\Phi_0)\xi^2 = \partial\eta + \eta\partial$. Hence the chain maps $X_H(\Phi_t)\xi^2 : \theta^2\Omega_H(A) \to X_H(B)$ for $t = 0, 1$ are homotopic.*

Using the map Φ we obtain an equivariant homotopy $A \hat{\otimes} \mathcal{K}_H \to (B \hat{\otimes} \mathcal{K}_H)[0,1]$ which induces an equivariant homomorphism $\mathcal{T}(A \hat{\otimes} \mathcal{K}_H) \to \mathcal{T}((B \hat{\otimes} \mathcal{K}_H)[0,1])$. Together with the obvious homomorphism $\mathcal{T}((B \hat{\otimes} \mathcal{K}_H)[0,1]) \to \mathcal{T}(B \hat{\otimes} \mathcal{K}_H)[0,1]$ this yields an equivariant homotopy $\Psi : \mathcal{T}(A \hat{\otimes} \mathcal{K}_H) \to \mathcal{T}(B \hat{\otimes} \mathcal{K}_H)[0,1]$. Since $\mathcal{T}(A \hat{\otimes} \mathcal{K}_H)$ is equivariantly quasifree we can apply proposition 8.2 and lemma 8.3 to obtain $[\Phi_0] = [\Phi_1] \in HP^H_0(A, B)$. This finishes the proof of theorem 8.1.

Homotopy invariance has several important consequences. Let us call an extension $0 \to J \to R \to A \to 0$ of pro-H-algebras with equivariant pro-linear splitting a universal locally nilpotent extension of A if J is locally nilpotent and R is equivariantly quasifree. In particular, $0 \to \mathcal{J}A \to \mathcal{T}A \to A \to 0$ is a universal locally nilpotent extension of A. Using homotopy invariance one shows that HP^H_* can be computed using arbitrary universal locally nilpotent extensions.

Let us next study stability. One has to be slightly careful to formulate correctly the statement of the stability theorem since the tensor product of two H-algebras is no longer an H-algebra in general.

Let H be a bornological quantum group and assume that we are given an essential H-module V together with an equivariant bilinear pairing $b : V \times V \to \mathbb{C}$. Moreover let A be an H-algebra. Recall from section 3 that $l(b; A) = V \hat{\otimes} A \hat{\otimes} V$ is the H-algebra with multiplication

$$(v_1 \otimes a_1 \otimes w_1)(v_2 \otimes a_2 \otimes w_2) = b(w_1, v_2)\, v_1 \otimes a_1 a_2 \otimes w_2$$

and the diagonal H-action. We call the pairing b admissible if there exists an H-invariant vector $u \in V$ such that $b(u, u) = 1$. In this case the map $\iota_A : A \to l(b; A)$ given by $\iota(a) = u \otimes a \otimes u$ is an equivariant homomorphism.

Theorem 8.4. *Let H be a bornological quantum group and let A be an H-algebra. For every admissible equivariant bilinear pairing $b : V \times V \to \mathbb{C}$ the map $\iota : A \to l(b; A)$ induces an invertible element $[\iota_A] \in H_0(\mathrm{Hom}_{\mathsf{A}(H)}(X_H(\mathcal{T}A), X_H(\mathcal{T}(l(b; A)))))$.*

Proof. The canonical map $l(b; A) \to l(b; \mathcal{T}A)$ induces an equivariant homomorphism $\lambda_A : \mathcal{T}(l(b; A)) \to l(b; \mathcal{T}A)$. Define the map $tr_A : X_H(l(b; \mathcal{T}A)) \to X_H(\mathcal{T}A)$ by

$$tr_A(x \otimes (v_0 \otimes a_0 \otimes w_0)) = b(S^{-1}(x_{(1)}) \cdot w_0, v_0)\, x_{(2)} \otimes a_0$$

and

$$tr_A(x \otimes (v_0 \otimes a_0 \otimes w_0)\, d(v_1 \otimes a_1 \otimes w_1)) = b((S^{-1}(x_{(1)}) \cdot w_1, v_0) b(w_0, v_1)\, x_{(2)} \otimes a_0 da_1.$$

In these formulas we implicitly use the twisted trace $tr_x : l(b) \to \mathbb{C}$ for $x \in H$ defined by $tr_x(v \otimes w) = b((S^{-1}(x) \cdot w, v)$. The twisted trace satisfies the relation

$$tr_x(T_0 T_1) = tr_{x_{(2)}}((S^{-1}(x_{(1)}) \cdot T_1) T_0)$$

for all $T_0, T_1 \in l(b)$. Using this relation one verifies that tr_A defines a chain map. It is clear that tr_A is $\hat{H}$-linear and it is straightforward to check that tr_A is H-linear.

Let us define $t_A = tr_A \circ X_H(\lambda_A)$ and show that $[t_A]$ is an inverse for $[\iota_A]$. Using the fact that u is H-invariant one computes $[\iota_A]\cdot[t_A] = 1$. We have to prove $[t_A]\cdot[\iota_A] = 1$. Consider the following equivariant homomorphisms $l(b;A) \to l(b;l(b;A))$ given by

$$i_1(v \otimes a \otimes w) = u \otimes v \otimes a \otimes w \otimes u$$
$$i_2(v \otimes a \otimes w) = v \otimes u \otimes a \otimes u \otimes w$$

As above we see $[i_1]\cdot[t_{l(b;A)}] = 1$ and we determine $[i_2]\cdot[t_{l(b;A)}] = [t_A]\cdot[\iota_A]$. Let h_t be the linear map from $l(b;A)$ into $l(b;l(b;A))$ given by

$$h_t(v \otimes a \otimes w) = \cos(\pi t/2)^2 u \otimes v \otimes a \otimes w \otimes u + \sin(\pi t/2)^2 v \otimes u \otimes a \otimes u \otimes w$$
$$- i\cos(\pi t/2)\sin(\pi t/2) u \otimes v \otimes a \otimes u \otimes w + i\sin(\pi t/2)\cos(\pi t/2) v \otimes u \otimes a \otimes w \otimes u$$

The family h_t depends smoothly on t and we have $h_0 = i_1$ and $h_1 = i_2$. Since u is invariant the map h_t is in fact equivariant and one checks that h_t is a homomorphism. Hence we have indeed defined a smooth homotopy between i_1 and i_2. This yields $[i_1] = [i_2]$ and hence $[t_A]\cdot[\iota_A] = 1$. □

We derive the following general stability theorem.

Proposition 8.5 (Stability)**.** *Let H be a bornological quantum group and let A be an H-algebra. Moreover let V be an essential H-module and let $b : V \times V \to \mathbb{C}$ be a nonzero equivariant bilinear pairing. Then there exists an invertible element in $HP_0^G(A, l(b;A))$. Hence there are natural isomorphisms*

$$HP_*^H(l(b;A), B) \cong HP_*^H(A, B) \qquad HP_*^H(A, B) \cong HP_*^H(A, l(b;B))$$

for all H-algebras A and B.

Proof. Let us write $\beta : H \times H \to \mathbb{C}$ for the canonical equivariant bilinear pairing introduced in section 3. Moreover we denote by b_τ the pairing $b : V_\tau \times V_\tau \to \mathbb{C}$ where V_τ is the space V equipped with the trivial H-action. We have an equivariant isomorphism $\gamma : l(b_\tau; l(\beta;A)) \cong l(\beta; l(b;A))$ given by

$$\gamma(v \otimes (x \otimes a \otimes y) \otimes w) = x_{(1)} \otimes x_{(2)} \cdot v \otimes a \otimes y_{(1)} \cdot w \otimes y_{(2)}$$

and using $\beta(x,y) = \psi(S(y)x)$ as well as the fact that ψ is right invariant one checks that γ is an algebra homomorphism. Now we can apply theorem 8.4 to obtain the assertion. □

We deduce a simpler description of equivariant periodic cyclic homology in certain cases. A bornological quantum group H is said to be of compact type if the dual algebra $\hat{H}$ is unital. Moreover let us call H of semisimple type if it is of compact type and the value of the integral for $\hat{H}$ on $1 \in \hat{H}$ is nonzero. For instance, the dual of a cosemisimple Hopf algebra $\hat{H}$ is of semisimple type.

Proposition 8.6. *Let H be a bornological quantum group of semisimple type. Then we have*

$$HP_*^H(A, B) \cong H_*(\mathrm{Hom}_{\mathsf{A}(H)}(X_H(\mathcal{T}A), X_H(\mathcal{T}B)))$$

for all H-algebras A and B.

Proof. Under the above assumptions the canonical bilinear pairing $\beta : \hat{H} \times \hat{H} \to \mathbb{C}$ is admissible since the element $1 \in \hat{H}$ is invariant. □

Finally we discuss excision in equivariant periodic cyclic homology. Consider an extension

$$K \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} Q$$

of H-algebras equipped with an equivariant linear splitting $\sigma : Q \to E$ for the quotient map $\pi : E \to Q$.

Let $X_H(\mathcal{T}E : \mathcal{T}Q)$ be the kernel of the map $X_H(\mathcal{T}\pi) : X_H(\mathcal{T}E) \to X_G(\mathcal{T}Q)$ induced by π. The splitting σ yields a direct sum decomposition $X_H(\mathcal{T}E) = X_H(\mathcal{T}E : \mathcal{T}Q) \oplus X_H(\mathcal{T}Q)$ of AYD-modules. The resulting extension

$$X_H(\mathcal{T}E : \mathcal{T}Q) \rightarrowtail X_H(\mathcal{T}E) \twoheadrightarrow X_H(\mathcal{T}Q)$$

of paracomplexes induces long exact sequences in homology in both variables. There is a natural covariant map $\rho : X_H(\mathcal{T}K) \to X_H(\mathcal{T}E : \mathcal{T}Q)$ of paracomplexes and we have the following generalized excision theorem.

Theorem 8.7. *The map $\rho : X_H(\mathcal{T}K) \to X_H(\mathcal{T}E : \mathcal{T}Q)$ is a homotopy equivalence.*

This result implies excision in equivariant periodic cyclic homology.

Theorem 8.8 (Excision)**.** *Let A be an H-algebra and let $(\iota, \pi) : 0 \to K \to E \to Q \to 0$ be an extension of H-algebras with a bounded linear splitting. Then there are two natural exact sequences*

$$\begin{array}{ccccc}
HP_0^H(A,K) & \longrightarrow & HP_0^H(A,E) & \longrightarrow & HP_0^H(A,Q) \\
\uparrow & & & & \downarrow \\
HP_1^H(A,Q) & \longleftarrow & HP_1^H(A,E) & \longleftarrow & HP_1^H(A,K)
\end{array}$$

and

$$\begin{array}{ccccc}
HP_0^H(Q,A) & \longrightarrow & HP_0^H(E,A) & \longrightarrow & HP_0^H(K,A) \\
\uparrow & & & & \downarrow \\
HP_1^H(K,A) & \longleftarrow & HP_1^H(E,A) & \longleftarrow & HP_1^H(Q,A)
\end{array}$$

The horizontal maps in these diagrams are induced by the maps in the extension.

In theorem 8.8 we only require a linear splitting for the quotient homomorphism $\pi : E \to Q$. Taking double crossed products of the extension given in theorem 8.8 yields an extension

$$K \hat{\otimes} \mathcal{K}_H \rightarrowtail E \hat{\otimes} \mathcal{K}_H \twoheadrightarrow Q \hat{\otimes} \mathcal{K}_H$$

of H-algebras with a linear splitting. Using lemma 3.3 one obtains an equivariant linear splitting for this extension. Now we can apply theorem 8.7 to this extension and obtain the claim by considering long exact sequences in homology.
For the proof of theorem 8.7 one considers the left ideal $\mathfrak{L} \subset \mathcal{T}E$ generated by $K \subset \mathcal{T}E$. The natural projection $\tau_E : \mathcal{T}E \to E$ induces an equivariant homomorphism $\tau : \mathfrak{L} \to K$ and one obtains an extension

$$N \rightarrowtail \mathfrak{L} \stackrel{\tau}{\twoheadrightarrow} K$$

of pro-H-algebras. The pro-H-algebra N is locally nilpotent and theorem 8.7 follows from the following assertion.

Theorem 8.9. *With the notations as above we have*

a) The pro-H-algebra $\mathfrak{L}$ is equivariantly quasifree.
b) The inclusion map $\mathfrak{L} \to \mathcal{T}E$ induces a homotopy equivalence $\psi : X_H(\mathfrak{L}) \to X_H(\mathcal{T}E : \mathcal{T}Q)$.

In order to prove the first part of theorem 8.9 one constructs explicitly a projective resolution of $\mathfrak{L}$ of length one.

9. Comparison to previous approaches

In this section we discuss the relation of the theory developped above to the previous approaches due to Akbarpour and Khalkhali [1], [2] as well as Neshveyev and Tuset [22]. As a natural domain in which all these theories are defined we choose to work with actions of cosemisimple Hopf algebras over the complex numbers. Note that a cosemisimple Hopf algebra H is a bornological quantum group with the fine bornology. In the sequel all vector spaces are equipped with the fine bornology when viewed as bornological vector spaces.
In their study of cyclic homology of crossed products [1] Akbarpour and Khalkhali define a cyclic module $C^H_*(A)$ of equivariant chains associated to a unital H-module algebra A. The space $C^H_n(A)$ in degree n of this cyclic module is the coinvariant space of $H \otimes A^{\otimes n+1}$ with respect to a certain action of H. Recall that the coinvariant space V_H associated to an H-module V is the quotient of V by the linear subspace generated by all elements of the form $t \cdot v - \epsilon(t)v$. Using the natural identification

$$\Omega^n_H(A) = H \otimes A^{\otimes n+1} \oplus H \otimes A^{\otimes n}$$

the action considered by Akbarpour and Khalkhali corresponds precisely to the action of H on the first summand in this decomposition. Hence $C^H_n(A)$ can be identified as a direct summand in the coinvariant space $\Omega^n_H(A)_H$. Moreover, the cyclic module structure of $C^H_*(A)$ reproduces the boundary operators b and B of $\Omega_H(A)_H$. We point out that the relation $T = \mathrm{id}$ holds on the coinvariant space $\Omega_H(A)_H$ which means that the latter is always a mixed complex.
It follows that there is a natural isomorphism of the cyclic type homologies associated to the cyclic module $C^H_*(A)$ and the mixed complex $\Omega_H(A)_H$, respectively. Note also that the complementary summand of $C^H_*(A)$ in $\Omega_H(A)_H$ is obtained from the bar complex of A tensored with H. Since A is assumed to be a unital H-algebra, this complementary summand is contractible with respect to the differential induced from the Hochschild boundary of $\Omega_H(A)_H$.
In the cohomological setting Akbarpour and Khalkhali introduce a cocyclic module $C^*_H(A)$ for every unital H-module algebra [2]. The definition of this cocyclic module is not exactly dual to the one of the cyclic module $C^H_*(A)$. In order to establish the connection to our constructions let first A be an arbitrary H-algebra. We define a modified action of H on $\Omega_H(A)$ by the formula

$$t \circ (x \otimes \omega) = t_{(2)} x S^{-1}(t_{(3)}) \otimes t_{(1)} \cdot \omega$$

and write $\Omega_H(A)^\mu$ for the space $\Omega_H(A)$ equipped with this action. Let us compare the modified action with the original action

$$t \cdot (x \otimes \omega) = t_{(3)} x S(t_{(1)}) \otimes t_{(2)} \cdot \omega$$

introduced in section 6. In the space $\Omega_H(A)_H$ of coinvariants with respect to the original action we have

$$\begin{aligned} t \circ (x \otimes \omega) &= t_{(2)} x S^{-1}(t_{(3)}) \otimes t_{(1)} \cdot \omega = t_{(4)} x S^{-1}(t_{(5)}) t_{(1)} S(t_{(2)}) \otimes t_{(3)} \cdot \omega \\ &= t_{(2)} \cdot (x S^{-1}(t_{(3)}) t_{(1)} \otimes \omega) = x S^{-1}(t_{(2)}) t_{(1)} \otimes \omega = \epsilon(t)\, x \otimes \omega \end{aligned}$$

which implies that the canonical projection $\Omega_H(A) \to \Omega_H(A)_H$ factorizes over the coinvariant space $\Omega_H(A)^\mu_H$ with respect to the modified action. Similarly, in the coinvariant space $\Omega_H(A)^\mu_H$ we have

$$\begin{aligned} t \cdot (x \otimes \omega) &= t_{(3)} x S(t_{(1)}) \otimes t_{(2)} \cdot \omega = t_{(3)} x S(t_{(1)}) t_{(5)} S^{-1}(t_{(4)}) \otimes t_{(2)} \cdot \omega \\ &= t_{(2)} \circ (x S(t_{(1)}) t_{(3)} \otimes \omega) = x S(t_{(1)}) t_{(2)} \otimes \omega = \epsilon(t)\, x \otimes \omega. \end{aligned}$$

As a consequence we see that the identity map on $\Omega_H(A)$ induces an isomorphism

$$\Omega_H(A)_H \cong \Omega_H(A)^\mu_H$$

between the coinvariant spaces.

We may view a linear map $\Omega_H(A) \to \mathbb{C}$ as a linear map $\Omega(A) \to F(H)$ where $F(H)$ denotes the linear dual space of H. Under this identification an element in $\mathrm{Hom}_H(\Omega_H(A)^\mu, \mathbb{C})$ corresponds to a linear map $f : \Omega(A) \to F(H)$ satisfying the equivariance condition

$$f(t \cdot \omega)(x) = f(\omega)(S(t_{(2)})xt_{(1)})$$

for all $t \in H$ and $\omega \in \Omega(A)$. In a completely analogous fashion to the case of homology discussed above, a direct inspection using the canonical isomorphisms

$$\mathrm{Hom}_H(\Omega_H(A)^\mu, \mathbb{C}) \cong \mathrm{Hom}(\Omega_H(A)^\mu_H, \mathbb{C}) \cong \mathrm{Hom}(\Omega_H(A)_H, \mathbb{C})$$

shows that the cyclic type cohomologies of the cocyclic module $C^*_H(A)$ agree for every unital H-algebra A with the ones associated to the mixed complex $\Omega_H(A)_H$. In particular, the definition in [2] is indeed obtained by dualizing the construction given in [1].

The main difference between the cocyclic module used by Akbarbour and Khalkhali and the definition in [22] is that Neshveyev and Tuset work with right actions instead of left actions. It is explained in [22] that the two approaches lead to isomorphic cocyclic modules and hence to isomorphic cyclic type cohomologies.

Now assume that H is a semisimple Hopf algebra. Then the coinvariant space $\Omega_H(A)_H$ is naturally isomorphic to the space $\Omega_H(A)^H$ of invariants. If A is a unital H-algebra then theorem 6.9 and proposition 8.6, together with the above considerations, yield a natural isomorphism

$$HP_*(C^H_*(A)) \cong HP^H_*(\mathbb{C}, A)$$

which identifies the periodic cyclic homology of the cyclic module $C^H_*(A)$ with the equivariant cyclic homology of A in the sense of definition 7.1. Similarly, for a semisimple Hopf algebra H one obtains a natural isomorphism

$$HP^*(C^*_H(A)) \cong HP^H_*(A, \mathbb{C})$$

for every unital H-algebra A.

Both of these isomorphisms fail to hold more generally, even in the classical setting of group actions. Let Γ be a discrete group and consider the group ring $H = \mathbb{C}\Gamma$. For the H-algebra $\mathbb{C}$ with the trivial action one easily obtains

$$HP^*(C^*_H(\mathbb{C})) \cong \mathrm{Hom}_H(H_{\mathsf{ad}}, \mathbb{C})$$

located in degree zero where H_{ad} is the space $H = \mathbb{C}\Gamma$ viewed as an H-module with the adjoint action. On the other hand, a result in [25] shows that there is a natural isomorphism

$$HP^H_*(\mathbb{C}, \mathbb{C}) \cong HP^*(H)$$

which identifies the H-equivariant theory of the complex numbers with the periodic cyclic cohomology of $H = \mathbb{C}\Gamma$. This is the result one should expect from equivariant KK-theory [16]. Hence, roughly speaking, the theory in [2], [22] only recovers the degree zero part of the group cohomology. A similar remark applies to the homology groups defined in [1].

We mention that Akbarbour and Khalkhali have obtained an analogue of the Green-Julg isomorphism

$$HP^H_*(\mathbb{C}, A) \cong HP_*(A \rtimes H)$$

if H is semisimple and A is a unital H-algebra [1]. This result holds in fact more generally in the case that H is the convolution algebra of a compact quantum group and A is an arbitrary H-algebra. Similarly, there is a dual version

$$HP^H_*(A, \mathbb{C}) \cong HP^*(A \rtimes H)$$

of the Green-Julg theorem for the convolution algebras of discrete quantum groups. The latter generalizes the identification of equivariant periodic cyclic cohomology for discrete groups mentioned above. We will discuss these results in detail in a separate paper.

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