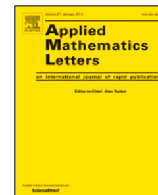




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On the proof of Taylor’s conjecture in multiply connected domains

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ABSTRACT

In this Letter we extend the proof, by Faraco and Lindberg (2020), of Taylor’s conjecture in multiply connected domains to cover arbitrary vector potentials and remove the need to impose restrictions on the magnetic field to ensure gauge invariance of the helicity integral. This extension allows us to treat general magnetic fields in closed domains that are important in laboratory plasmas and brings closure to a conjecture whose resolution has been open for almost 50 years.
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1. Introduction

The Woltjer–Taylor theory of magnetic relaxation is remarkable in the sense that it is one of the few examples of turbulence theory for which the end result can be written down explicitly. In short, the theory predicts that for a magnetically-closed plasma, a magnetic field will evolve through turbulent relaxation to the equilibrium of a linear force-free field. This relaxation is based on two key results: Woltjer’s theorem [1] and Taylor’s conjecture [2,3]. Woltjer’s theorem states that in a fixed and magnetically-closed domain, the magnetic field that minimizes the magnetic energy whilst keeping the global magnetic helicity fixed is a linear force-free field. Taylor combined this theorem with the conjecture that in a non-ideal plasma (one with small but non-zero resistivity) the global helicity of the entire domain is approximately conserved. Related to this, Taylor also assumed that the magnetic helicities of subdomains (sub-helicities) are not important in determining the final relaxed state since the magnetic field undergoes significant field line reconnection during turbulent relaxation, thus destroying the sub-helicities.

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1 Despite some good experimental evidence for Woltjer–Taylor relaxation, see [3–5], there also exists
 2 research which suggests that the assumption that only the global magnetic helicity is conserved is too
 3 restrictive in general. Simulations, such as [6], show that magnetic fields do not always relax to linear force-
 4 free fields. This deviation from the classic Woltjer–Taylor theory has been attributed to the influence of
 5 other topological invariants besides the global magnetic helicity [7]. It has also been suggested that field line
 6 helicities (sub-helicities of individual field lines) can play an important role in magnetic relaxation, and that
 7 they are redistributed by reconnection rather than destroyed. See [8] for a summary of these issues.

8 Although modifications to the classic Woltjer–Taylor theory seem to be needed for a more comprehensive
 9 theory of magnetic relaxation, one aspect of the theory that has proved to be robust is Taylor’s conjecture
 10 itself. Berger [9] showed that strict limits on magnetic helicity dissipation can be found, which depend on the
 11 plasma energy, the energy dissipation rate and the magnetic diffusion. In both astrophysical and laboratory
 12 plasmas, the conditions are often such that magnetic helicity dissipation, following Berger’s bounds, is very
 13 small and so magnetic helicity can be treated as an invariant in non-ideal (resistive) plasmas as well as ideal
 14 plasmas. For a list of other relevant theoretical works on helicity conservation, we direct the reader to [10,
 15 Sect. 1].

16 Recently, Faraco and Lindberg [10] provided a rigorous proof of Taylor’s conjecture in a magnetically-
 17 closed turbulent plasma. They described the turbulent plasma using Onsager’s theory of turbulence [11,12]
 18 and proved the conjecture by considering limits of Leray–Hopf solutions of non-ideal magnetohydrodynamics
 19 (MHD). The proof considers both simply connected and multiply connected domains. In the former, the
 20 choice of vector potential in the magnetic helicity is arbitrary, but in the latter this is not the case. The fact
 21 that only certain vector potentials are suitable in multiply connected domains stems from [10] considering
 22 the classical form of magnetic helicity,

$$23 \quad H = \int_{\Omega} \mathbf{A} \cdot \mathbf{B}, \quad (1)$$

24 where \mathbf{B} is the magnetic field, \mathbf{A} is the vector potential with $\text{curl } \mathbf{A} = \mathbf{B}$ and Ω is the magnetically-closed
 25 domain. Eq. (1) represents the gauge invariant form of helicity in simply connected domains. Thus, the
 26 problem of confirming Taylor’s conjecture fully without reference to specific vector potentials remains to be
 27 resolved.

28 MacTaggart and Valli [13] showed that the definition of magnetic helicity H needs to be extended for
 29 multiply connected domains in order to accommodate arbitrary vector potentials that do not impose any
 30 restrictions on the magnetic field. For example, it can be shown that a consequence of imposing gauge
 31 invariance with Eq. (1) in multiply connected domains is that the magnetic flux through internal cuts of
 32 the domain must be zero, e.g., the toroidal magnetic flux in a tokamak would be zero.

33 In this Letter, we bring together the results of Faraco and Lindberg [10] and MacTaggart and Valli [13] to
 34 provide a complete proof of Taylor’s conjecture for arbitrary vector potentials of closed magnetic fields with
 35 no restrictions on the magnetic field imposed by gauge invariance. We proceed by only describing extensions
 36 and modifications to specific parts of the proof in [10] that are relevant for expanding its scope in the manner
 37 described above. For full details of the many technical results involved in the original proof, the reader is
 38 directed to the work by Faraco and Lindberg [10]. Before presenting the main result, we now introduce some
 39 preliminary material that will be key to understanding the extension to their proof.

40 2. Preliminaries

41 2.1. Geometric set-up

42 We consider a multiply connected domain $\Omega \subset \mathbb{R}^3$ that is a bounded open connected set with Lipschitz
 43 continuous boundary $\partial\Omega$ and unit outer normal \mathbf{n} . Let $g > 0$ be the first Betti number, or genus, of Ω . Then

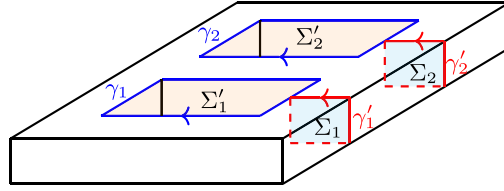


Fig. 1. An illustration of cutting surfaces (shaded areas) and cycles (coloured paths) for a 2-fold torus. The notation is as described in the main text. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

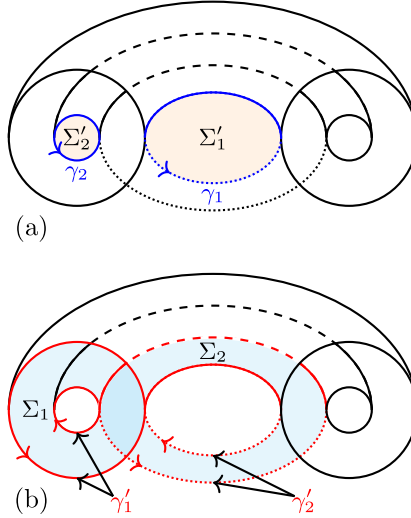


Fig. 2. An illustration of cutting surfaces (shaded areas) and cycles (coloured paths) for a toroidal shell (cut in half). (a) shows the cutting surfaces of Ω' and their associated cycles. (b) shows the same but for Ω . (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

the first Betti number of $\partial\Omega$ is $2g$. We can consider $2g$ non-bounding cycles on $\partial\Omega$, $\{\gamma_i\}_{i=1}^g \cup \{\gamma'_i\}_{i=1}^g$, that represent the generators of the first homology group of $\partial\Omega$. The $\{\gamma_i\}_{i=1}^g$ represent the generators of the first homology group of $\overline{\Omega}$. The tangent vector on γ_i is denoted \mathbf{t}_i . Similarly, the $\{\gamma'_i\}_{i=1}^g$ represent the generators of the first homology group of $\overline{\Omega}'$, where $\Omega' = B \setminus \overline{\Omega}$ and B is an open ball containing $\overline{\Omega}$. The tangent vector on γ'_i is denoted \mathbf{t}'_i .

In Ω there exist g ‘cutting surfaces’ $\{\Sigma_i\}_{i=1}^g$ that are connected orientable Lipschitz surfaces satisfying $\Sigma_i \subset \Omega$. Each cutting surface in Ω satisfies $\partial\Sigma_i = \gamma'_i$ and $\overline{\Sigma}_i$ ‘cuts’ the corresponding cycle γ_i in exactly one point. Each cutting surface Σ_i has a unit normal vector \mathbf{n}_{Σ_i} oriented as γ_i . An equivalent definition applies for the cutting surfaces $\{\Sigma'_i\}_{i=1}^g$ of Ω' . In particular, $\partial\Sigma'_i = \gamma_i$.

Two examples of the geometric set-up are displayed in Fig. 1 for a 2-fold torus and in Fig. 2 for a toroidal shell.

2.2. Required function spaces

Using the basic Hilbert spaces listed in Alonso Rodríguez et al. [14, Sect. 2], the space of harmonic Neumann fields is defined by

$$\mathcal{K}_T := H(\text{curl}^0; \Omega) \cap H_0(\text{div}^0; \Omega).$$

1 Remember that if $\mathbf{v} \in H_0(\operatorname{div}^0; \Omega)$ satisfies $\mathbf{v} \perp \mathcal{K}_T$ (where orthogonality has to be intended in $(L^2(\Omega))^3$)
 2 then it follows that $\int_{\Sigma_i} \mathbf{v} \cdot \mathbf{n}_{\Sigma_i} dS = 0$ for $i = 1, \dots, g$, and viceversa. Thus, if we denote

3
$$\mathcal{Q} := \nabla H^1(\Omega) \overset{\perp}{\oplus} \mathcal{K}_T,$$

we have

$$\begin{aligned} \mathcal{Q} &= H(\operatorname{curl}^0; \Omega), \\ \mathcal{Q}^\perp &= \left\{ \mathbf{v} \in H_0(\operatorname{div}^0; \Omega) : \int_{\Sigma_i} \mathbf{v} \cdot \mathbf{n}_{\Sigma_i} = 0 \text{ for } i = 1, \dots, g \right\}, \\ H_0(\operatorname{div}^0; \Omega) &= \mathcal{Q}^\perp \overset{\perp}{\oplus} \mathcal{K}_T. \end{aligned}$$

4 Analogous spaces can be defined for Ω' , e.g., $\mathcal{K}'_T := H(\operatorname{curl}^0; \Omega') \cap H_0(\operatorname{div}^0; \Omega')$. For the ease of the reader, let
 5 us also recall that the notation in Faraco and Lindberg [10] is different: precisely, $L^2_\sigma(\Omega; \mathbb{R}^3) = H_0(\operatorname{div}^0; \Omega)$,
 6 $L^2_H(\Omega; \mathbb{R}^3) = \mathcal{K}_T$, $L^2_\Sigma(\Omega; \mathbb{R}^3) = \mathcal{Q}^\perp$.

7 *2.3. Integral identities*

8 Let the basis functions of \mathcal{K}_T and \mathcal{K}'_T be denoted by $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}'_i$, $i = 1 \dots g$, respectively. These functions
 9 are defined by imposing that $\oint_{\gamma_k} \boldsymbol{\rho}_i \cdot \mathbf{t}_k = \delta_{ki}$ and $\oint_{\gamma'_k} \boldsymbol{\rho}'_i \cdot \mathbf{t}'_k = \delta_{ki}$.

10 Let us recall that, given $\mathbf{v} \in H_0(\operatorname{div}^0; \Omega)$, it holds that

11
$$\int_{\Omega} \mathbf{v} \cdot \boldsymbol{\rho}_i = \int_{\Sigma_i} \mathbf{v} \cdot \mathbf{n}_{\Sigma_i}. \tag{2}$$

12 Given $\mathbf{w} \in \mathcal{X} := \{\mathbf{w} \in H(\operatorname{curl}; \Omega) : \operatorname{curl} \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$, as in Alonso Rodríguez et al. [14] we define

13
$$\oint_{\gamma_i} \mathbf{w} \cdot \mathbf{t}_i := - \int_{\partial\Omega} \mathbf{n} \times \mathbf{w} \cdot \boldsymbol{\rho}'_i, \quad \oint_{\gamma'_i} \mathbf{w} \cdot \mathbf{t}'_i := \int_{\partial\Omega} \mathbf{n} \times \mathbf{w} \cdot \boldsymbol{\rho}_i$$

14 (as in [14] the generators of the first homology group of $\overline{\Omega}$ are denoted γ'_i and those of the first homology
 15 group of $\overline{\Omega'}$ are denoted γ_i). Note that, with these definitions and taking into account (2), Stokes' theorem
 16 holds on Σ_i , i.e., $\oint_{\gamma'_i} \mathbf{w} \cdot \mathbf{t}'_i = \int_{\Sigma_i} \operatorname{curl} \mathbf{w} \cdot \mathbf{n}_{\Sigma_i}$.

17 As in Lemma 7 of [14], for $\mathbf{w}, \mathbf{v} \in \mathcal{X}$ we find

18
$$\begin{aligned} \int_{\partial\Omega} (\mathbf{w} \times \mathbf{n}) \cdot \mathbf{v} &= \sum_{i=1}^g \left(\oint_{\gamma_i} \mathbf{w} \cdot \mathbf{t}_i \right) \left(\oint_{\gamma'_i} \mathbf{v} \cdot \mathbf{t}'_i \right) \\ &\quad - \sum_{i=1}^g \left(\oint_{\gamma'_i} \mathbf{w} \cdot \mathbf{t}'_i \right) \left(\oint_{\gamma_i} \mathbf{v} \cdot \mathbf{t}_i \right). \end{aligned} \tag{3}$$

19 For more details on the derivation of this result, the reader is directed to [14].

20 *2.4. Leray–Hopf solutions of viscous, resistive MHD*

21 In [10], the domains studied have a $C^{1,1}$ regular boundary. This smoothness condition implies, in
 22 particular, that $X_T(\Omega) := H(\operatorname{curl}; \Omega) \cap H_0(\operatorname{div}; \Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^6(\Omega)$, and this chain of embeddings
 23 allows one to prove the existence of Leray–Hopf solutions for the velocity $\mathbf{u} \in L^\infty(0, T; H_0(\operatorname{div}^0(\Omega))) \cap$
 24 $L^2(0, T; H^1_0(\Omega))$ and magnetic field $\mathbf{B} \in L^\infty(0, T; H_0(\operatorname{div}^0(\Omega))) \cap L^2(0, T; H^1(\Omega))$ by standard arguments
 25 (see [10, Appendix A]).

In fact, Lipschitz regularity of $\partial\Omega$ is also a strong enough assumption for the existence of Leray–Hopf solutions, as long as the definition of Leray–Hopf solutions is modified in the manner described below. First recall that in bounded Lipschitz domains, the weaker embedding result $X_T(\Omega) \hookrightarrow H^{1/2}(\Omega) \hookrightarrow L^3(\Omega)$ holds. We relax the smoothness requirements on the magnetic field to $\mathbf{B} \in L^\infty(0, T; H_0(\text{div}^0(\Omega))) \cap L^2(0, T; X_T(\Omega))$. Whereas in $C^{1,1}$ regular domains the weak formulation of the Cauchy momentum equation says that, for viscosity ν , $\langle \partial_t \mathbf{u}, \varphi \rangle + \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{B} \cdot \nabla \mathbf{B}) \cdot \varphi + \nu \int_\Omega \nabla \mathbf{u} : \nabla \varphi = 0$ holds for almost all $t \in [0, T]$ for every $\varphi \in H(\text{div}^0(\Omega)) \cap H_0^1(\Omega)$, in bounded Lipschitz domains the formula $\langle \partial_t \mathbf{u}, \varphi \rangle + \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u} - \text{curl } \mathbf{B} \times \mathbf{B}) \cdot \varphi + \nu \int_\Omega \nabla \mathbf{u} : \nabla \varphi = 0$ needs to be used instead; note that $\|\text{curl } \mathbf{B} \times \mathbf{B} \cdot \varphi\|_{L^1} \leq \|\text{curl } \mathbf{B}\|_{L^2} \|\mathbf{B}\|_{L^3} \|\varphi\|_{L^6}$ for almost all $t \in [0, T]$ by Hölder’s inequality. Under these modifications, the main result [10, Theorem 1.5] holds and the proof only requires minor adjustments.

3. Main result

Suppose that $\mathbf{B}, \mathbf{u} \in L^\infty(0, T; H_0(\text{div}^0; \Omega))$ (therefore, with $\mathbf{B} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and no other boundary condition) form a weak solution of ideal MHD with initial data $\mathbf{B}_0, \mathbf{u}_0 \in H_0(\text{div}^0; \Omega)$, where \mathbf{u} is the velocity. Assume, furthermore, that (\mathbf{B}, \mathbf{u}) arises as an ideal (inviscid, non-resistive) limit of Leray–Hopf solutions $(\mathbf{B}_k, \mathbf{u}_k)$ (in particular, satisfying the boundary conditions $\mathbf{u}_k = \mathbf{0}$, $\mathbf{B}_k \cdot \mathbf{n} = 0$ and $\text{curl } \mathbf{B}_k \times \mathbf{n} = \mathbf{0}$). Using the notation from [10] we denote

$$\mathbf{B} = \mathbf{B}^\Sigma + \mathbf{B}^H, \quad \mathbf{B}^\Sigma \in \mathcal{Q}^\perp \text{ and } \mathbf{B}^H \in \mathcal{K}_T. \tag{4}$$

Let us recall that for almost all time, \mathbf{B}^H is the unique element of \mathcal{K}_T such that $\int_{\Sigma_i} \mathbf{B}^H \cdot \mathbf{n}_{\Sigma_i} = \int_{\Sigma_i} \mathbf{B} \cdot \mathbf{n}_{\Sigma_i}$ for each $i = 1, \dots, g$. Further, we decompose the vector potential as

$$\mathbf{A} = \mathbf{A}^\Sigma + \mathbf{A}^H, \tag{5}$$

where, for almost all time, \mathbf{A}^H is the unique element of \mathcal{Q}^\perp such that $\text{curl } \mathbf{A}^H = \mathbf{B}^H$ (for these existence results see, e.g., [15, Sect. 2.2, and in particular Remark 1]). Similar notation is used for \mathbf{B}_0 and \mathbf{A}_0 .

The generalized expression for magnetic helicity in multiply connected domains is defined as

$$\Upsilon(\mathbf{B}) := \int_\Omega \mathbf{A} \cdot \mathbf{B} - \sum_{i=1}^g \oint_{\gamma_i} \mathbf{A} \cdot \mathbf{t}_i \int_{\Sigma_i} \mathbf{B} \cdot \mathbf{n}_{\Sigma_i} \tag{6}$$

(see MacTaggart and Valli [13]). It is important to note that this definition is gauge invariant, i.e., it is independent of the vector potential $\mathbf{A} \in \mathcal{X}$.

Using this gauge invariant expression for magnetic helicity we are now in a position to write an adapted version of Theorem 1.5 of Faraco and Lindberg [10], which describes the convergence of magnetic helicity; clearly, no dependence on the vector potential appears.

Theorem 1. *In a domain $\Omega \subset \mathbb{R}^3$ with the properties described previously, suppose that (\mathbf{u}, \mathbf{B}) is a weak ideal limit of Leray–Hopf solutions $(\mathbf{u}_k, \mathbf{B}_k)$, $k \in \mathbb{N}$. The initial data of \mathbf{B} and \mathbf{B}_k are denoted \mathbf{B}_0 and $\mathbf{B}_{k,0}$, respectively. Then it holds that*

$$\Upsilon(\mathbf{B}_k, t) - \Upsilon(\mathbf{B}_{k,0}) = -2\eta_k \int_0^t \int_\Omega \mathbf{B}_k \cdot \text{curl } \mathbf{B}_k, \tag{7}$$

where $\eta_k > 0$ is the resistivity, for all $k \in \mathbb{N}$ and $t \in [0, T]$. Furthermore

$$\Upsilon(\mathbf{B}, t) - \Upsilon(\mathbf{B}_0) = 0, \tag{8}$$

for almost all $t \in [0, T]$.

1 **Proof.** Let us define

$$2 \quad Z(\mathbf{B}_k, \mathbf{A}_k; t) := \int_{\Omega} \mathbf{A}_k(t) \cdot \mathbf{B}_k(t) - \int_{\partial\Omega} \mathbf{A}_k^{\Sigma}(t) \times \mathbf{n} \cdot \mathbf{A}_k^H(t)$$

3 and similarly for $\mathbf{B}_{k,0}$ and $\mathbf{A}_{k,0}$, \mathbf{B} and \mathbf{A} , \mathbf{B}_0 and \mathbf{A}_0 . In [10, Lemma 4.3] it is shown that for Leray–Hopf
4 solutions (and their weak limits) the harmonic part of the magnetic field is stationary, i.e.,

$$5 \quad \mathbf{B}_k^H(t) = \mathbf{B}_{k,0}^H, \quad \mathbf{B}^H(t) = \mathbf{B}_0^H,$$

and consequently $\mathbf{A}_k^H(t) = \mathbf{A}_{k,0}^H$, $\mathbf{A}^H(t) = \mathbf{A}_0^H$. Therefore, Theorem 1.5 in [10] (having corrected for a minus
6 sign that is a misprint) says that

$$7 \quad Z(\mathbf{B}_k, \mathbf{A}_k; t) - Z(\mathbf{B}_{k,0}, \mathbf{A}_{k,0}) = -2\eta_k \int_0^t \int_{\Omega} \mathbf{B}_k \cdot \operatorname{curl} \mathbf{B}_k, \quad (9a)$$

$$8 \quad Z(\mathbf{B}, \mathbf{A}; t) - Z(\mathbf{B}_0, \mathbf{A}_0) = 0. \quad (9b)$$

Let us compute $\int_{\partial\Omega} \mathbf{A}_k^{\Sigma}(t) \times \mathbf{n} \cdot \mathbf{A}_k^H(t)$. By (3) we have

$$\begin{aligned} \int_{\partial\Omega} \mathbf{A}_k^{\Sigma}(t) \times \mathbf{n} \cdot \mathbf{A}_k^H(t) &= \sum_{i=1}^g \left(\oint_{\gamma_i} \mathbf{A}_k^{\Sigma}(t) \cdot \mathbf{t}_i \right) \left(\oint_{\gamma'_i} \mathbf{A}_{k,0}^H \cdot \mathbf{t}'_i \right) \\ &\quad - \sum_{i=1}^g \left(\oint_{\gamma'_i} \mathbf{A}_k^{\Sigma}(t) \cdot \mathbf{t}'_i \right) \left(\oint_{\gamma_i} \mathbf{A}_{k,0}^H \cdot \mathbf{t}_i \right). \end{aligned}$$

6 Using the decomposition in (4), we know that

$$7 \quad \int_{\Sigma_i} \mathbf{B}_k^{\Sigma}(t) \cdot \mathbf{n}_{\Sigma_i} = \int_{\Sigma_i} \mathbf{B}_{k,0}^{\Sigma} \cdot \mathbf{n}_{\Sigma_i} = 0. \quad (10)$$

8 Since $\operatorname{curl} \mathbf{A}_k^{\Sigma}(t) = \mathbf{B}_k^{\Sigma}(t)$, $\operatorname{curl} \mathbf{A}_{k,0}^H = \mathbf{B}_{k,0}^H$ and $\gamma'_i = \partial\Sigma_i$, applying Stokes' theorem gives

$$\begin{aligned} 9 \quad \oint_{\gamma'_i} \mathbf{A}_k^{\Sigma}(t) \cdot \mathbf{t}'_i &= \int_{\Sigma_i} \mathbf{B}_k^{\Sigma}(t) \cdot \mathbf{n}_{\Sigma_i} = 0, \\ \oint_{\gamma'_i} \mathbf{A}_{k,0}^H \cdot \mathbf{t}'_i &= \int_{\Sigma_i} \mathbf{B}_{k,0}^H \cdot \mathbf{n}_{\Sigma_i} = \int_{\Sigma_i} \mathbf{B}_{k,0} \cdot \mathbf{n}_{\Sigma_i}, \end{aligned}$$

10 and also

$$\begin{aligned} 11 \quad \int_{\Sigma_i} \mathbf{B}_k(t) \cdot \mathbf{n}_{\Sigma_i} &= \int_{\Sigma_i} \mathbf{B}_k^H(t) \cdot \mathbf{n}_{\Sigma_i} \\ &= \oint_{\gamma'_i} \mathbf{A}_k^H(t) \cdot \mathbf{t}'_i = \oint_{\gamma'_i} \mathbf{A}_{k,0}^H \cdot \mathbf{t}'_i = \int_{\Sigma_i} \mathbf{B}_{k,0} \cdot \mathbf{n}_{\Sigma_i}, \end{aligned} \quad (11)$$

12 for $t \in [0, T]$, $i = 1, \dots, g$, $k \in \mathbb{N}$. In conclusion, repeating the same computation for $\int_{\partial\Omega} \mathbf{A}_k^{\Sigma} \times \mathbf{n} \cdot \mathbf{A}_{k,0}^H$, we
13 find

$$\begin{aligned} Z(\mathbf{B}_k, \mathbf{A}_k; t) - Z(\mathbf{B}_{k,0}, \mathbf{A}_{k,0}) &= \int_{\Omega} \mathbf{A}_k(t) \cdot \mathbf{B}_k(t) - \int_{\Omega} \mathbf{A}_{k,0} \cdot \mathbf{B}_{k,0} \\ &\quad - \sum_{i=1}^g \left(\oint_{\gamma_i} [\mathbf{A}_k^{\Sigma}(t) - \mathbf{A}_{k,0}^{\Sigma}] \cdot \mathbf{t}_i \right) \left(\int_{\Sigma_i} \mathbf{B}_{k,0} \cdot \mathbf{n}_{\Sigma_i} \right) \\ 14 \quad &= \int_{\Omega} \mathbf{A}_k(t) \cdot \mathbf{B}_k(t) - \int_{\Omega} \mathbf{A}_{k,0} \cdot \mathbf{B}_{k,0} \\ &\quad - \sum_{i=1}^g \left(\oint_{\gamma_i} [\mathbf{A}_k(t) - \mathbf{A}_{k,0}] \cdot \mathbf{t}_i \right) \left(\int_{\Sigma_i} \mathbf{B}_{k,0} \cdot \mathbf{n}_{\Sigma_i} \right) \\ &= \Upsilon(\mathbf{B}_k, t) - \Upsilon(\mathbf{B}_{k,0}), \end{aligned}$$

15 having used again that $\mathbf{A}_k^H(t) = \mathbf{A}_{k,0}^H$ and moreover (11). By combining the computation above with (9a)
16 we get (7).

17 By analogous reasoning, $Z(\mathbf{B}, \mathbf{A}; t) - Z(\mathbf{B}_0, \mathbf{A}_0) = \Upsilon(\mathbf{B}, t) - \Upsilon(\mathbf{B}_0)$. Now (9b) leads to (8), and Theorem 1
18 is proved. \square

19 **Remark 1.** The right-hand side of (7) is bounded by $\frac{3}{2}\sqrt{t\eta_k}(\|\mathbf{u}_{k,0}\|_{L^2}^2 + \|\mathbf{B}_{k,0}\|_{L^2}^2)$ for all $k \in \mathbb{N}$ and
20 $t \in [0, T)$. This is seen exactly as in [10, Lemma 4.6] by using Young's inequality and then appealing to the
21 energy inequality.

4. Conclusion

This extension to the proof in Faraco and Lindberg [10] of Taylor’s conjecture allows for the extra consideration of defining magnetic helicity in multiply connected domains with arbitrary vector potentials, rather than a specific subset which may impose unwanted conditions on the magnetic field, e.g., forcing zero flux through cutting surfaces. This closes a long-standing conjecture that was posed by Taylor in [2] (see also [3, p. 744] where the multiply connected case is discussed explicitly).

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References

[1] L. Woltjer, A theorem on force-free magnetic fields, *Proc. Natl. Acad. Sci. USA* 44 (1958) 489–491.
 [2] J.B. Taylor, Relaxation of toroidal plasma and generation of reverse magnetic fields, *Phys. Rev. Lett.* 33 (1974) 1139–1141.
 [3] J.B. Taylor, Relaxation and magnetic reconnection in plasmas, *Rev. Modern Phys.* 56 (1986) 741–763.
 [4] E.P. Butt, A.A. Newton, A.J.L. Verhage, Experimental observations of the self reversal of toroidal magnetic field in pinches, in: D.E. Evans (Ed.), *Pulsed High Beta Plasmas*, Oxford, Pergamon, 1976, pp. 419–424.
 [5] H. Ji, S.C. Prager, J.S. Sarff, Conservation of magnetic helicity during plasma relaxation, *Phys. Rev. Lett.* 74 (1995) 2945–2948.
 [6] A.R. Yeates, A.J.B. Russell, G. Hornig, Physical role of topological constraints in localized magnetic relaxation, *Proc. Roy. Soc. A* 471 (2015) 20150012.
 [7] A.R. Yeates, G. Hornig, Dynamical constraints from field line topology in magnetic flux tubes, *J. Phys. A Math. Theor.* 44 (2011) 265501.
 [8] A.R. Yeates, Magnetohydrodynamic relaxation theory, in: D. MacTaggart, A. Hillier (Eds.), *Topics in Magnetohydrodynamic Topology, Reconnection and Stability Theory*, Springer, 2020, pp. 117–143.
 [9] M.A. Berger, Rigorous new limits on magnetic helicity dissipation in the solar corona, *Geophys. Astrophys. Fluid Dyn.* 30 (1984) 79–104.
 [10] D. Faraco, S. Lindberg, Proof of Taylor’s conjecture on magnetic helicity conservation, *Comm. Math. Phys.* 373 (2) (2020) 707–738.
 [11] L. Onsager, Statistical hydrodynamics, *Nuovo Cimento* 6 (1949) 279–287.
 [12] G.L. Eyink, K.R. Sreenivasan, Onsager and the theory of hydrodynamic turbulence, *Rev. Modern Phys.* 787–135 (2006) 279–287.
 [13] D. MacTaggart, A. Valli, Magnetic helicity in multiply connected domains, *J. Plasma Phys.* 85 (2019) 775850501.
 [14] A. Alonso Rodríguez, J. Camaño, R. Rodríguez, A. Valli, P. Venegas, Finite element approximation of the spectrum of the curl operator in a multiply connected domain, *Found. Comput. Math.* 18 (6) (2018) 1493–1533.
 [15] A. Alonso Rodríguez, E. Bertolazzi, A. Valli, The curl-div system: theory and finite element approximation, in: U. Langer, D. Pauly, S. Repin (Eds.), *Maxwell’s Equations*, De Gruyter, Berlin/Boston, 2019, pp. 1–43.