Core Dynamics: Magnetic Instabilities

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Introduction

The technique of considering the stability of some basic state to a perturbation is an essential one in fluid dynamics since it can give significant insight into fundamental mechanisms (see for example Chandrasekhar, 1961; Drazin and Reid, 1981). The classic example is that of Rayleigh-Bénard convection, in which fluid is contained between two horizontal flat plates. The lower plate is heated and the upper cooled. Since colder fluid is heavier than warmer fluid, the basic state consists of fluid at rest, with density ρ increasing with height z. This configuration is potentially unstable (heavy fluid lying over light fluid). However, convection (overturning of the fluid) will only take place if the density gradient $d\rho/dz$ is sufficiently large; the buoyancy force arising from the variation in density must be large enough to overcome the viscous drag of the fluid. An alternative way of thinking about this is to consider a small parcel of fluid. Consider a perturbation of the basic state that causes the parcel to move upwards. This results in the displacement of heavier fluid downwards, the whole process releasing gravitational potential energy stored in the basic state. If this energy is greater than the work done against viscous drag then the perturbation will grow. Otherwise it will decay. The density gradient that marks the borderline between these two situations is known as the *critical* density gradient. The non-dimensional number commonly used to measure the density gradient is known as the *Rayleigh number*

$$Ra \equiv g\alpha\beta\ell^4/\nu\kappa\,. \tag{1}$$

Here g is the gravitational acceleration, α the coefficient of volume expansion $(-\rho^{-1}\partial\rho/\partial T)$, β the temperature gradient $(-dT/dz)$, ℓ the distance between the plates, ν the kinematic viscosity and κ the thermal diffusivity of the fluid.

Much can be learned from linear theory which determines the critical value Ra_c of the Rayleigh number. Of fundamental interest to the geodynamo problem is the dependence of Ra_c on the rotation rate Ω and the magnetic field $\mathbf{B}$, (see Core Convection).

Here, since the topic of thermal instability (or convection) is dealt with elsewhere, we shall focus on other sources of instability, in particular magnetic instability. The principle is the same; the basic state stores energy and a perturbation may extract that energy. The perturbation grows (i.e. the basic state is unstable) if energy is extracted and is more than enough to overcome any diffusive losses.

Magnetic instability

Associated with any magnetic field $\mathbf{B}$ is its magnetic energy $\int_V (\mathbf{B}^2/2\mu)\, dV$ where μ is the magnetic permeability and V is the region containing the field. If a rearrangement of field lines would result in a lower magnetic energy (just as the interchange of heavy and light fluid, as discussed above, results in a lower gravitational potential energy), then there is the possibility of instability driven by the magnetic field. Of course, the field cannot be considered in isolation. In the Earth's core, the field permeates a conducting fluid and the evolution of the field and the motion of the core fluid are strongly coupled. This inevitably constrains what rearrangements of field lines are possible. If a permitted rearrangement results in a lower total energy, then instability will result if the energy released exceeds the diffusive losses resulting from the rearrangement.

Mean-field dynamo theory focuses on the generation of an axisymmetric (or mean) magnetic field by the action of a mean electromotive force (e.m.f.) and differential rotation. A topic that has received somewhat less attention is that of the stability of the field to non-axisymmetric perturbations. In mean-field dynamo theory, the field is *maintained* when the generation effect of the mean e.m.f. and differential rotation balance the decay due to ohmic diffusion. However, if the field is sufficiently strong and it satisfies certain other conditions then the field may be in an unstable configuration. Instability can extract energy from the mean field, so the generation mechanism may have a second sink of energy to counteract. Magnetic instabilities may therefore play an important role in determining what fields are observed and how strong they are. Linear theory has established that the minimum field strength required for instability (though depending on many factors) is comparable with estimates of the Earth's toroidal field strength. Also, a careful analysis (McFadden and Merrill, 1993) of the reversal data has concluded that "reversals are triggered by internal instabilities of the fluid motion of the core". Here, we review the various classes of magnetic instability and the conditions required for instability.

Classes of instability

Energy can be extracted from a basic magnetic field by a rearrangement of field lines in one of two ways; with or without reconnecting field lines. In a perfectly conducting fluid, field lines are frozen into the fluid (see Alfvén's Theorem) and field lines can neither be broken nor reconnected. This constrains what perturbations are possible. An instability is known as *ideal* if it can extract energy without reconnecting field lines and can therefore exist in a perfectly conducting fluid. Alternatively, if the existence of an instability depends on reconnecting field lines and is therefore absent in a perfectly conducting fluid, the instability is described as *resistive*.

A given magnetic field configuration $\mathbf{B}$ in an ideal fluid may be stable. Adding resistive effects can destabilise it. While initially somewhat counter-intuitive, the reason is quite clear; adding the effect of resistivity increases the number of degrees of freedom of the system by allowing field-line reconnection. A very similar situation is familiar in parallel shear flows where an inviscid flow may be stable, but can be destabilised by adding viscosity (see for example Drazin and Reid, 1981).

The key parameter; the Elsasser number

For both ideal and resistive modes of instability, resistive effects (otherwise known as ohmic diffusion) also play a more traditional role; diffusion acts to damp out instability if diffusion is sufficiently strong, just as is the case for viscosity in thermal convection, see (1). The key parameter in the case of magnetic instability in a rapidly rotating system (such as the Earth's core) is the Elsasser number

$$\Lambda \equiv \frac{\mathcal{B}^2}{2\Omega\mu_0\rho_0\eta} \left(= \frac{\sigma\mathcal{B}^2}{2\Omega\rho_0} = \frac{\tau_\eta}{\tau_s} \right), \tag{2}$$

where the ohmic diffusion time

$$\tau_\eta = \frac{\mathcal{L}^2}{\eta} \tag{3}$$

and the slow MHD time scale

$$\tau_s = \frac{2\Omega}{{\Omega_A}^2}, \qquad \text{where} \quad {\Omega_A}^2 = \frac{\mathcal{B}^2}{\mu_0\rho_0\mathcal{L}^2}. \tag{4}$$

In the above, $\mathcal{B}$ is the magnetic field strength, Ω is the rotation frequency of the Earth, μ_0 is the magnetic permeability of free space, ρ_0 is the core density, $\mathcal{L}$ is a characteristic length scale, for example the radius of the core, σ is the electrical conductivity, and $\eta = 1/(\mu_0\sigma)$ is the magnetic diffusivity.

The Elsasser number is a non-dimensional measure of the field strength. It can also be thought of as an inverse measure of the strength of magnetic diffusion; $\Lambda \to \infty$ is the perfectly conducting limit. The expression of Λ in terms of the ratio of τ_η and τ_s is instructive; τ_s is the time scale on which diffusionless magnetic waves evolve in a rapidly rotating system for which $\Omega \gg \Omega_A$, (Ω_A is the Alfvén frequency, see Alfvén Waves). When Λ is large ($\tau_\eta \gg \tau_s$) the time scale on which magnetic waves evolve is short compared with the time scale on which magnetic diffusion acts and we therefore expect diffusive damping to be negligible. By contrast, when Λ is of order unity we can expect diffusive damping to be important. When Λ is small, the magnetic field is weak; it will have insufficient energy to drive an instability and will not play a dominant role in the dynamics of the fluid.

Conditions for instability

As expected from the above argument, detailed model calculations (based on the simultaneous solution of the Navier-Stokes equations and the magnetic induction equation for a prescribed axisymmetric field in a given geometry, see for example Fearn, 1994) indeed show that a necessary condition for magnetic instability is

$$\Lambda > \Lambda_c \tag{5}$$

where the exact value of Λ_c of course depends on the choice of field used in the model. Values of order 10 are typical. This is significant, because, for the Earth's core, $\Lambda = 10$ corresponds to a field strength of some 5mT. Field strengths in the core are believed to be of this order so it is probable that magnetic instabilities are relevant to the dynamics of the Earth's core, and indeed may play an important role in constraining the field strength.

In addition to the condition (5) on the field strength, instability is also dependent on the field geometry (or shape). Condition (5) is about there being sufficient energy stored in the field. The geometric conditions described below are about whether that energy can be extracted.

The specific discussion here is tailored to the case of a rapidly-rotating fluid, as is appropriate for application to the Earth's core. Most of the qualitative ideas, though, apply equally well to systems that are not rotating or where rotational effects are much less important, for example in laboratory plasmas (eg see Davidson 2001) and the solar atmosphere (eg see Priest, 1982).

Ideal Instability

Motivated by the belief that the toroidal part of the core field dominates the poloidal part, early work focussed on purely azimuthal fields of the form

$$\mathbf{B} = B\mathbf{e}_\phi \tag{6}$$

where (s, ϕ, z) are cylindrical polar coordinates and $\mathbf{e}_\phi$ is the unit vector in the ϕ-direction and z is the axial direction. A local stability analysis for $B = B(s)$ (Acheson, 1983) has shown this to be unstable if B increases sufficiently rapidly with s somewhere in the core. The nature of this condition has led to the alternative name *field gradient instability.* If $B \propto s^\alpha$ then the condition for the field-gradient instability is $\alpha > 3/2$. More generally Acheson (1983) found instability if

$$\Delta \equiv \frac{2s^2}{B}\frac{d}{ds}\left(\frac{B}{s}\right) > m^2\,, \tag{7}$$

where m is the azimuthal wavenumber of the instability.

Numerical studies of field (6) have confirmed Acheson's prediction and then gone on to consider more complex fields (see Fearn, 1994 for a review). Where (7) is not satisfied everywhere, there is a tendency for the instability to be concentrated in the region where (7) is satisfied.

It is worth commenting at this stage on the field $B \propto s$ ($\alpha = 1$) that results from a uniform current in the z-direction. This choice leads to a particularly simple form of the Lorentz force. For this reason, it has often been used in studies of the effect of a magnetic field on thermally-driven convection. It is not ideally unstable in a rapidly rotating system, but *is* ideally unstable in a non-rotating system. Rotation is therefore seen to have an inhibiting effect. There are several studies that identify rather esoteric instabilities of magnetic origin for $B \propto s$. Fearn (1988) was able to link these to the presence of some additional effect, for example stable density stratification, counteracting the inhibiting effect of rotation. The message from this is that while $B \propto s$ is a perfectly adequate choice for the purpose of studying the effect of a magnetic field on convection, it is not a typical field for the study of magnetic instability.

Resistive Instability

Resistive instability is usually associated with so-called *critical levels* $\mathbf{k}\cdot\mathbf{B} = 0$ where $\mathbf{k}$ is the wavevector of the instability. For fields of the form (6), this condition reduces to $B = 0$, i.e. resistive instability is associated with there being a zero of the azimuthal field somewhere in the core. The condition $\mathbf{k} \cdot \mathbf{B} = 0$ is well known in the non-rotating plasma physics literature and the main effect of rotation is to modify the timescale on which the instability operates. Field-line curvature is unimportant; the instability has been found both for curved fields of the form (6) and for straight field lines.

Discussion

Further studies have looked at adding a z-dependence to B and also investigated poloidal fields. These have shown that the basic qualitative understanding derived from studying the field (6) is robust, see discussion and references in Fearn (1998). In applying these ideas to the core, we know that the mean toroidal field vanishes on the axis, so must increase with s somewhere, before decreasing again to zero at the core-mantle boundary. It is also likely that the resistive instability condition $\mathbf{k} \cdot \mathbf{B} = 0$ will be satisfied somewhere. To further investigate fields relevant to the core, Zhang and Fearn (1994, 1995) have investigated the stability of the toroidal and poloidal decay modes of the core (see Figures 1 and 2) and found them to be unstable, with Λ_c typically in the range 10-20. Given all this, it seems highly likely that any dynamo generated field will have a configuration that is unstable somewhere in the core. Whether or not it is actually unstable will then depend on the field strength.

Several studies have investigated the combined effects of thermal convection and magnetic instability. A recent study (Zhang and Gubbins, 2000) builds on the Zhang and Fearn work referred to above, incorporating a basic magnetic field that is a combination of toroidal and poloidal decay modes and also includes density stratification. In general, instability is a result of both thermal and magnetic forcing. One way of visualising this is through a graph of Ra_c versus Λ. This has a negative gradient (Ra_c decreases as Λ increases) with the graph cutting the horizontal axis ($Ra = 0$) at $\Lambda = \Lambda_c$. As Λ is decreased from this point the contribution from magnetic energy to the instability decreases and the thermal contribution must increase (Ra_c increases) to compensate. The slope of the graph depends on the Roberts number $q = \kappa/\eta$, approaching the vertical for small q.

Recent developments

The stability analyses leading to an understanding of the necessary conditions for instability (see above) have all been linear. Such analyses can say nothing about how the growth of the instability feeds back on the dynamo process generating $\mathbf{B}$.

The effect of the instability can be thought of as twofold. Firstly, and most simply, the instability extracts energy from the field so represents a drain on the field's energy in addition to ohmic diffusion. We therefore expect that for a given energy source driving the geodynamo, the field generated would be weakened by the presence of an instability. The big question is "by how much?" Secondly, there will be a mean e.m.f. associated with the instability. This will feed back on the mechanism generating the field. Recent work (Fearn & Rahman, 2004) has begun to investigate this and found that this feedback effect can be important and that magnetic instability can significantly constrain the strength of the mean field.

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Acknowledgements

Figures 1 and 2 are reproduced from Figure 1 of Zhang and Fearn (1994) and Figure 1 of Zhang and Fearn (1995) respectively with permission of the publishers Taylor and Francis (`http://www.tanf.co.uk`).

Bibliography

Acheson, D.J. (1983) Local analysis of thermal and magnetic instabilities in a rapidly rotating fluid. *Geophys. Astrophys. Fluid Dynam.* **27**, 123–136.

Chandrasekhar, S. (1961) *Hydrodynamic and Hydromagnetic Stability.* Oxford; Clarendon Press.

Davidson, P.A. (2001) *An Introduction to Magnetohydrodynamics.* Cambridge; Cambridge University Press.

Drazin, P.G. and Reid, W.H. (1981) *Hydrodynamic stability.* Cambridge; Cambridge University Press.

Fearn, D.R. (1988) Hydromagnetic waves in a differentially rotating annulus IV. Insulating boundaries. *Geophys. Astrophys. Fluid Dynam.* **44**, 55–75.

Fearn, D.R. (1994) Magnetic instabilities in rapidly rotating systems, in *Solar and Planetary Dynamos* (eds. M.R.E. Proctor, P.C. Matthews and A.M. Rucklidge). Cambridge; Cambridge University Press, pp. 59-68.

Fearn, D.R. (1998) Hydromagnetic flows in planetary cores. *Rep. Prog. Phys.* **61**, 175-235.

Fearn, D.R. and Rahman, M.M. (2004) Instability of non-linear α^2-dynamos. *Phys. Earth Planet. Inter.* **142**, 101-112.

McFadden, P.L. and Merrill, R.T. (1993) Inhibition and geomagnetic reversals. *J. Geophys. Res.* **98**, 6189-6199.

Priest, E.R. (1982) *Solar Magnetohydrodynamics.* Dordrecht; Reidel.

Zhang, K. and Fearn D.R. (1994) Hydromagnetic waves in rapidly rotating spherical shells generated by magnetic toroidal decay modes. *Geophys. Astrophys. Fluid Dynam.* **77**, 133-157.

Zhang, K. and Fearn D.R. (1995) Hydromagnetic waves in rapidly rotating spherical shells generated by poloidal decay modes. *Geophys. Astrophys. Fluid Dynam.* **81**, 193-209.

Zhang, K. and Gubbins, D. (2000) Is the geodynamo process intrinsically unstable? *Geophys. J. Int.* **140**, F1-F4.

Cross references

Core convection
Core field
Core flow

Core motions
Magnetoconvection
Magnetohydrodynamic waves
Time scales and Dimensional analysis of the geodynamo
Reversals

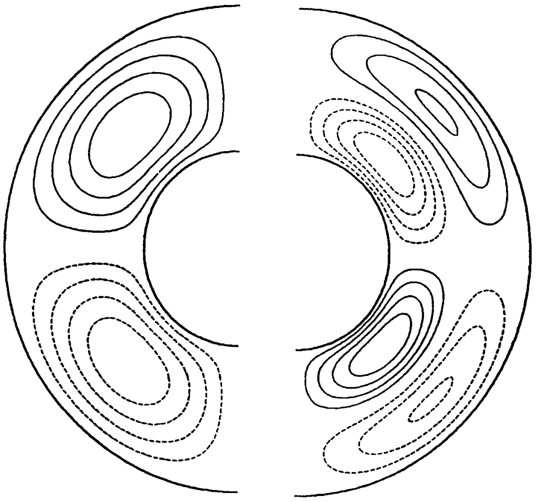

Figure 1: Contours of B_ϕ for two toroidal decay modes (one on the left and one on the right) studied by Zhang and Fearn 1994. Reproduced from that paper by permission.

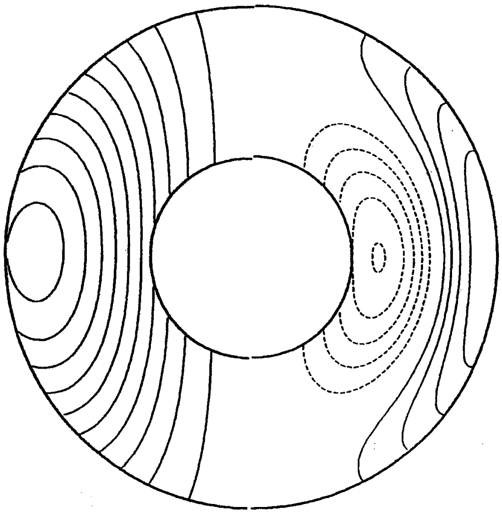

Figure 2: Field lines for two poloidal decay modes (one on the left and one on the right) studied by Zhang and Fearn 1995. Reproduced from that paper by permission.