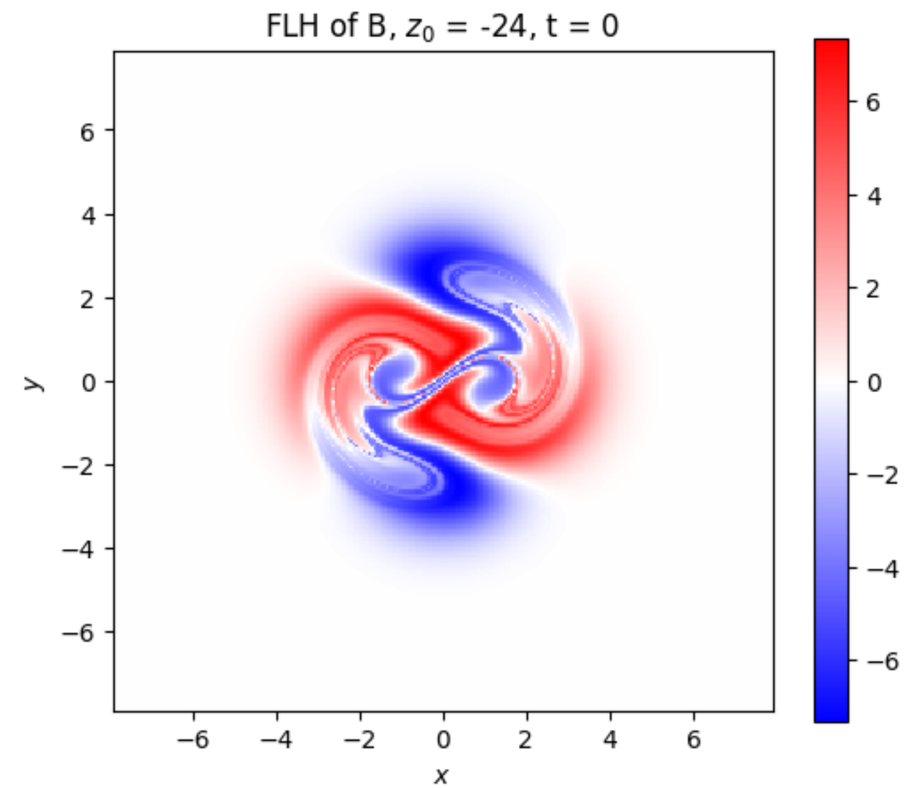
The topology of resistive magnetic relaxation



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Background
- magnetic relaxation and topology
- numerical experiments with braided magnetic field

Problem set-up
- unstirring
- variational method vs. magnetic relaxation

Results
- simple test + complex configurations

Summary

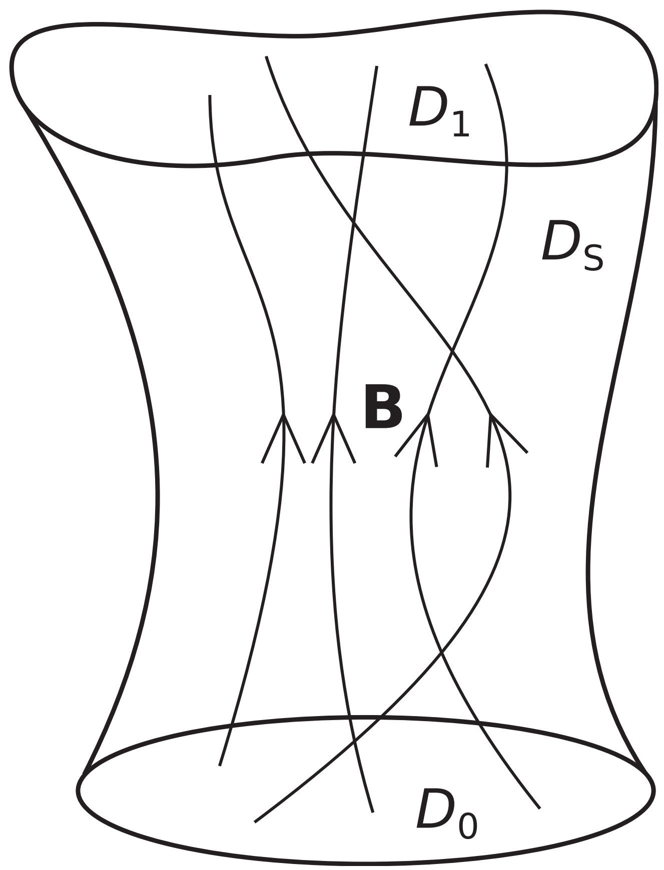
On the solar surface, the interaction of the magnetic field and surrounding plasmas creates complex structures—often seen as entangled bundles of field lines.

NASA/GSFC/Solar Dynamics Observatory

Coronal loops illuminated by charged particles spinning along the magnetic field lines.

Complex magnetic fields can relax into simpler patterns, sometimes but not always involving eruptions. This is in an interesting regime where the topology is quasi-conserved.

One way of characterising the topology in a mathematical language is to use the field line helicity.



Russell *et. al.* (2015)

Suppose the magnetic field **B** can be described by a vector potential **A**,

$$\mathbf{B} = \nabla \times \mathbf{A}$$

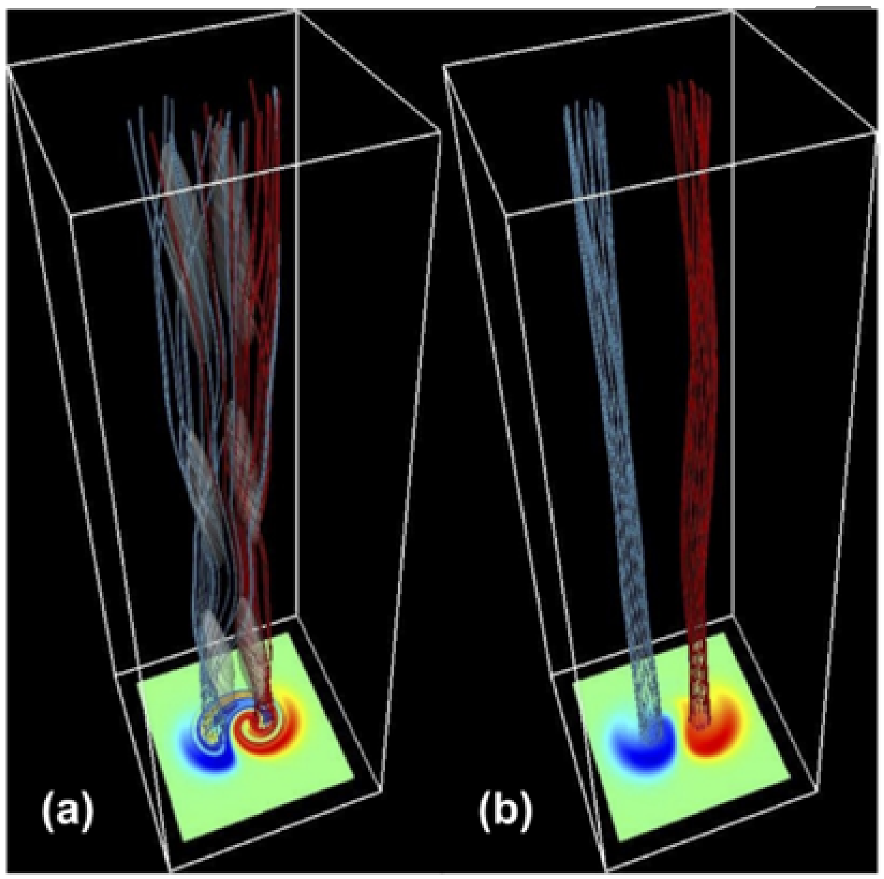
and the (magnetic) field line helicity (FLH) is

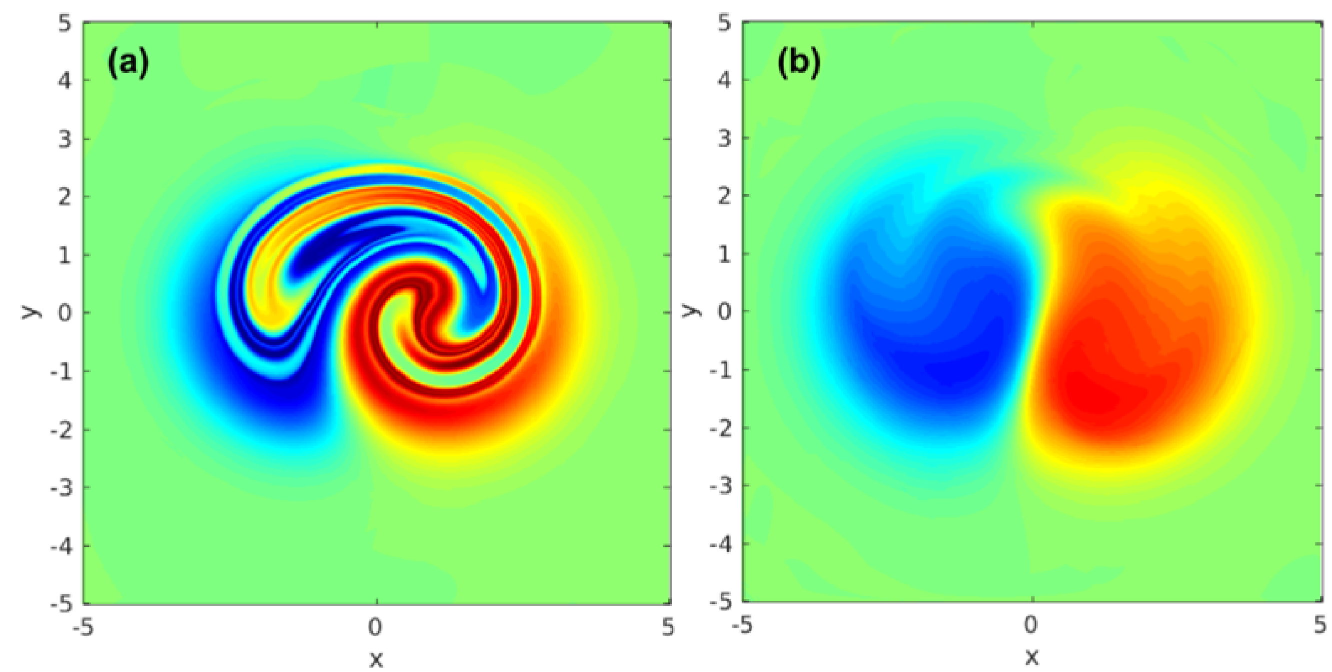
$$\mathcal{A}(\mathbf{x}) = \int_{F(\mathbf{x})} \mathbf{A} \cdot \mathrm{d}\mathbf{l}$$

where F(**x**) is the magnetic field line through point **x**.

We use the field line helicity to trace the evolution of magnetic structures.

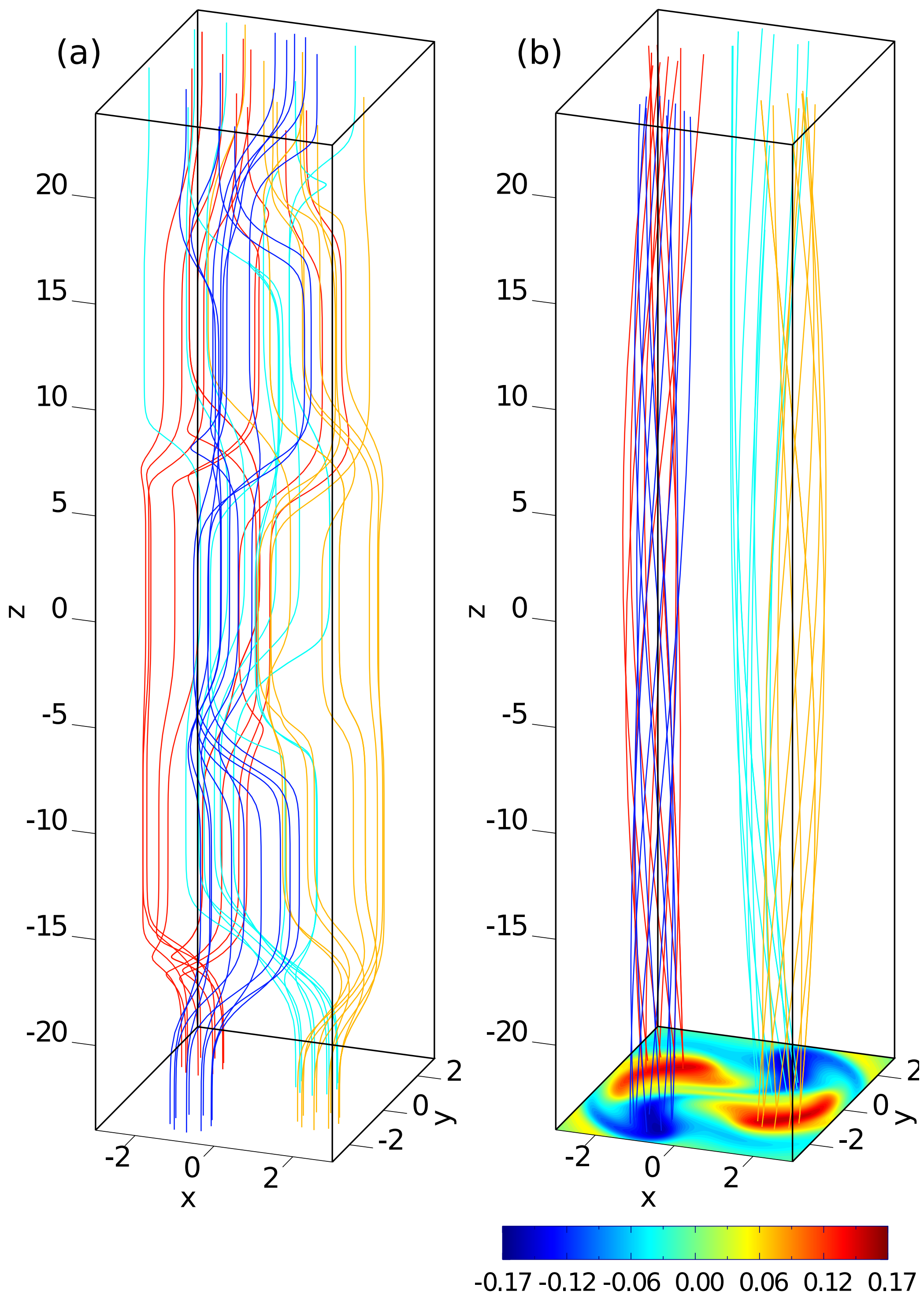
Conceptually, topological dissipation was discussed by Parker (1972) and reduction of complex structures to simple flux tubes by Parker (1983).



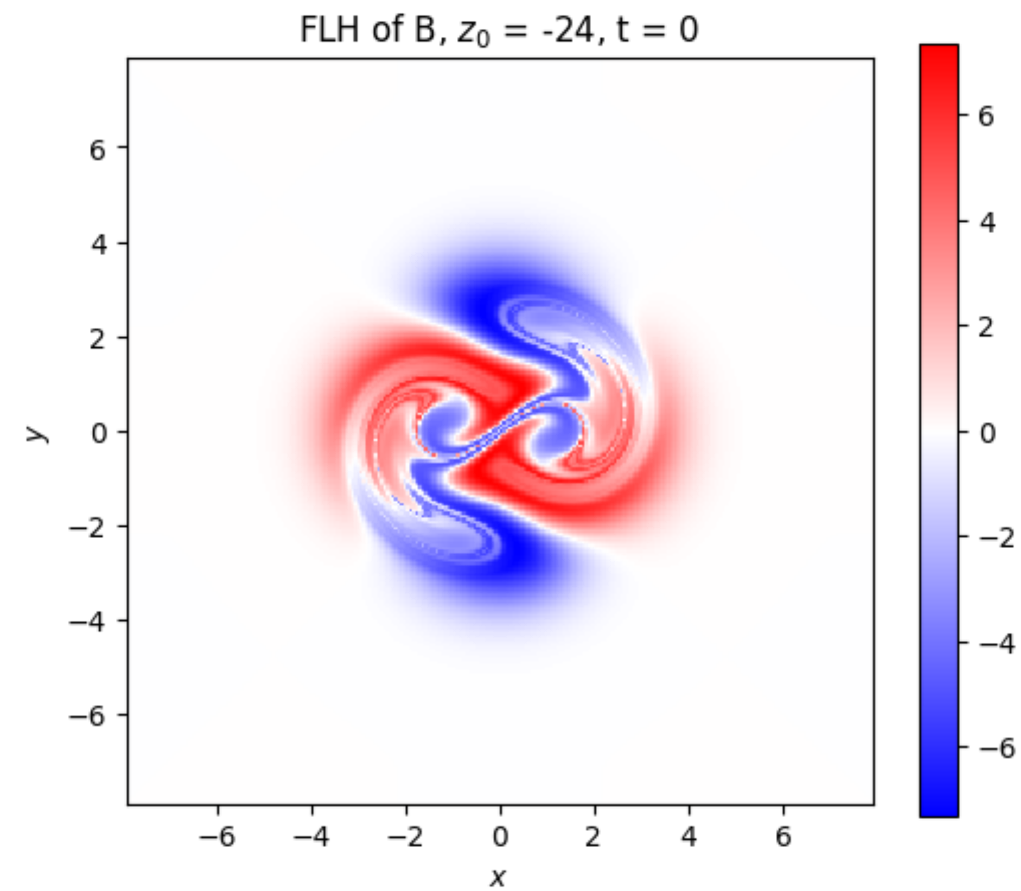


Numerical experiments of braided magnetic field show the simplification of FLH as complexity of the magnetic field reduces (Yeates *et al.* 2010, 2015).

e.g, E3 case which relaxes to two flux tubes (Russell *et al.* 2015).

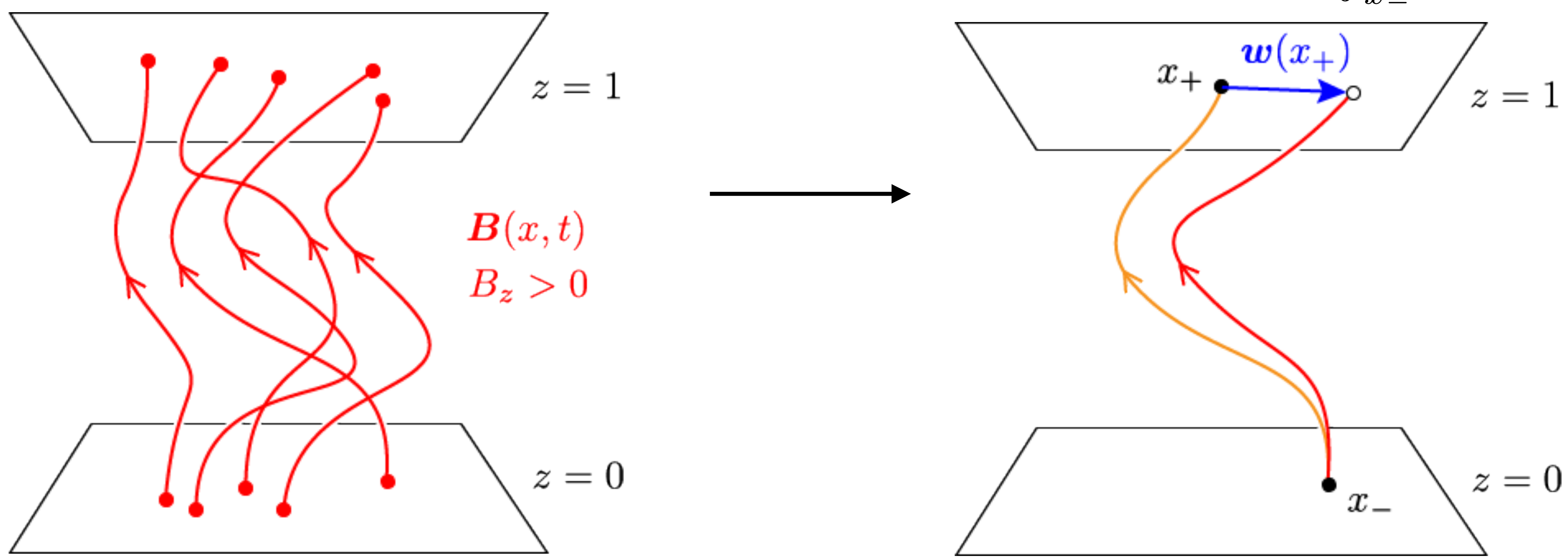


T=3 model with four regions of twists relaxes into four tubes (Yeates *et al.* 2010).



We can translate the dynamics to a reduced model in order to better understand the topological evolution.

$$\psi(x_+) = \int_{x_-}^{x_+} \eta \boldsymbol{j} \cdot \mathrm{d}\boldsymbol{l}$$



(Yeates 2017 research notes)

$$\begin{aligned}\boldsymbol{E} &= -\,\boldsymbol{v} \times \boldsymbol{B} + \eta \boldsymbol{j} \\ &= -\underbrace{(\boldsymbol{v} + \boldsymbol{u})}_{\boldsymbol{w}} \times \boldsymbol{B} + \nabla \psi\end{aligned}$$

Separate current into parallel and perpendicular components when there is a small resistivity.

$$\frac{\partial A}{\partial t} = -\boldsymbol{E} - \nabla \phi$$

Then use the (uncurled) induction equation with a scalar potential ϕ

Use the definition of field line helicity $\mathcal{A} = \int \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$,
and set $\phi = 0$ for a fixed gauge

$$\frac{\partial \mathcal{A}}{\partial t} + \boldsymbol{w} \cdot \nabla \mathcal{A} = \cancel{\boldsymbol{w} \cdot \boldsymbol{A}} - \cancel{\psi}$$

Hypothesis: to first order approximation the evolution of FLH behaves mostly as if it is being advected by a fictitious flow

Problem set-up

Define a Dirichlet functional to measure the complexity of a 2D density function

$$E(t) = \langle |\nabla f|^2 \rangle$$

where $\langle \cdots \rangle = A^{-1} \iint_D \cdots \mathrm{dxdy}$

Suppose it is being advected by an ideal incompressible fluid

$$\frac{\partial f}{\partial t} + \boldsymbol{w} \cdot \nabla f = 0,$$

$$\nabla \cdot \boldsymbol{w} = 0$$

We want to find a flow field $\boldsymbol{w}$ such that it minimises $E(T)$,
i.e., optimally reduce complexity to match with the relaxed state.

With some algebra (Moffatt 1990, Arnold & Khesin 1998, they studied similar fluid models) one can show the optimal state satisfies:

$$\nabla f_T \times \nabla \left(\nabla^2 f_T \right) = \mathbf{0}$$

Two optimal approaches:

1. Variational method (VM)

adjust norm

$$\mathcal{L} = \langle |\nabla^{\theta} f_T|^2 \rangle + \mathrm{sgn}(\theta) \langle \Pi \nabla \cdot \boldsymbol{w} \rangle + \mathrm{sgn}(\theta) \int_0^T \left\langle \Gamma \left(\frac{\partial f}{\partial t} + \boldsymbol{w} \cdot \nabla f \right) \right\rangle \, \mathrm{d}t,$$

Objective

Constraints

Solve a coupled system of PDE until the optimal solution is found

$$\delta \mathcal{L} = \left\langle \frac{\delta \mathcal{L}}{\delta \boldsymbol{w}} \cdot \delta \boldsymbol{w} \right\rangle + \left\langle \frac{\delta \mathcal{L}}{\delta \Pi} \delta \Pi \right\rangle + \int_0^T \left\langle \delta \Gamma \frac{\delta \mathcal{L}}{\delta \Gamma} \right\rangle \mathrm{d}t + \int_0^T \left\langle \delta f \frac{\delta \mathcal{L}}{\delta f} \right\rangle \mathrm{d}t + \left\langle \frac{\delta \mathcal{L}}{\delta f_T} \delta f_T \right\rangle \to 0$$

2. Magnetic relaxation method (MR)

Define a 2D magnetic field from the flux function $\widetilde{\boldsymbol{B}} = \nabla \times f(x, y, t)\hat{\boldsymbol{z}}$

evolve as if it is being relaxed resistively

$$\mu \nabla^2 \boldsymbol{w} + (\nabla \times \widetilde{\boldsymbol{B}}) \times \widetilde{\boldsymbol{B}} - \nabla P = 0,$$

coupled with the advection equation by an incompressible flow

$$\frac{\partial f}{\partial t} + \boldsymbol{w} \cdot \nabla f = 0,$$

$$\nabla \cdot \boldsymbol{w} = 0$$

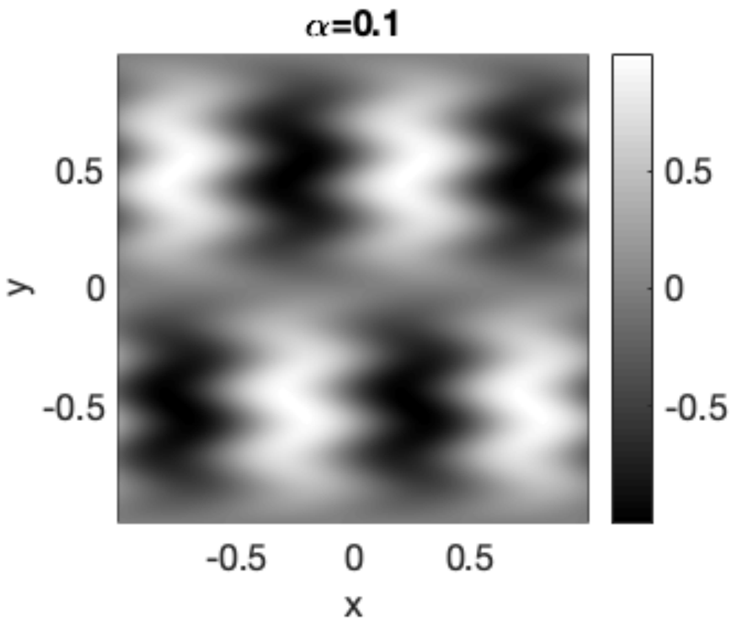
After some manipulation, one can show the energy is monotonically decreasing

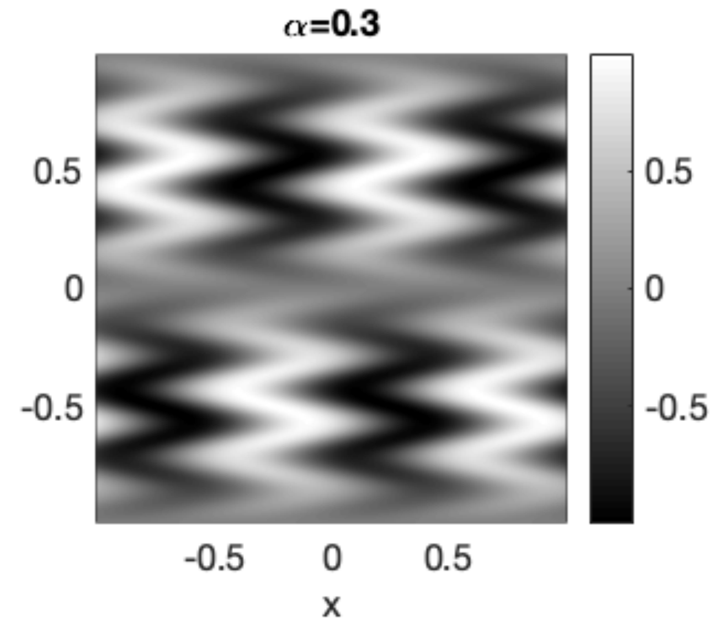
$$\frac{A}{2} \frac{\partial E(t)}{\partial t} = -\mu \iint_D |\nabla \times \boldsymbol{w}|^2 \mathrm{dxdy} \leq 0$$

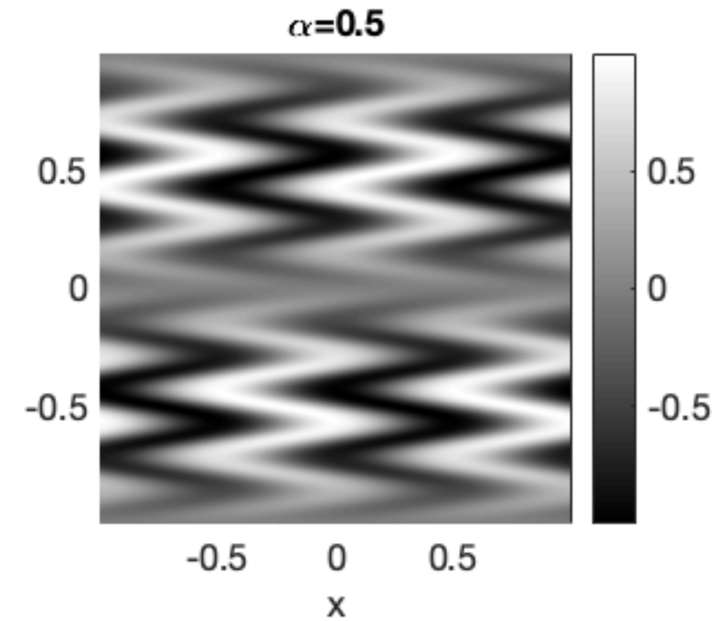
Similar methods have been mentioned by Moffatt (1990), Linardatos (1993), Moffatt & Dormy (2019). We set $\mu = 1$.

Simple test results

Initial state

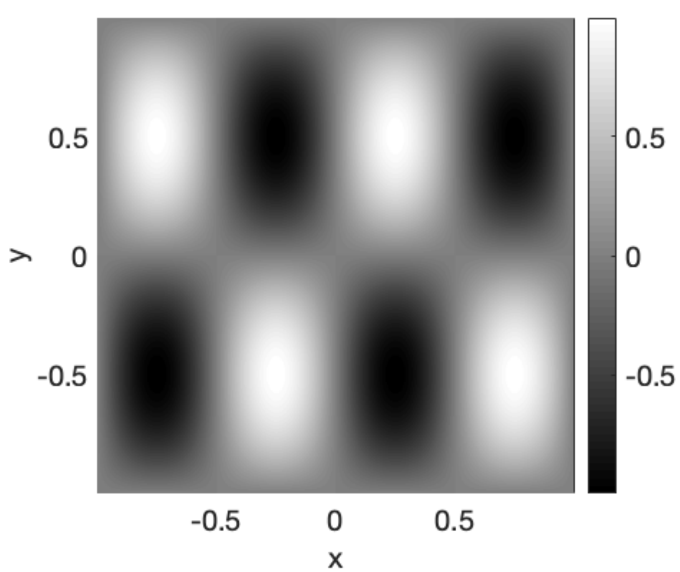




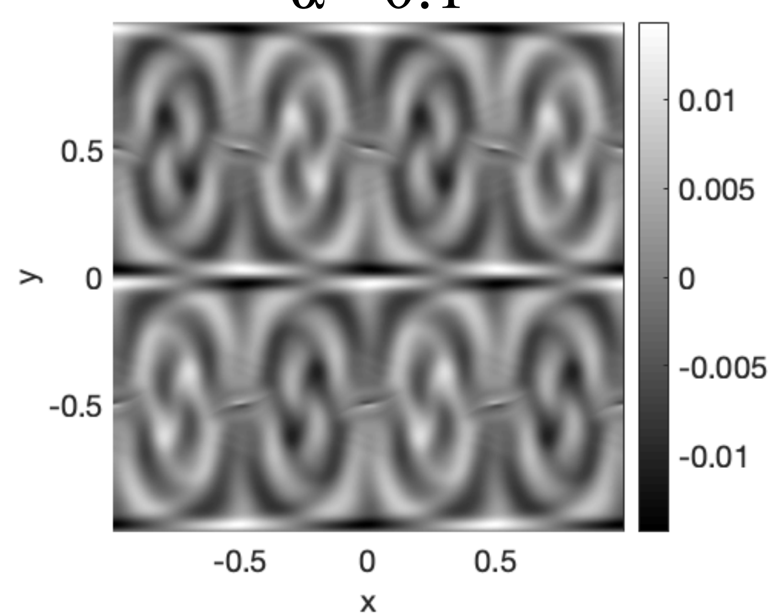


v

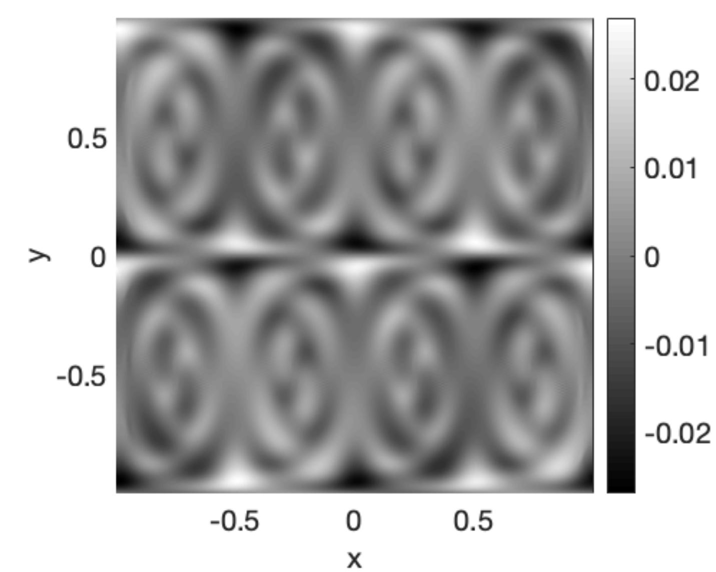
Expected final state for all values of α



error VM $\theta = -1$
α =0.1



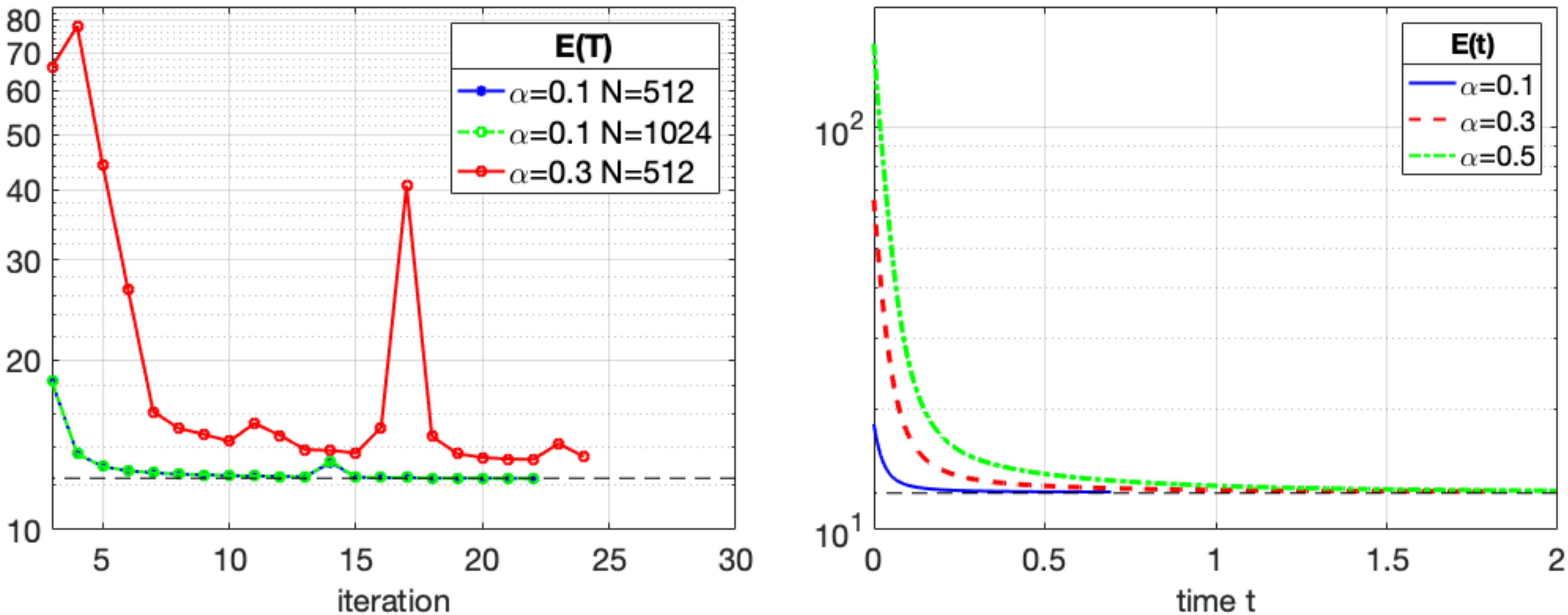
error MR
α =0.1



(Chen *et al.* 2020. To be submitted to JFM)

Simple test results

VR $\theta = -1$ MR



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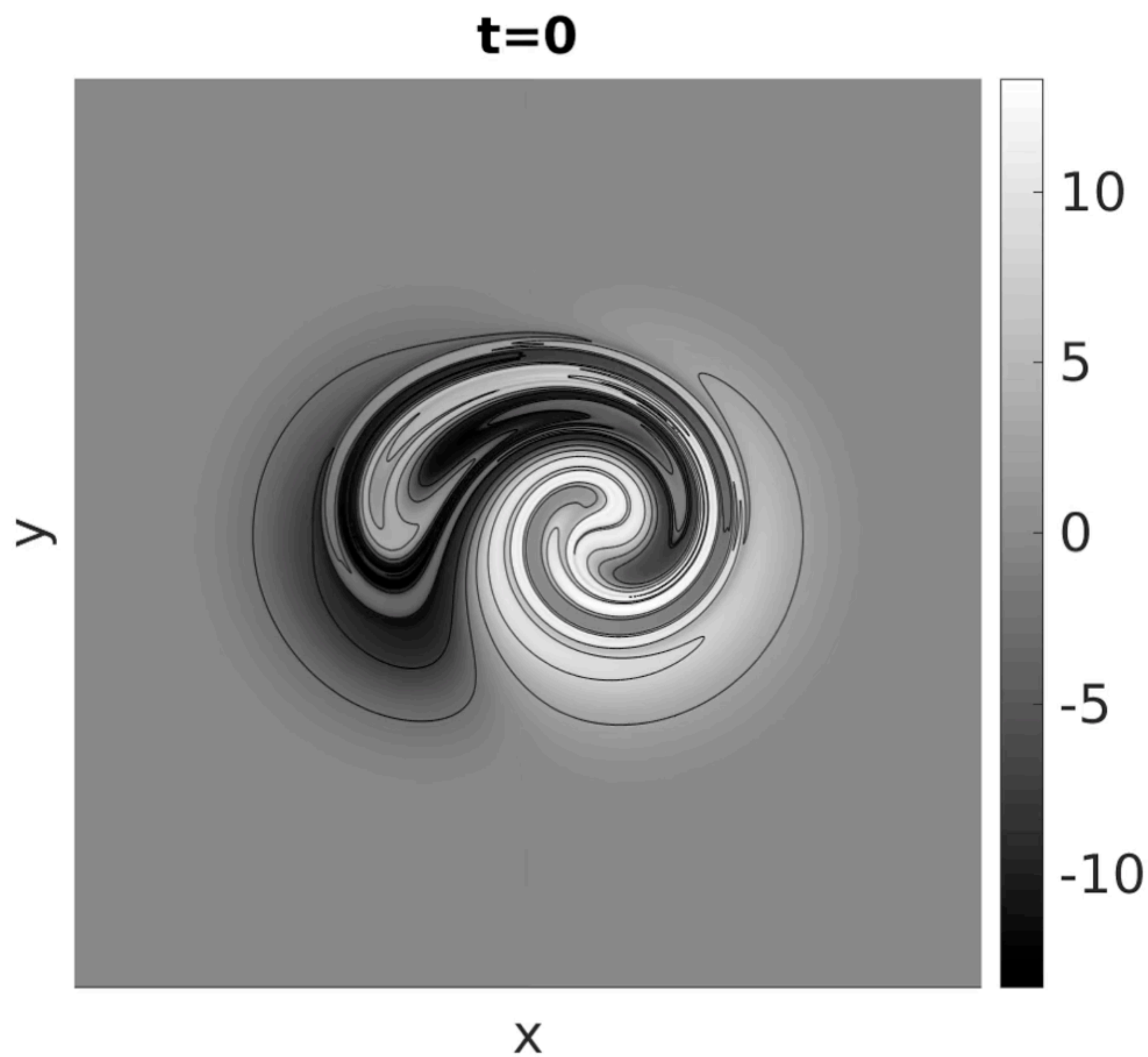
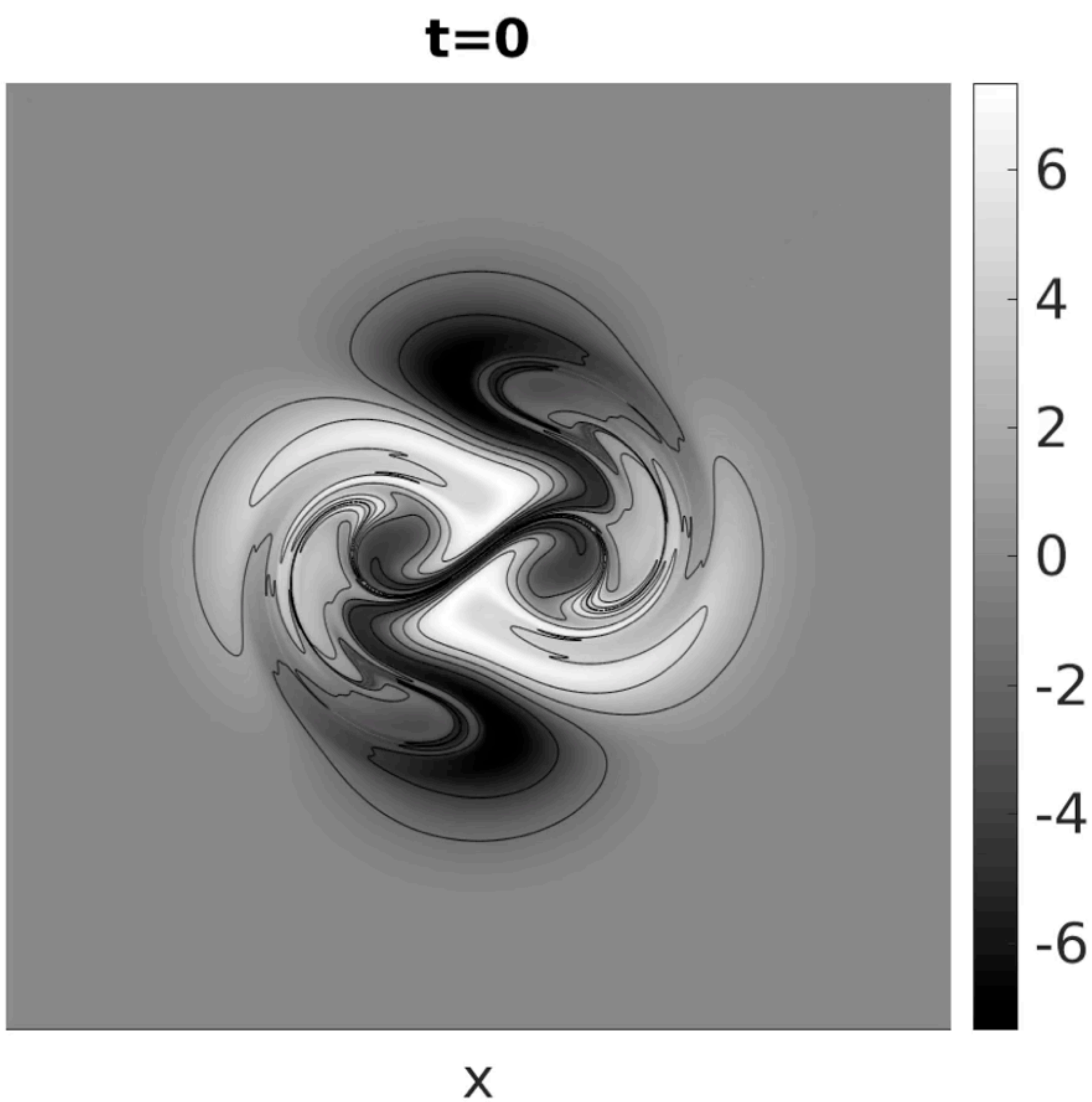
Complex cases

E3 braid



T=3 model



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Complex cases

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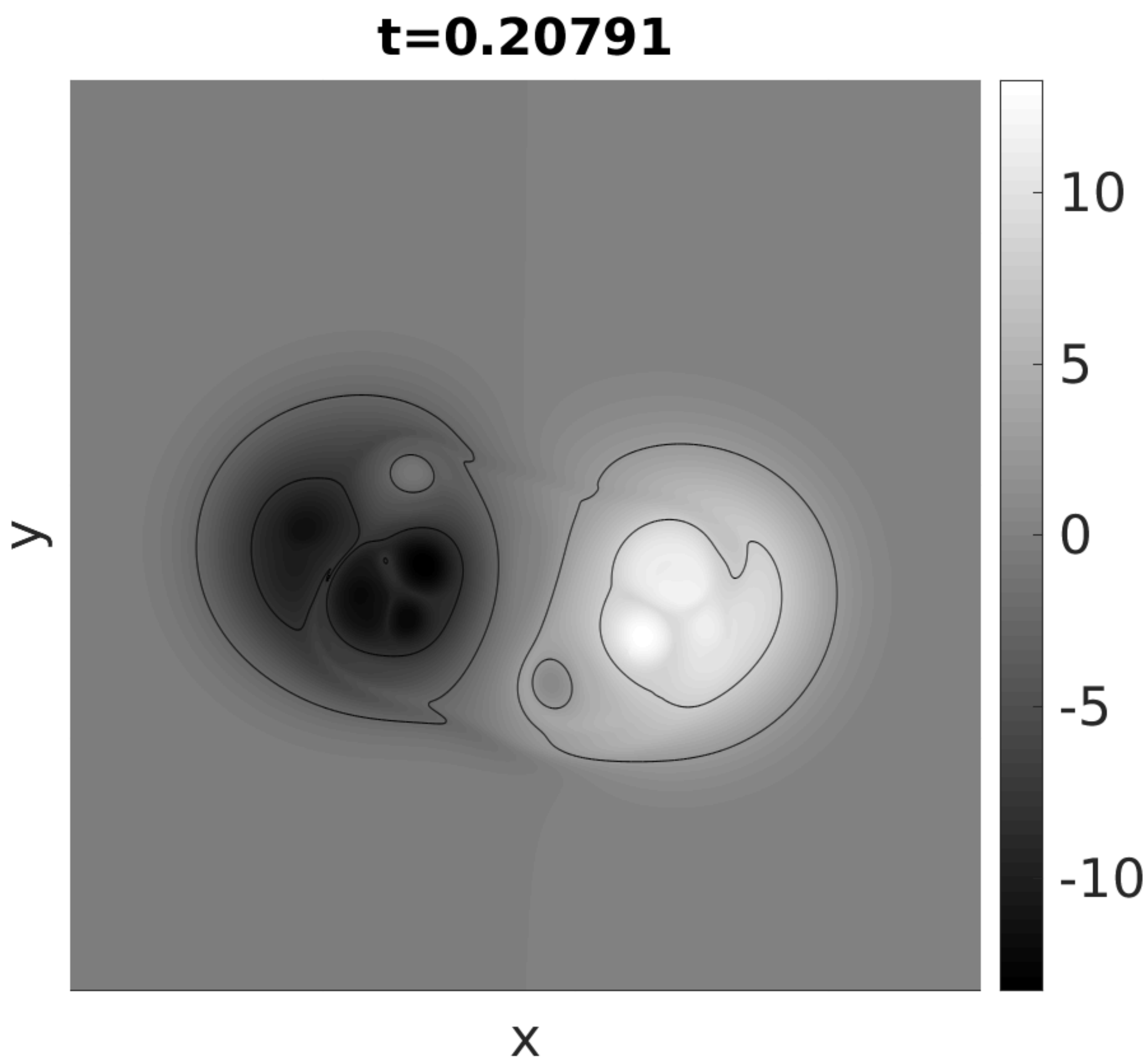

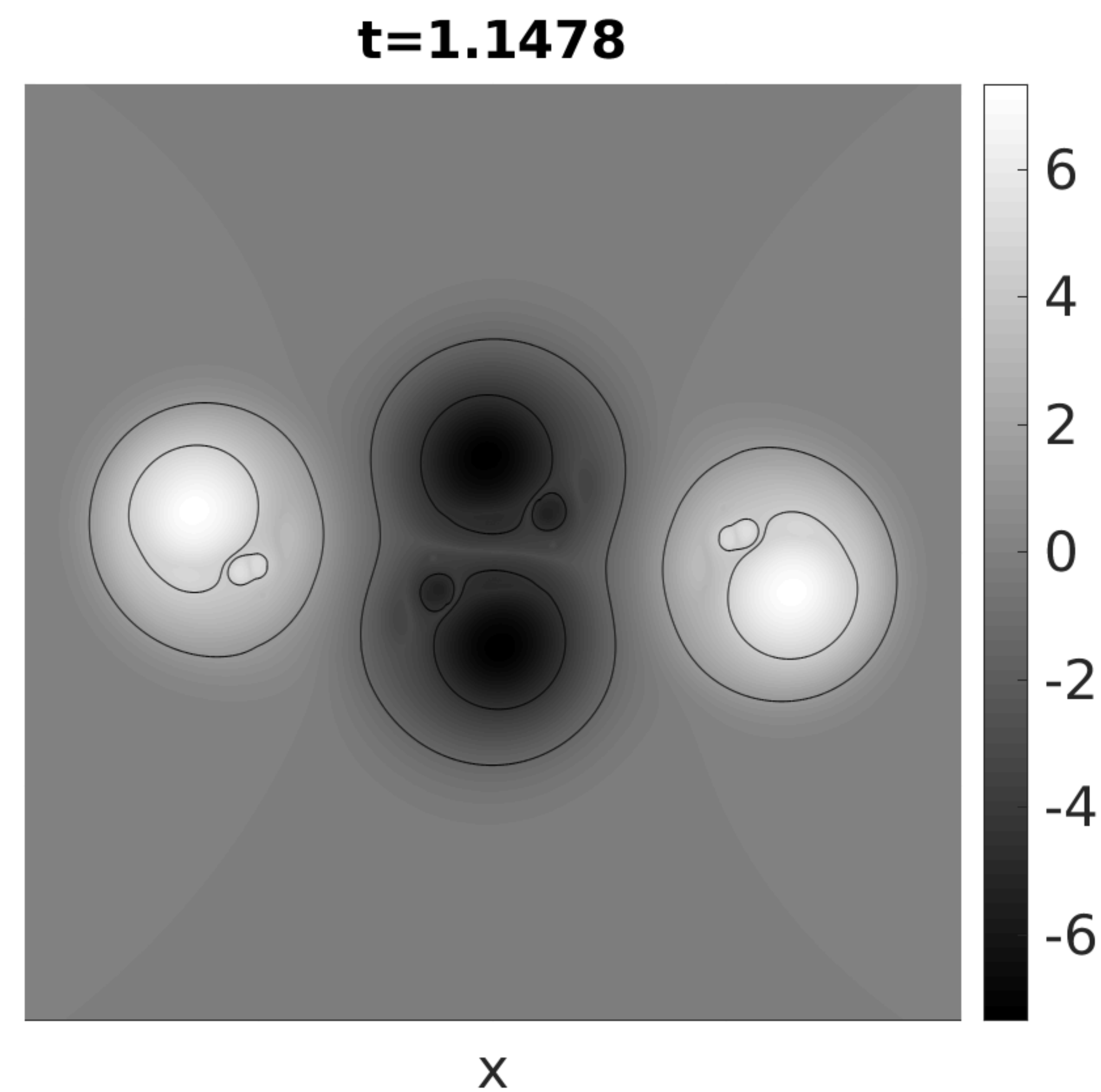

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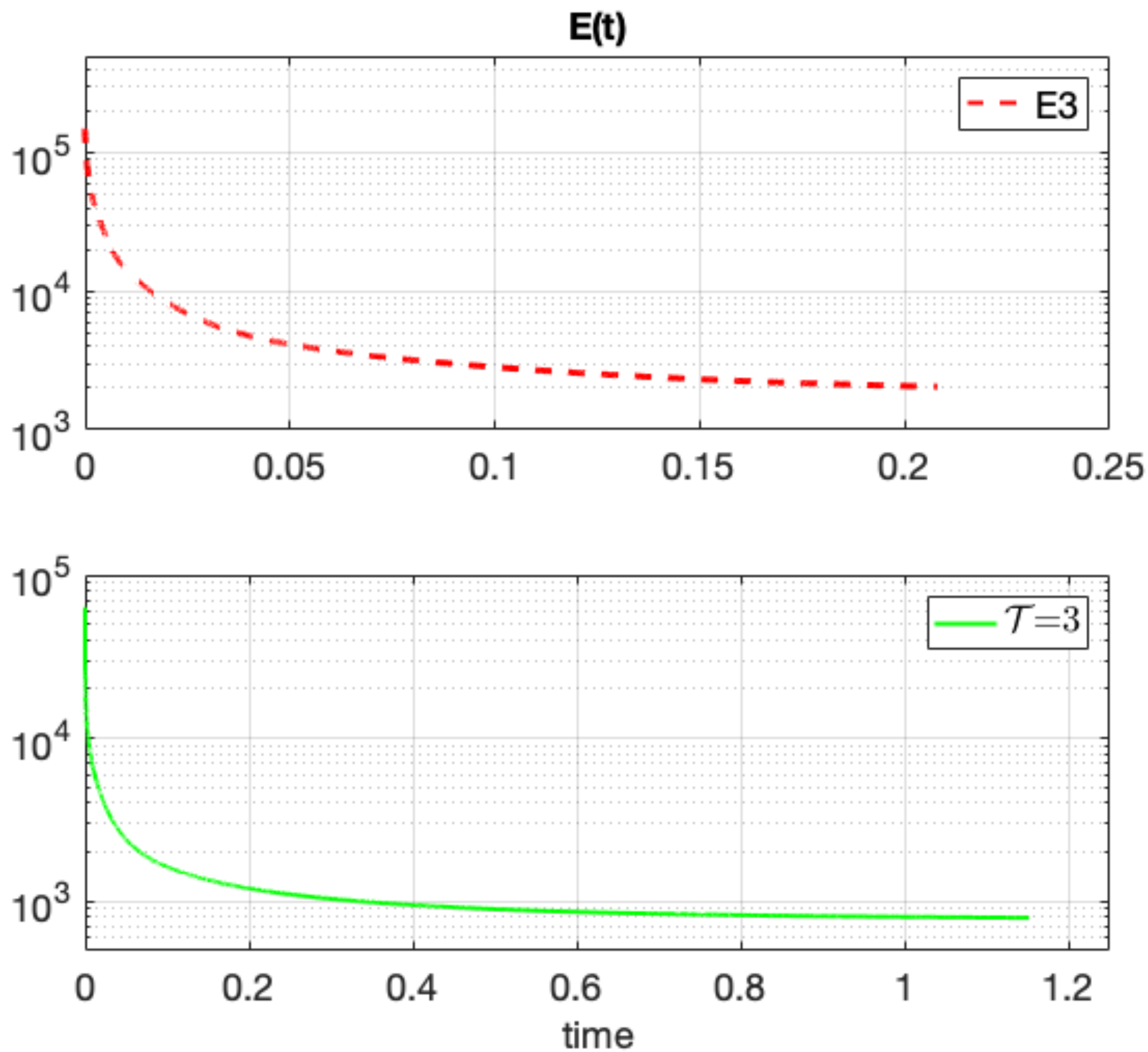
Convergence of the energy



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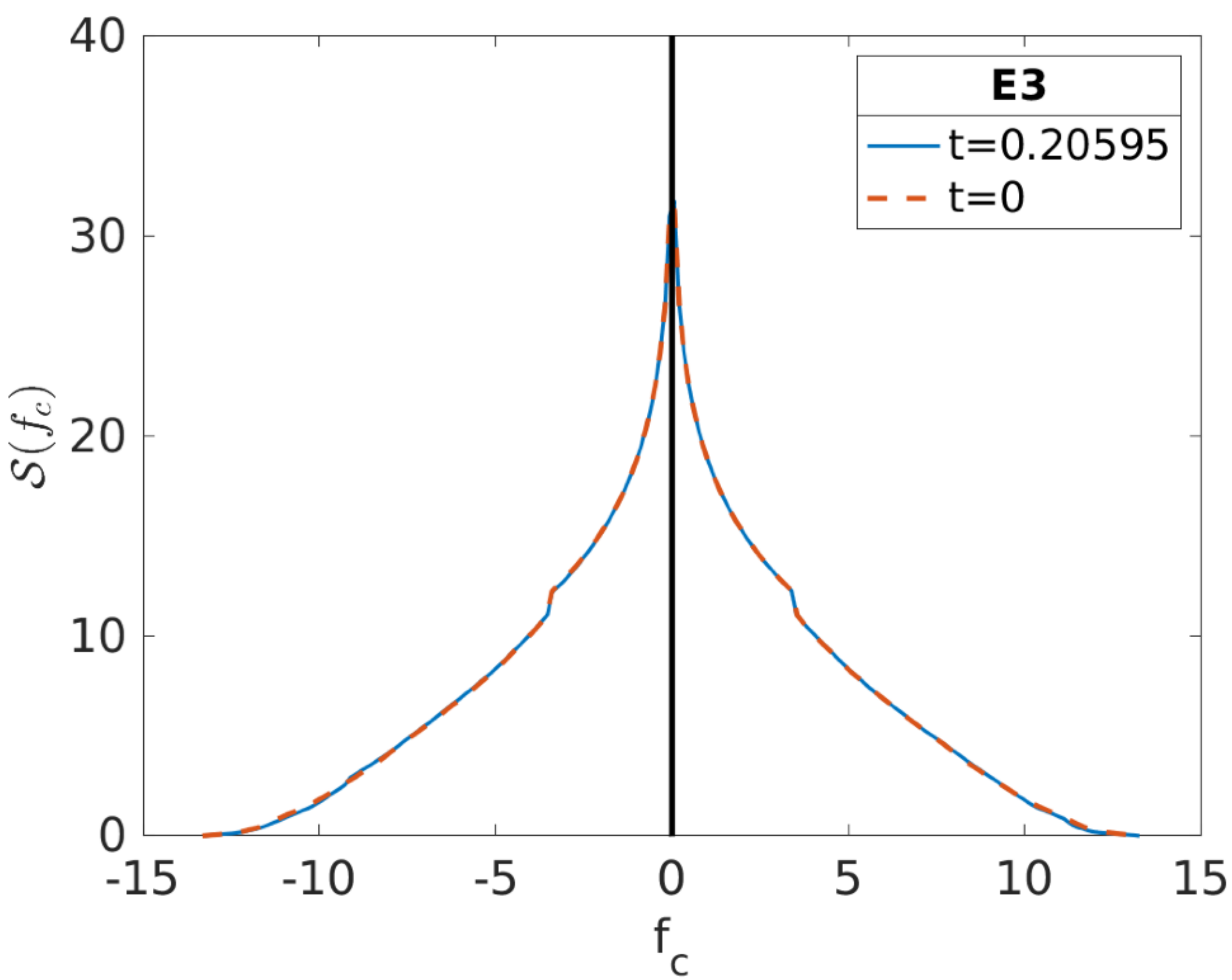
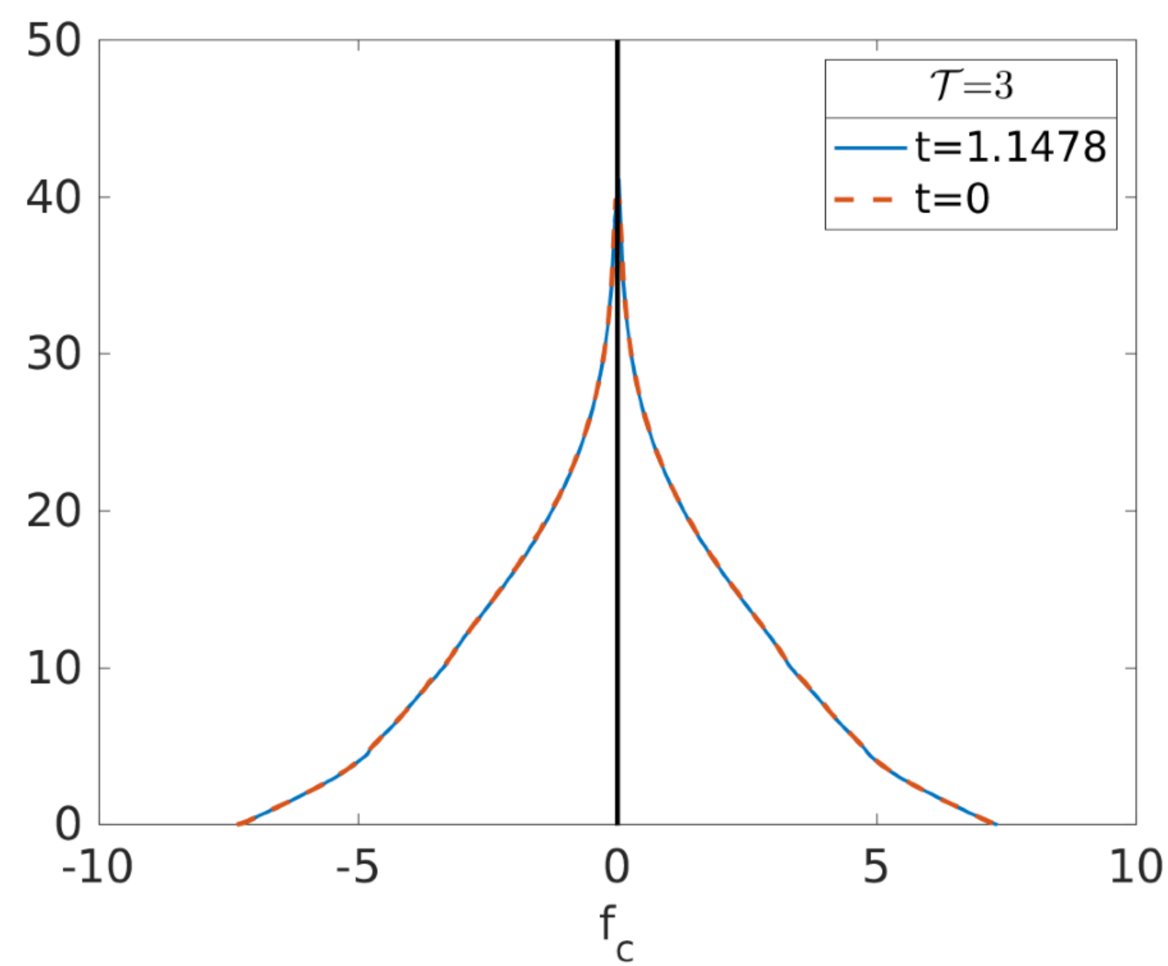
Conservation of topology:

A consequence of advection by an incompressible fluid is that the area enclosed by each iso-contour remains constant. This is measured by the signature function

$$\mathcal{S}(f_c) = \begin{cases} \iint_{f(x,y)\geqslant f_c} \mathrm{d}^2x, & \text{if } f_c > 0 \\ A, & \text{if } f_c = 0, \\ \iint_{f(x,y)\leqslant f_c} \mathrm{d}^2x, & \text{if } f_c < 0, \end{cases}$$

where $\min f(x,y) \leq f_c \leq \max f(x,y)$





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Summary

Topology of magnetic field can be tracked through field line helicity during magnetic relaxation. It behaves as if it's mainly been advected by a fictitious fluid.

Two approaches have been developed to find the optimal flow field in order to predict the final state: variational method (VM) and magnetic relaxation method (MR). We find MR is numerically more stable.

With MR, we tested the E3 and T=3 cases of braided magnetic field, we find the overall topology of the final state matches the 3D results. This then validates our hypothesis.

Next step: test more configurations, study detailed configuration of the final state (with "w.A" term)?

Thank you