Taylor's conjecture on magnetic helicity conservation in magnetohydrodynamics

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The mathematical setting of the talk

The **(homogeneous,) incompressible, viscous, resistive MHD equations** consist of

$$\partial_t \mathrm{u} + (\mathrm{u} \cdot \nabla)\mathrm{u} + \nabla p - \nu \Delta u - (\nabla \times \mathrm{B}) \times \mathrm{B} = 0, \tag{1}$$

$$\nabla \cdot \mathrm{u} = 0, \tag{2}$$

$$\partial_t \mathrm{B} + \nabla \times (\mathrm{B} \times \mathrm{u}) + \eta \nabla \times (\nabla \times \mathrm{B}) = 0, \tag{3}$$

$$\nabla \cdot \mathrm{B} = 0, \tag{4}$$

where $\nu, \eta > 0$.

In **ideal MHD**, $\nu = \eta = 0$.

Given a bounded, simply connected domain $\mathcal{V} \subset \mathbb{R}^3$ with $\partial \mathcal{V} = \mathcal{S}$ we set the boundary conditions $\mathrm{u}|_{\mathcal{S}} = 0$; $\mathrm{B} \cdot \mathrm{n}|_{\mathcal{S}} = 0$; $\mathrm{E} \times \mathrm{n}|_{\mathcal{S}} = 0$, which leads to $(\nabla \times \mathrm{B}) \times \mathrm{n}|_{\mathcal{S}} = 0$.

In ideal MHD, we set $\mathrm{u} \cdot \mathrm{n}|_{\mathcal{S}} = \mathrm{B} \cdot \mathrm{n}|_{\mathcal{S}} = 0$.

Conserved integral quantities of ideal MHD

Continuously differentiable solutions u, B of *ideal* MHD conserve

$$\int_{\mathcal{V}} \frac{|u|^2 + |B|^2}{2} \, dx \qquad (\textbf{total energy}),$$

$$\int_{\mathcal{V}} u \cdot B \, dx \qquad (\textbf{cross helicity}),$$

$$\int_{\mathcal{V}} A \cdot B \, dx \qquad (\textbf{magnetic helicity}),$$

where $\nabla \times A = B$.

However, simulations point towards **anomalous dissipation** of total energy: when viscosity and resistivity tend to zero, the energy dissipation rate tends to a *positive* constant:

- Mininni-Pouquet (Phys. Rev. Lett. 2009),
- Dallas-Alexakis (Astrophys. J. Lett. 2014),
- Linkmann-Berera-McComb-McKay (Phys. Rev. E 2015).

Taylor's conjecture (Phys. Rev. Lett. 1974): magnetic helicity is approximately conserved in $\mathcal{V}$ for very small resistivities $\eta > 0$.

Berger's solution

Berger (Geophys. Astrophys. Fluid Dynamics 1984) solved physically Taylor's conjecture by showing that for small resistivities $\eta > 0$, magnetic helicity dissipates much slower than magnetic energy.

When $\nu, \eta > 0$, the dissipation rates of total energy and magnetic helicity are

$$\partial_t \mathcal{H}(t) = \partial_t \int_{\mathcal{V}} \mathrm{A} \cdot \mathrm{B}\, \mathrm{dx} = -2\eta \int_{\mathcal{V}} \mathrm{B} \cdot \nabla \times \mathrm{B}\, \mathrm{dx},$$
$$\partial_t \mathcal{E}(t) = \partial_t \int_{\mathcal{V}} (|\mathrm{B}|^2 + |\mathrm{u}|^2)/2\, \mathrm{dx} = -\nu \int_{\mathcal{V}} |\nabla \times \mathrm{u}|^2\, \mathrm{dx} - \eta \int_{\mathcal{V}} |\nabla \times \mathrm{B}|^2\, \mathrm{dx}.$$

The part of Berger's argument most relevant for this talk is the use of the Cauchy-Schwarz inequality $\left|\int_{\mathcal{V}} f \cdot g\right|^2 \leq \int_{\mathcal{V}} |f|^2 \int_{\mathcal{V}} |\mathrm{B}|^2$ on $f = \sqrt{\eta}\mathrm{B}$ and $g = \sqrt{\eta}\nabla \times \mathrm{B}$:

$$\begin{aligned} |\partial_t \mathcal{H}(t)|^2 &= 4 \left| \int_{\mathcal{V}} \eta \mathrm{B} \cdot \nabla \times \mathrm{B}\, \mathrm{dx} \right|^2 \\ &\leq 4 \int_{\mathcal{V}} \eta\, |\mathrm{B}|^2\, \mathrm{dx} \int_{\mathcal{V}} \eta\, |\nabla \times \mathrm{B}|^2\, \mathrm{dx} \\ &\leq 4\eta \int_{\mathcal{B}} |\mathrm{B}|^2\, \mathrm{dx}\, |\partial_t \mathcal{E}(t)|\,. \end{aligned}$$

A mathematical programme

The following mathematical version of Taylor's conjecture was presented in Caflisch-Klapper-Steele (Comm. Math. Phys. 1997):

Conjecture

Magnetic helicity does not dissipate in the ideal limit.

At $\eta = 0$, does there exist a natural class of solutions of ideal MHD that

- dissipate total energy,
- conserve magnetic helicity,
- arise as limits of solutions of resistive MHD when $\nu, \eta \searrow 0$ (e.g. when keeping initial datas u_0 and B_0 fixed)?

Fix u_0 and B_0. When $\nu, \eta > 0$, denote a solution of viscous, resistive MHD with initial data $(\mathrm{u}_0, \mathrm{B}_0)$ by $(\mathrm{u}^{\nu,\eta}, \mathrm{B}^{\nu,\eta})$. If the dissipation rate

$$\nu \int_{\mathcal{V}} |\nabla \mathrm{u}^{\nu,\eta}|^2 \, \mathrm{dx} + \eta \int_{\mathcal{V}} |\nabla \mathrm{B}^{\nu,\eta}|^2 \, \mathrm{dx} \longrightarrow \epsilon_* > 0,$$

one expects

$$\int_{\mathcal{V}} |\nabla \mathrm{u}^{\nu,\eta}|^2 \, \mathrm{dx} \sim \frac{1}{\nu} \longrightarrow \infty, \qquad \int_{\mathcal{V}} |\nabla \mathrm{B}^{\nu,\eta}|^2 \, \mathrm{dx} \sim \frac{1}{\eta} \longrightarrow \infty.$$

Onsager's theory of turbulence

Onsager (Nuovo Cimento 1949) suggested *singular/weak solutions* of Euler equations as a model of hydrodynamic turbulence at the limit $\mathrm{Re} \to \infty$ ($\nu \searrow 0$).

The singularity of the solutions would cause kinetic energy dissipation even in the absence of viscosity(!).

In order to define weak solutions, suppose now u is a solution of Euler equations and $\varphi \in C_0^\infty(\mathcal{V} \times [0, T))$. Integrating by parts,

$$\int_0^T \int_{\mathcal{V}} (\partial_t \mathrm{u} + \nabla \cdot (\mathrm{u} \otimes \mathrm{u}) + \nabla p) \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t = \tag{5}$$

$$- \int_0^T \int_{\mathcal{V}} (\mathrm{u} \cdot \partial_t \varphi + (\mathrm{u} \otimes \mathrm{u}) \cdot \nabla \varphi + p \, \nabla \cdot \varphi) \, \mathrm{d}x \, \mathrm{d}t - \int_{\mathcal{V}} \mathrm{u}_0 \cdot \varphi(\cdot, 0) \, \mathrm{d}x = 0, \tag{6}$$

$$\int_0^T \int_{\mathcal{V}} \mathrm{u} \cdot \nabla \varphi \, \mathrm{d}x = 0. \tag{7}$$

If $\int_0^T \int_{\mathcal{V}} |\mathrm{u}|^2 \, \mathrm{d}x \, \mathrm{d}t < \infty$ and u satisfies (6)–(7) for all φ, then u is called a **weak solution** of Euler equations.

For reviews of Onsager's theory and modern work on it see Eyink-Sreenivasan (Rev. Modern Phys. 2008), Eyink (Physica D 2008), Eyink (arXiv 2018).

'Turbulent' weak solutions of viscous, resistive MHD

Given $\nu, \eta > 0$ and smooth initial datas u_0 and B_0, it is wide open whether the Cauchy problem for MHD has a *smooth* solution.

However, a weak 'Leray-Hopf solution' or 'turbulent solution' exists, see Sermange-Temam (Commun. Pure Appl. Math. 1984). For Navier-Stokes, see Leray (Acta Math. 1934) and Hopf (Math. Nachr. 1950/1951).

Definition

Suppose $\int_{\mathcal{V}} |\mathrm{u}_0|^2 + \int_{\mathcal{V}} |\mathrm{B}_0|^2 < \infty$, $\nabla \cdot \mathrm{u}_0 = \nabla \cdot \mathrm{B}_0 = 0$ and $\mathrm{u}_0 \cdot \mathrm{n}|_{\mathcal{S}} = \mathrm{B}_0 \cdot \mathrm{n}|_{\mathcal{S}} = 0$.

Then (u, B) is called a **Leray-Hopf solution** if

- (u, B) is a (weak) solution of viscous, resistive MHD with initial datas $\mathrm{u}_0, \mathrm{B}_0$,
- u and B satisfy the **energy inequality**

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{V}} (|\mathrm{u}(\mathrm{x}, t)|^2 + |\mathrm{B}(\mathrm{x}, t)|^2) \, \mathrm{dx} + \int_0^t \int_{\mathcal{V}} (\nu \, |\nabla \mathrm{u}(\mathrm{x}, \tau)|^2 + \eta \, |\nabla \mathrm{B}(\mathrm{x}, \tau)|^2) \, \mathrm{dx} \, \mathrm{d}\tau \\ \leq \; & \frac{1}{2} \int_{\mathcal{V}} (|\mathrm{u}_0(\mathrm{x})|^2 + |\mathrm{B}_0(\mathrm{x})|^2) \, \mathrm{dx}. \end{aligned}$$

If a smooth solution exists, then it coincides with the Leray-Hopf solution.

Uniqueness of Leray-Hopf solutions is open for general initial datas.

Solution of (the mathematical version of) Taylor's conjecture

We say that $\mathrm{v}_j \rightharpoonup \mathrm{v}$ in $L^2([0,T) \times \mathcal{V})$ if $\int_0^T \int_{\mathcal{V}} \mathrm{v}_j \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_{\mathcal{V}} \mathrm{v} \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t$ for every $\varphi \in L^2([0,T) \times \mathcal{V})$.

Theorem (Faraco-L. (Comm. Math. Phys. 2020))

Suppose

- $\nu_j, \eta_j \searrow 0$ *when* $j \to \infty$.
- *At each* j, $(\mathrm{u}_j, \mathrm{B}_j)$ *is a Leray-Hopf solution with initial data* $(\mathrm{u}_{0,j}, \mathrm{B}_{0,j})$,
- $\mathrm{u}_j \rightharpoonup \mathrm{u}$, $\mathrm{B}_j \rightharpoonup \mathrm{B}$ *in* $L^2([0,T) \times \mathcal{V})$,
- $\mathrm{u}_{0,j} \rightharpoonup \mathrm{u}_0$, $\mathrm{B}_{0,j} \rightharpoonup \mathrm{B}_0$ *in* $L^2(\mathcal{V})$.

Then (u, B) *conserves magnetic helicity in time.*

Other mathematical results on magnetic helicity conservation in ideal MHD:

- u and B in suitable Besov spaces: Caflisch-Klapper-Steele (Comm. Math. Phys. 1997),
- $\int_0^T \int_{\mathbb{T}^3} (|\mathrm{u}|^3 + |\mathrm{B}|^3) \, \mathrm{d}x \, \mathrm{d}t < \infty$: Kang-Lee (Nonlinearity 2007) and Aluie (Ph.D. dissertation 2009) with Eyink.
- Magnetic helicity is *not* in general conserved when $\int_0^T \int_{\Omega} (|\mathrm{u}|^2 + |\mathrm{B}|^2) \, \mathrm{d}x \, \mathrm{d}t < \infty$: Beekie-Buckmaster-Vicol (arXiv 2019).

The difference between second and third moments

Consider a solution (u, B) of ideal MHD on $\mathbb{T}^3 = [0, 2\pi]^3$ (2π-periodic in x_1, x_2, x_3) with $\int_0^T \int_{\mathcal{V}} (|\mathrm{u}|^3 + |\mathrm{B}|^3)\,\mathrm{dx}\,\mathrm{d}t < \infty$. The time derivative $\partial_t \int_{\mathbb{T}^3} \mathrm{A} \cdot \mathrm{B}\,\mathrm{dx} = \int_{\mathbb{T}^3} \mathrm{A} \cdot \partial_t \mathrm{B} + \int_{\mathbb{T}^3} \partial_t \mathrm{A} \cdot \mathrm{B}\,\mathrm{dx}$ does not make immediate sense (since $\partial_t \mathrm{B} = -\nabla \times (\mathrm{B} \times \mathrm{u})$).

For every $\ell > 0$, consider regularisations B_ℓ of B (e.g. mollifications $\mathrm{B}_\ell = \mathrm{B} * g_\ell$ with a filtering kernel $g \in C_0^\infty(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} g(\mathrm{x})\,d\mathrm{x} = 1$, $g_\ell(\mathrm{x}) := \ell^{-3} g(\mathrm{x}/\ell)$).

Now $\partial_t \mathrm{B}_\ell + \nabla \times (\mathrm{B} \times \mathrm{u})_\ell = 0$ implies that $\partial_t \mathrm{A}_\ell + (\mathrm{B} \times \mathrm{u})_\ell = \nabla g^\ell$, and

$$\partial_t \int_{\mathbb{T}^3} \mathrm{A}_\ell \cdot \mathrm{B}_\ell\,\mathrm{dx} = 2 \int_{\mathbb{T}^3} \partial_t \mathrm{A}_\ell \cdot \mathrm{B}_\ell\,\mathrm{dx} = \int_{\mathbb{T}^3} (\nabla g^\ell - (\mathrm{B} \times \mathrm{u})_\ell) \cdot \mathrm{B}_\ell$$
$$= -\int_{\mathbb{T}^3} (\mathrm{B} \times \mathrm{u})_\ell \cdot \mathrm{B}_\ell\,\mathrm{dx} \to -\int_{\mathbb{T}^3} \mathrm{B} \times \mathrm{u} \cdot \mathrm{B} = 0.$$

Furthermore, $\partial_t \int_{\mathbb{T}^3} A_\ell \cdot \mathrm{B}_\ell\,\mathrm{dx} \to \partial_t \int_{\mathbb{T}^3} \mathrm{A} \cdot \mathrm{B}\,\mathrm{dx}$ e.g. in the sense of distributions.

However, the limit of the integrals $\int_{\mathbb{T}^3} (\mathrm{B} \times \mathrm{u})_\ell \cdot \mathrm{B}_\ell$ need not exist if only $\int_0^T \int_{\mathcal{V}} (|\mathrm{u}|^2 + |\mathrm{B}|^2)\,\mathrm{dx}\,\mathrm{d}t < \infty$!

Some ideas of our proof

The proof of the Faraco-L. Theorem has two parts: 1) the dissipation rates $\partial_t \int_{\mathcal{V}} \mathrm{A}_j \cdot \mathrm{B}_j \, \mathrm{dx}$ tend to zero when $\nu_j, \eta_j \searrow 0$, 2) the dissipation rate of the limit is the limit of the dissipation rates.

1): Recall that $\partial_t \int_{\mathcal{V}} \mathrm{A}_j \cdot \mathrm{B}_j \, \mathrm{dx} = -2\eta \int_{\mathcal{V}} \mathrm{B}_j \cdot \nabla \times \mathrm{B}_j \, \mathrm{dx}$. By the inequality $cd \leq c^2/2 + d^2/2$ with $c = |\mathrm{B}_j|$ and $d = \sqrt{\mu}\,|\nabla \times \mathrm{B}_j|$ and the energy inequality,

$$
\begin{aligned}
&\mu_j \int_0^T \int_\Omega |\mathrm{B}_j(\mathrm{x},t) \cdot \nabla \times \mathrm{B}_j(\mathrm{x},t)| \, \mathrm{dx} \, \mathrm{d}t \\
&\leq \sqrt{\eta_j} \int_0^T \int_\Omega (|\mathrm{B}_j(\mathrm{x},t)|^2 + \mu_j \, |\nabla \times \mathrm{B}_j(\mathrm{x},t)|^2) \, \mathrm{dx} \, \mathrm{d}t \\
&\leq \sqrt{\eta_j}(T+1)(\|\mathrm{B}_{j,0}\|_{L^2}^2 + \|\mathrm{u}_{j,0}\|_{L^2}^2) \longrightarrow 0,
\end{aligned}
$$

since $\eta_j \to 0$, $\mathrm{B}_{j,0} \rightharpoonup \mathrm{B}_0$ and $\mathrm{u}_{j,0} \rightharpoonup \mathrm{u}_0$.

2): $\mathrm{B}_j \rightharpoonup \mathrm{B}$ in $L^2([0,T) \times \mathcal{V})$ and the Aubin-Lions Lemma are used to show that $\int_0^T \int_{\mathcal{V}} |\mathrm{A}_j - A|^2 \, \mathrm{dx} \, \mathrm{d}t \to 0$ for suitable potentials A_j and A.

Thus, $\mathrm{A}_j \cdot \mathrm{B}_j \rightharpoonup \mathrm{A} \cdot \mathrm{B}$ in $L^1([0,T) \times \mathcal{V})$. It follows that the magnetic helicity dissipation rate of the limit is the (distributional) limit of the dissipation rates.

Dissipation of total energy in ideal MHD

Energy dissipative solutions of ideal MHD have been constructed recently:

- Bronzi-Lopes-Nussenzveig (Commun. Math. Sci. 2015), solutions 'not genuinely 3D'.
- Beekie-Buckmaster-Vicol (arXiv 2019): $\int_0^T \int_\Omega (|u|^2 + |B|^2)\,dx\,dt < \infty$,
- Faraco-L.-Székelyhidi (to appear in Arch. Ration. Mech Anal.): $|u|$ and $|B|$ bounded in space and time (in particular $\int_0^T \int_\Omega (|u|^3 + |B|^3)\,dx\,dt < \infty$).

Onsager's theory of turbulence II

Let $0 < \alpha < 1$. We say that u is C^α (Hölder) continuous if

$$|u(x,t) - u(y,t)| \leq C\,|x-y|^\alpha \qquad \text{for all } x, y \in \mathcal{V},\ t \in [0,T).$$

Conjecture (Onsager, 1949)

1. *Suppose $\alpha > 1/3$. Then every C^α solution conserves kinetic energy.*
2. *Suppose $\alpha < 1/3$. Then there exist C^α solutions dissipating kinetic energy.*

Part 1: Eyink (Phys. D 1994), Constantin-E-Titi (Comm. Math. Phys. 1994).

Part 2:

- Scheffer (J. Geom. Anal. 1993): the first solutions violating energy conservation,
- de Lellis & Székelyhidi (Ann. Math. 2009): solutions via John Nash's method of **convex integration**,
- Isett (Ann. Math. 2018): solutions violating energy conservation for every $\alpha < 1/3$,
- Buckmaster-de Lellis-Székelyhidi-Vicol (CPAM 2018): energy dissipating solutions for every $\alpha < 1/3$.

Thank you for your attention!