The alternating-hyperbolic sawtooth

Or: How I Learned To Stop Worrying About 1/1 And Love 2/3

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Overview

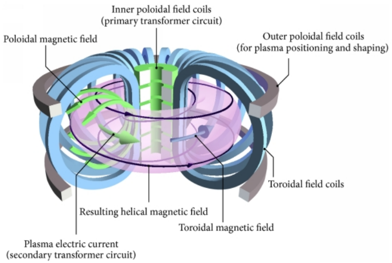
- Observations on the sawtooth crash.
- X-points, O-points and the group that binds them.
- An ideal mode when $q_0 = 2/3$.
- Transition to chaos.
- The alternating-hyperbolic sawtooth model.

The Tokamak fusion reactor creates an equilibrium using current in the plasma

- A tokamak fusion reactor confines plasma with an axisymmetric magnetic field.
- The toroidal field is generated by external coils.
- The poloidal field is generated by currents in the plasma itself.
- The safety factor q quantifies the winding of the field lines

$$q = 1/\iota = \frac{\#\ \text{toroidal rotations}}{\#\ \text{poloidal rotations}}$$

- One might think this is incredibly unstable. One would be right (see: stellarators)
- The sawtooth crash is one of these instabilities.



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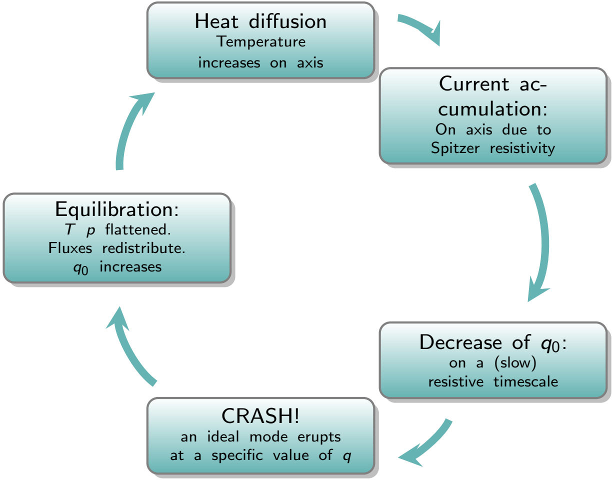
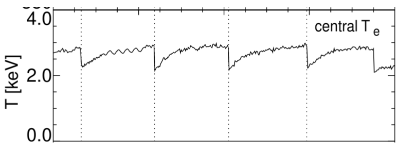
The Sawtooth Oscillation consists of a repeated fast crash



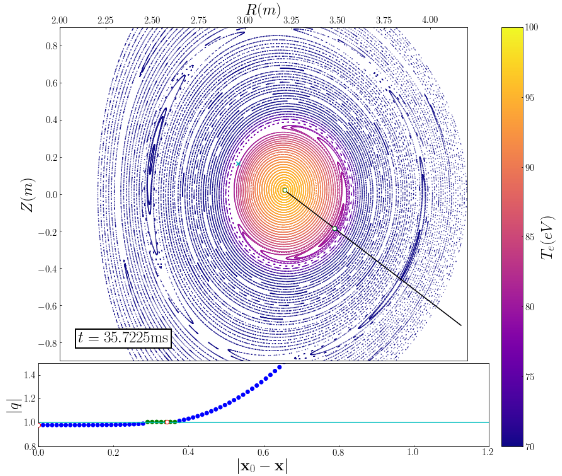


- First observed in 1974.
- Observed in every tokamak built since.
- The precise cause has been debated since 1976.

In the Kadomtsev model the crash is caused by a 1/1 mode

- q_0 reaches slightly below 1, and an intact $q = 1$ surface exists.
- This is unstable to the 1/1 internal kink mode, resistive tearing instability of the $q = 1$ surface, or both[a].
- Hot core plasma reconnects with cold plasma outside the $q = 1$ surface and is deposited in a growing $q/1$ island. After the crash the q-profile is very flat.
- The Kadomtsev model predicts a **crash when** $q_0 \sim 1$, and a **re-set to** $q_0 = 1$ after the crash.
- Experiments often observe exactly this[b,c,d].
- But not always . . .

Kadomtsev-like reconnection in M3DC^1 simulations



[a] Coppi, B., et al. Fizika Plazmy 2 (1976): 961-966.
[b] Weller, Phys. Rev. Lett. 59 2303 (1987)
[c] Wroblewski Phys. Fluids B 3, 2877 (1991)
[d] Nam, Y et al, NF 58 066009 (2018)

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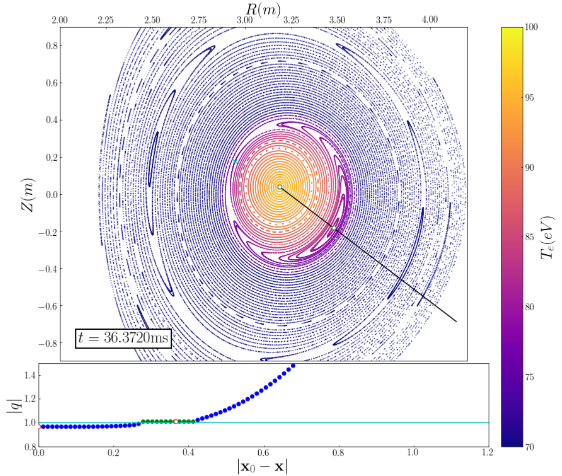

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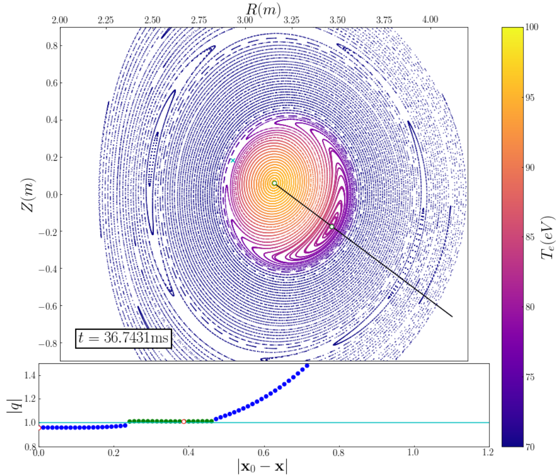

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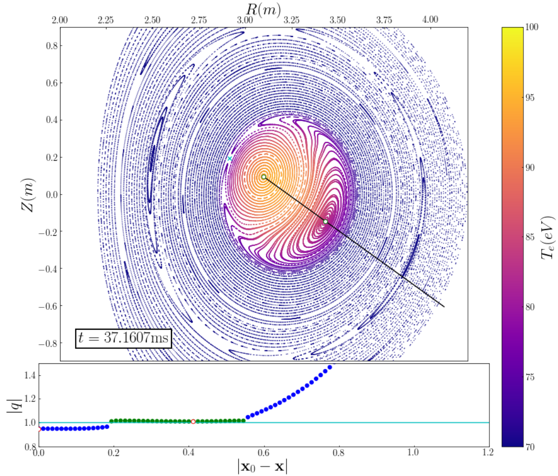

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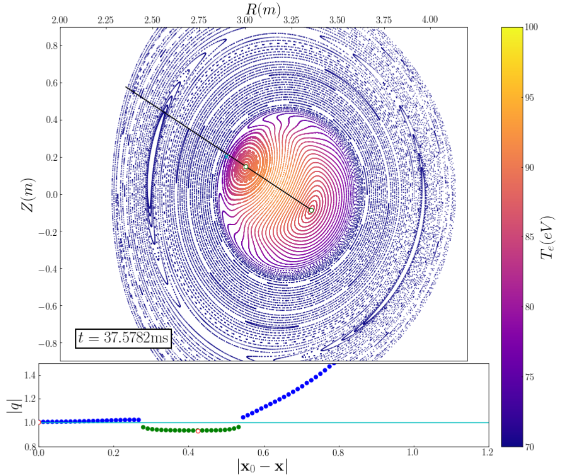

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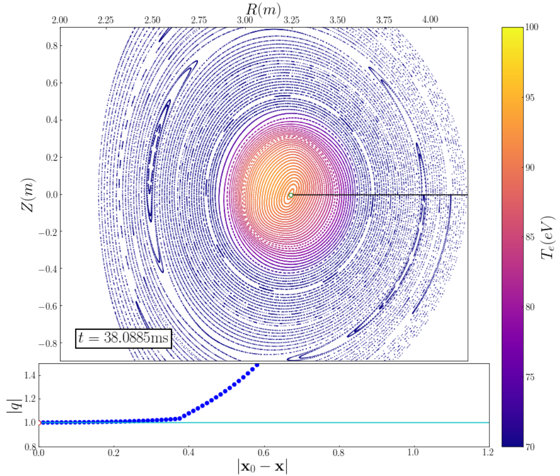


Direct measurements can also find $q_0 \simeq 0.7$

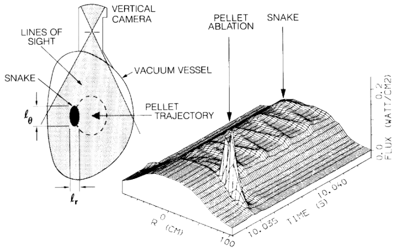
- Direct measurements of q_0 have been made with Motional Stark Effect (MSE), Faraday rotation imaging (FR), and Lithium Fluorescence imaging (Li).
- Experiments have consistently been measuring crashes when $q_0 = 0.7$, and $q < 1$ for the entire cycle.
- Though every measurement constitutes a difficult problem of translating pitch angle or polarization rotation to safety factor, the agreement, using different techniques, on many different machines, cannot be ignored.
- These measurements are irreconcileable with the Kadomtsev model.

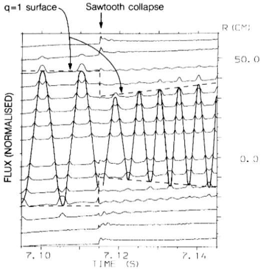
Device	**q_0 at crash**	**Method**
TEXTOR[a,b]	0.7 ± 0.1	FR
TFTR[c,d]	0.7 ± 0.1	MSE
TEXT[e]	$0.7 - 0.8$	Li
JET[f]	$0.7 - 0.8 \pm 0.1$	FR & MSE

[a] Soltwisch, Rev. Sci. Instrum. 57, 1939 (1986)
[b] Soltwisch, Rev. Sci. Inst. 59, 1599 (1988)
[c] Levinton, F. et al Phys. Fluids B. 5 2554 (1993)
[d] Yamada, Phys. Plasmas 1 3269 (1994)
[e] West, Phys. Rev. Lett. 58 2758 (1987)
[f] Wolf, Nuclear Fusion, 33 663 (1993)

Snakes show the $q = 1$ surface is not removed

- Upon pellet injection in JET, persistent 1/1 density perturbations were observed in soft x-rays.
- These snakes can appear double, occur where the $q = 1$ surface is expected, and are postulated to be islands on the 1/1 surface.
- The snakes can survive a sawtooth crash.
- Such a crash can thus not remove the 1/1 surface, which is a central prediction of the Kadomtsev model.



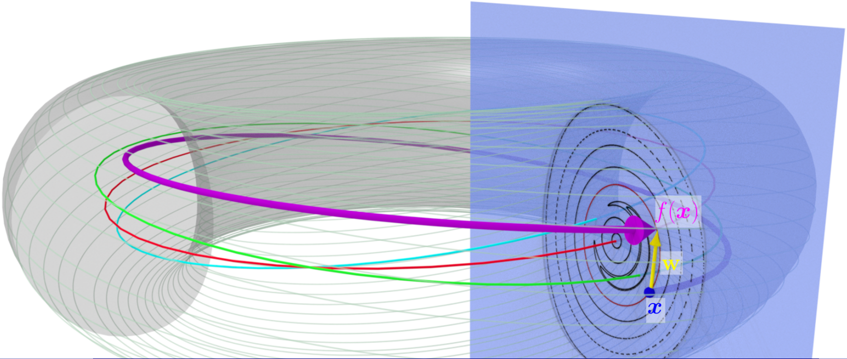


Another model is needed

- There is no question that we see crashes that involve $1/1$ (reset to $q_0 = 1$ and and a crash occurring when $q \sim 1$) that indicate a Kadomtsev-like process.
- A large set of other measurements show $q_0 \sim 0.7$ at time of crash, $q < 1$ for the entire cycle, and crashes that do not remove the $q = 1$ surface.
- These observations cannot be reconciled with the Kadomtsev model, and should be caused by a different mechanism.
- The *alternating-hyperbolic* sawtooth model predicts a crash caused by a fast ideal mode when $q_0 = 2/3$, that $q < 1$ for the entire cycle, and that the $q = 1$ surface is not removed.

The magnetic field in a tokamak defines a mapping

- Following a magnetic field line from the point $\boldsymbol{x} = (R_0, Z_0)$ once around to the same poloidal cross section defines the mapping $f(R_0, Z_0) = (R_1, Z_1)^T$.
- *Brouwer's fixed point theorem:* Any map from the disk to the disk contains a fixed point $f(\boldsymbol{x}) = \boldsymbol{x}$.
- The magnetic axis is an example of a closed field line.

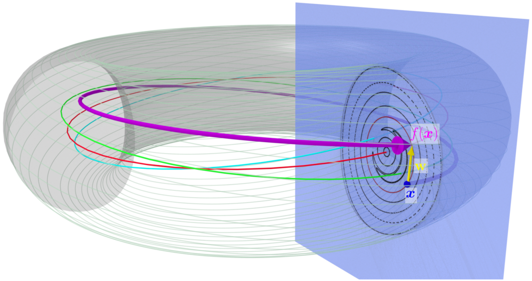


The mapping around a fixed point corresponds to an element of a Lie group

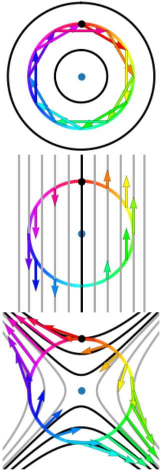
- Around a closed field line (where $\mathbf{f}(R_0, Z_0) = (R_1, Z_1) = (R_0, Z_0)$), we construct the matrix of partial derivatives
  $$\mathrm{M} = \begin{pmatrix} \frac{\partial R_1}{\partial R_0}, & \frac{\partial R_1}{\partial Z_0} \\ \frac{\partial Z_1}{\partial R_0}, & \frac{\partial Z_1}{\partial R_0} \end{pmatrix}.$$
- The mapping has to be area preserving: $\det(\mathrm{M}) = 1$.
  M has real coefficients.
- The set of all 2×2 real matrices with unit determinant forms the Lie group $\mathrm{M} \in SL_2(\mathbb{R})$.
- Every possible configuration of the field around a fixed point corresponds with a matrix of this group.
- This matrix describes to first order the mapping around the fixed point: $\boldsymbol{f}(\boldsymbol{x}_0 + \delta\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \mathrm{M} \cdot \delta\boldsymbol{x}$.

As the magnetic field changes continuously, so does the element of $\mathrm{SL}(2, \mathbb{R})$

- When the magnetic field changes continuously in time, field line map changes continuously, as it is the integration of the field over a finite distance.
- Every entry in the matrix M therefore changes continuously in time.
- The matrix M, describing the structure of the field, traces a continuous path through the group $\mathrm{SL}(2, \mathbb{R})$.



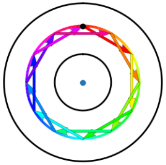
The Lie group $\mathrm{SL}(2,\mathbb{R})$ has three subsets

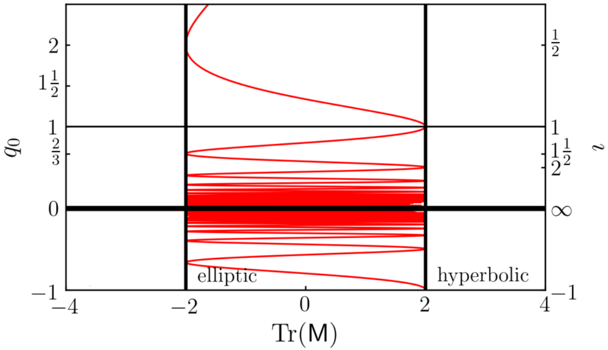


- Elements of $\mathrm{SL}(2,\mathbb{R})$ can be classified into three subsets by their action as linear transformation on the Euclidean plane:
  - The *elliptic* subset constitutes rotations (magetic axis).
  - The *parabolic* subset constitutes shear mappings (intact q-surface).
  - The *hyperbolic* subset constitutes squeeze mappings (x-point).
- The subset a matrix is a part of is determined by its trace.

The elliptic subset consists of matrices with $|\mathrm{Tr}(\mathsf{M})| < 2$

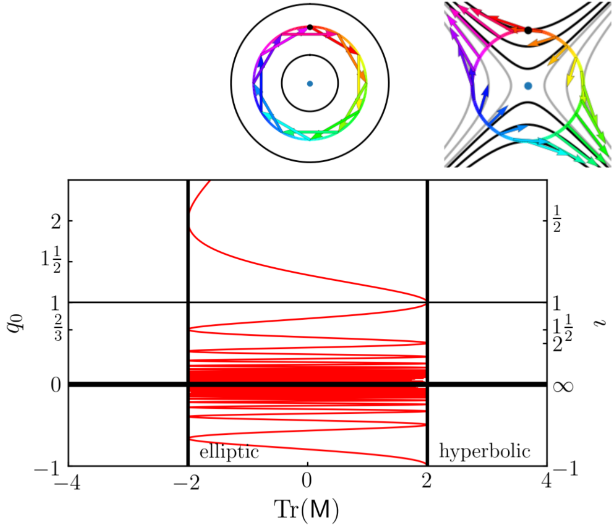
- When $|\mathrm{Tr}(\mathsf{M})| < 2$, the eigenvalues and eigenvectors are complex: $\lambda_{\pm} = (\mathrm{Tr}(\mathsf{M}) \pm \sqrt{\mathrm{Tr}(\mathsf{M})^2 - 4})/2$.
- Field lines stay on ellipses (which form closed magnetic surfaces), and rotate over a certain angle.
- The safety factor is related to the trace via: $\cos(2\pi/q_0) = \frac{1}{2}\mathrm{Tr}(\mathsf{M})$.
- Because of their structure in a Poincaré plot, fixed points like this are called O-points.





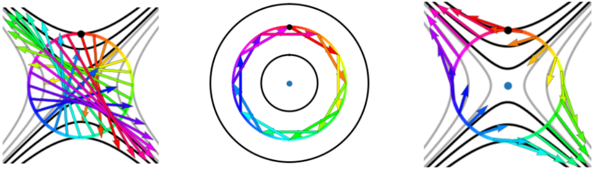
The hyperbolic subset consists of matrices with $\mathrm{Tr}(\mathsf{M}) > 2$

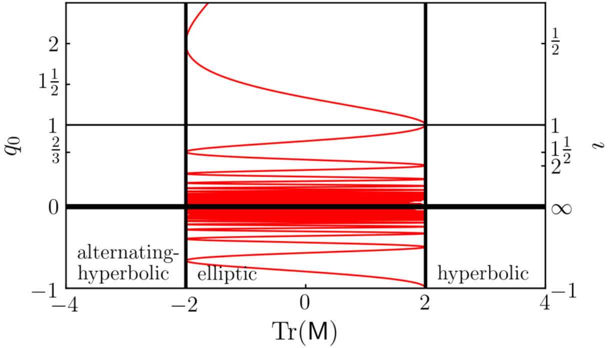
- When $\mathrm{Tr}(\mathsf{M}) > 2$, the eigenvalues and eigenvectors are real and positive: $\lambda_{\pm} = (\mathrm{Tr}(\mathsf{M}) \pm \sqrt{\mathrm{Tr}(\mathsf{M})^2 - 4})/2$.
- These two real vectors determine the direction field lines approach and leave the x-point ($\boldsymbol{f}(\boldsymbol{x}_0 + \delta \boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \mathsf{M} \cdot \delta \boldsymbol{x}$).
- Because of ther structure in a Poincaré plot, fixed points of this type are called X-points.



Alternating-hyperbolic points have $\mathrm{Tr}(\mathrm{M}) < -2$

- When $\mathrm{Tr}(\mathrm{M}) < -2$ both eigenvectors are real and negative .
- The mapping is a combination of a pointwise reflection in the fixed point $(-\mathbb{I})$ and a squeeze mapping. We call these points *alternaging-hyperbolic* fixed points.
- When the field lines rotate exactly an integer *and a half* times around the axis ($q_0 = 1/(2+n)$) the mapping becomes a pointwise reflection in the fixed point $-\mathbb{I}$.
- An infinitesimal perturbation to this mapping can then change the structure to alternating-hyperbolic.
- When field lines wind around 1.5 times,
  $q_0 = 2/3 = 0.66\ldots \simeq 0.7$





Ideal modes in the core region

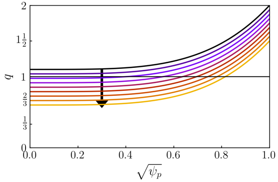
- We use the NOVA code to analyze the ideal modes in the core region.
- The NOVA code solves normal mode formulation of the ideal MHD stability equation:

$$-\omega^2 \rho \boldsymbol{\xi} = \boldsymbol{F}[\boldsymbol{\xi}], \quad (1)$$

where ω^2 is the mode frequency squared, ρ is the plasma density, $\boldsymbol{F}$ is the linerized MHD force operator, and $\boldsymbol{\xi}$ is the displacement vector.

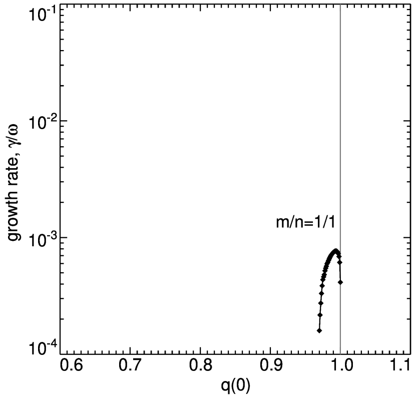
- Solutions $\boldsymbol{\xi}$ with negative ω^2 correspond to ideally unstable, exponentially growing modes.

parameter	value
B_T	$1\mathrm{T}$
R_0	$3\mathrm{m}$
a	$1\mathrm{m}$
q	$q_0 + 0.9\psi^2$
β_0	3%



Ideal modes in the core region

- q_0 is scanned from $q_0 = 1.1$ to $q_0 = 0.6$
- When $q_0 \sim 1$ we see the $1/1$ internal kink instability.



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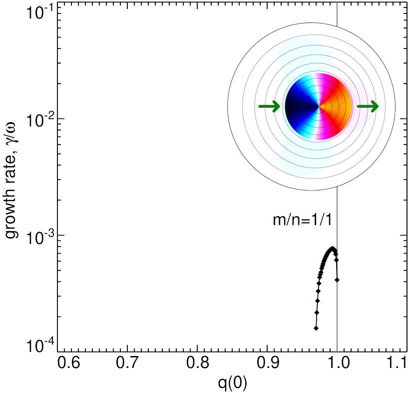
- The displacement vector $\boldsymbol{\xi}$ of the $1/1$ mode shows the expected translation of the core plasma corresponding with the internal kink.



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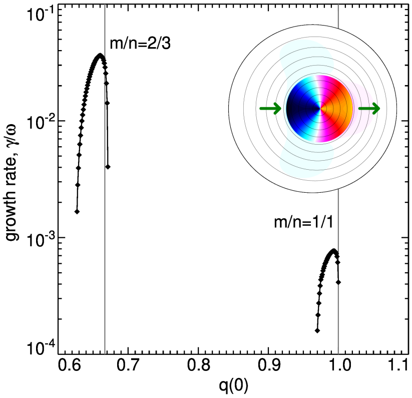
- When $q_0 = 2/3$ we see a new ideal 2/3 instability with an order of magnitude higher growth rate.



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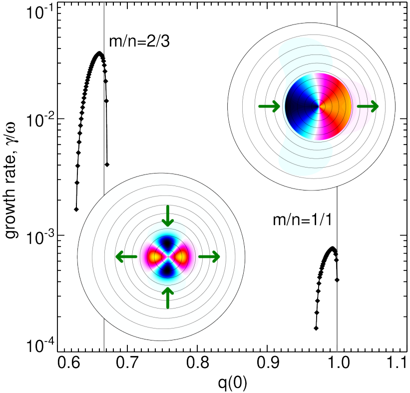
- The displacement vector $\boldsymbol{\xi}$ of the $1/1$ mode shows the expected translation of the core plasma corresponding with the internal kink.
- When $q_0 = 2/3$ we see a new ideal $2/3$ instability with an order of magnitude higher growth rate.
- $\boldsymbol{\xi}$ of the $2/3$ mode indicates a displacement where the plasma is forced onto the axis in two directions, and away from it in two others.



The displacement produces a perturbed magnetic field

- In the normal mode formulation of the ideal MHD equations, the perturbed field $\delta\boldsymbol{B}$ caused by a displacement $\boldsymbol{\xi}$ is given by:

$$\delta\boldsymbol{B} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B_0}) \tag{2}$$

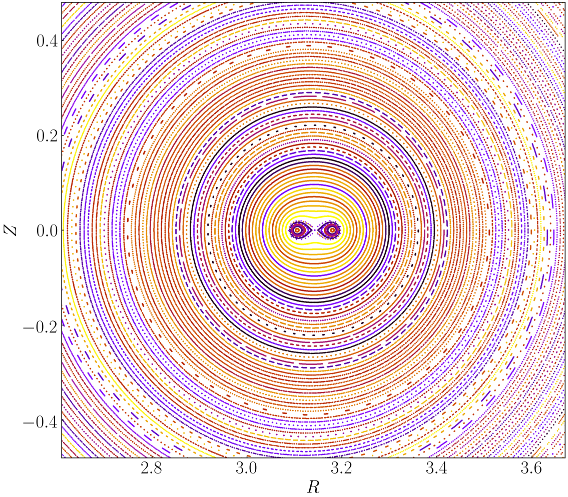
- We use an analytical perturbation $\boldsymbol{\xi}$ with $\xi_R = -(1/R)\partial_Z\Psi$ and $\xi_Z = (1/R)\partial_R\Psi$.

$$\Psi = A \exp\left(-\frac{\psi_p}{\sigma}\right)\cos(2\theta - 3\phi) \tag{3}$$

- We add this to a Grad-Shafranov equilibrium with $\beta_0 = 3\%$ and $q = 2/3 + 2.3333 * \psi_p$.
- We construct a Poincaré plot of the perturbed field $\boldsymbol{B}_0 + \delta\boldsymbol{B}$ by tracing the orbits of low-energy electrons $(1\mathrm{keV})$ using the SPIRAL code.

The mode drives the axis to an alternating-hyperbolic point

- At a small amplitude of the perturbation $A = 4 \times 10^{-4}$ the perturbed field $|\delta \boldsymbol{B}|/|\boldsymbol{B}_0| \sim 1 \times 10^{-3}$.
- The axis becomes an alternating hyperbolic fixed point



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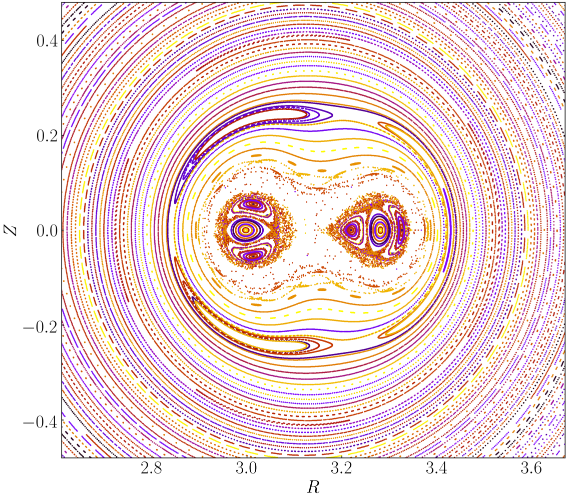
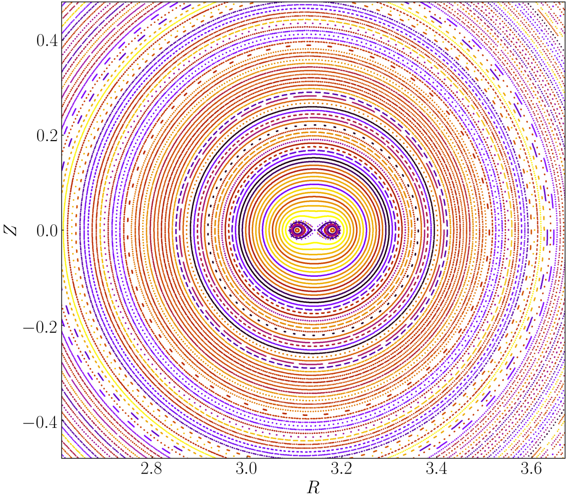
- At a small amplitude of the perturbation $A = 4 \times 10^{-4}$ the perturbed field $|\delta \boldsymbol{B}|/|\boldsymbol{B}_0| \sim 1 \times 10^{-3}$.
- The axis becomes an alternating hyperbolic fixed point
- At higher amplitude of the perturbation At a higher amplitude $A = 0.01$ ($|\delta \boldsymbol{B}|/|\boldsymbol{B}_0| \sim 2.5 \times 10^{-2}$) a stochastic region is created in the core.
- The $q = 1$ surface is broken up into a 3/3 island chain.



Beyond linearized ideal MHD

- The change in topology magnetically connects the highest-pressure plasma on the axis with surfaces further out.
- Rapid parallel transport can induce flow along the field lines and cause further perturbations.



Beyond linearized ideal MHD

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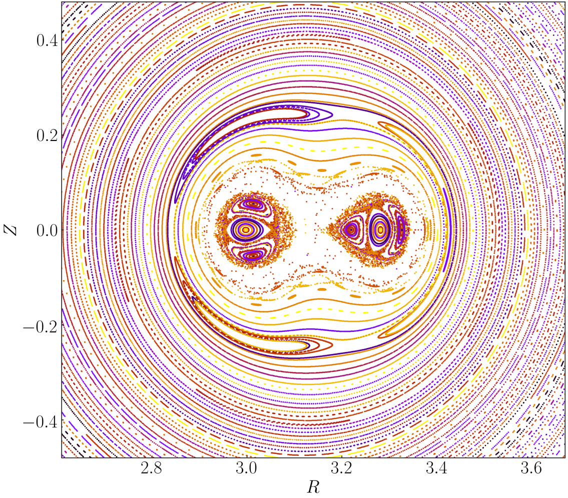
- Rapid parallel transport can induce flow along the field lines and cause further perturbations.
- At high amplitude of the perturbation, the stochastic region is magnetically connected.
- The temperature can equilibrate in the connected region.
- Magnetic chaos leads to rapid reconnection in this regioin. This redistributes poloidal and toroidal fluxes, and a higher q_0



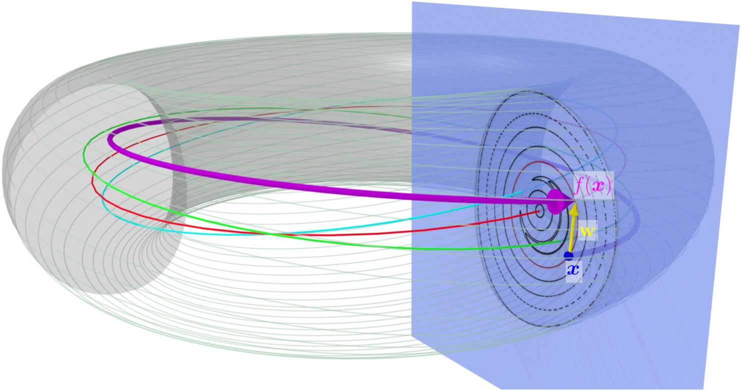
The Alternating Hyperbolic sawtooth model

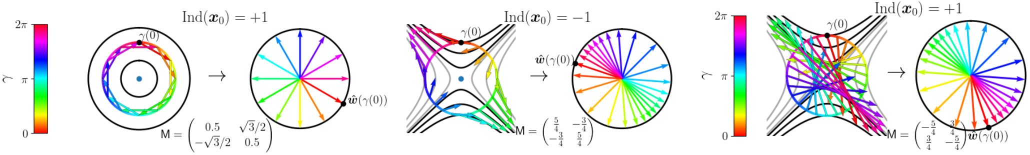
1. The internal kink is stabilized (through toroidal rotation, fast particles or another mechanism), and q_0 decreases through slow current diffusion.
2. When q_0 reaches $2/3$, the ideal mode causes the axis to transition to an alternating-hyperbolic fixed point.
3. This mode (possibly with other modes that occur when the topology changes) creates a stochastic region in the core.
4. The temperature equilibrates in the core region, and poloidal and toroidal fluxes are redistributed leading to an increased q_0.
5. The field shifts out of resonance with the $2/3$ mode, the flux surfaces in the core heal, and the cycle begins anew.

Conclusions

- The observations of $q_0 \sim 0.7$ at time of crash and below 1 for the entire cycle, are irreconcilable with the Kadomtsev and Wesson models.
- By identifying the magnetic structure around the magnetic axis with elements of $\mathrm{SL}(2, \mathbb{R})$, we show that exactly when when $q_0 = 2/3$ the magnetic axis can continuously change to become alternating hyperbolic.
- MHD stability calculations show that there is a high growth rate ideal mode exactly when this is the case.
- The displacement associated with this mode is directed onto and away from the axis in a pattern that drives the transition to the alternating-hyperbolic configuration; the magnetic field associated with this perturbation directly causes it.
- The alternating-hyperbolic sawtooth model fits with the low q_0 measurements, allows for a cycle where q_0 remains below 1, and does not remove the $q = 1$ surface.

Thank you for your attention

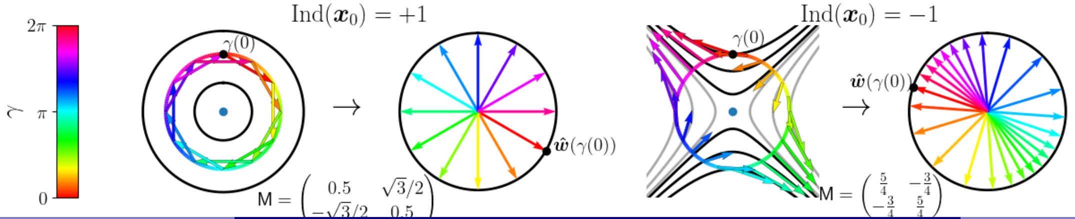




Bonus slide: Topological index of a fixed point

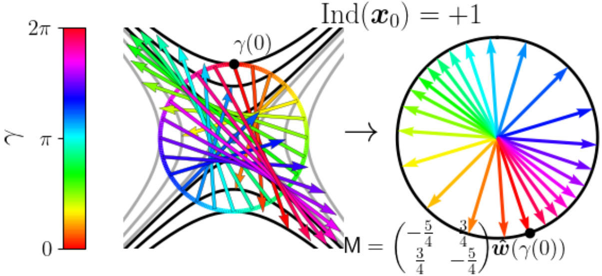
- Define the vector field $\boldsymbol{w} = (R_0, Z_0)^T - \boldsymbol{f}(R_0, Z_0)$, which is zero at the fixed points and let $\gamma(\tau)$ be a closed curve, parametrized by τ that encloses an isolated zero $\boldsymbol{x}_0$ of $\boldsymbol{w}$.
- The function $g : \gamma \to S^1$ sends every point on γ to the unit vector $g : \gamma(\tau) \mapsto \boldsymbol{w}(\gamma)/|\boldsymbol{w}(\gamma)|$ on the unit circle S^1.
- The index of the isolated zero $\boldsymbol{x}_0$ is then the *degree* of this mapping: $\mathrm{Ind}(\mathrm{x}_0) = \mathrm{deg}(\mathrm{g})$
- The degree is also sometimes called the winding number, and is positive if the image of $\gamma(\tau)$ is traversed in the same (co/counter-clockwise) direction as $\gamma(\tau)$.
- In case γ encloses more than one isolated zero, the degree of g is the sum of the indices of all zeros enclosed.

The sum of the indices of fixed points in a tokamak is always $+1$



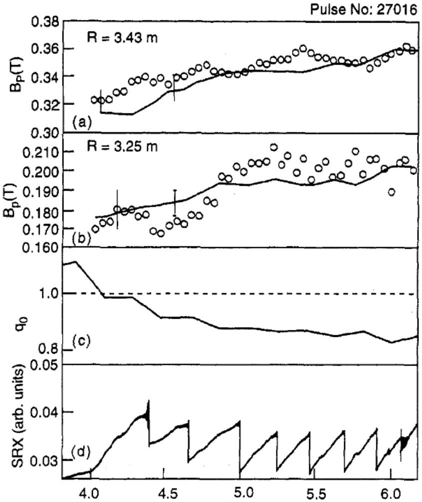
Bonus slide: An alternating-hyperbolic fixed point has index $+1$

- The alernating-hyperbolic fixed point maps points to the opposite hyperbolic branch.
- when $\gamma(\tau)$ is traversed clockwise, the image on S^1 is also traversed clockwise.
- The only fixed point in the tokamak can become alternating-hyperbolic.



JET: Circular plasmas in D shaped devices show crashes at 1 and below

- Circular cross-section discharges with combined MSE and Faraday rotation measurents were studied in JET. Both techniques yield $q_0 \sim 0.7$.
- First crash occurs when $q_0 \sim 1$ is reconstructed.
- Subsequently there is a significant increase in poloidal field.
- Even if absolute value is of q is debatable, the delta is clear.
- Note: q_0 represents cycle-average and therefore higher.



Hints from TFTR

- Yamada *et al.* measured sawteeth with line ECE and MSE on TFTR.
- The heat flow contours in the core region show a $n = 2$ structure at time of crash.
- The poloidal temperature distribution was inferred from toroidal rotation ($120\mu s$).