RATIONAL CATALAN NUMBERS, AIM.
EXERCISES

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In this exercise, the idea is to try and develop a better understanding some aspects of the proof by Gordon and Griffeth, [2], of the positivity of the $m$th $q$-Fuss-Catalan polynomial.

The rational Cherednik algebra associated to a complex reflection group $(G, V)$ (at ”equal parameters”) is the quotient of the algebra $T(V \oplus V^*) \times G$ by the relations
\[
[x, x'] = [y, y'] = 0, \quad \forall \ x, x' \in V^*, \ y, y' \in V,
\]
and
\[
[y, x] = \langle x, y \rangle - 2c \sum_{s \in S} \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} s,
\]
where $S$ is the set of pseudo-reflections\footnote{Recall that an element $s \in G$ is said to be a pseudo-reflection if $\text{rk}(1 - s) = 1$.} in $G$ and $\alpha_s$ (resp. $\alpha_s^\vee$) spans the one-dimensional space $\text{Im}(1 - s)|_{V^*}$ (resp. $\text{Im}(1 - s)|_V$). The relations (1) are independent of the choice of $\alpha_s$ and $\alpha_s^\vee$. The Euler operator
\[
eu_c = \sum_{i=1}^{n} x_i y_i + \sum_{s \in S} \frac{2c}{1 - \lambda_s} (1 - s)
\]
plays a key role in the representation theory of $H_c(G)$ because (exercise!)
\[
[\eu_c, x] = x, \quad [\eu_c, w] = 0, \quad [\eu_c, y] = -y, \quad \forall \ x \in V^*, \ y \in V, \ w \in G.
\]

Recall that the generalized eigenspaces of $\eu_c$ define a grading on the simple $H_c(G)$-modules $L_c(\lambda)$ and Verma modules $M_c(\lambda)$. More generally, for each $w \in G$, we can define
\[
\text{ch}(L_c(\lambda); w, q) = \sum_{\kappa \in \mathbb{C}} (\text{Tr}(w, L_c(\lambda)_\kappa)) q^\kappa,
\]
where $L_c(\lambda)_\kappa = \{ l \in L_c(\lambda) \mid (\eu_c - \kappa)^l \cdot l = 0 \text{ for } l \gg 0 \}$. This makes sense for any $M \in \mathcal{O}_c$ because the space $M_\kappa$ is a finite dimensional $G$-module, for all $\kappa \in \mathbb{C}$.

**Exercise 1.** For simplicity, we assume that $G$ is well-generated; this implies that $\Psi(V^*) = V^*$. Let $c = (1 + mh)/h$, where $m \in \mathbb{Z}_{\geq 1}$ and is the $h$ the Coxeter number for $G$.

1. Check that $\text{ch}(M_c(\lambda); q) = \frac{1}{(1-q)^n}$. 

1
(2) Using the fact that \( L_c(triv) \) is the quotient of \( M_c(triv) \) by an ideal \( I \) generated by a homogeneous regular sequence \( R \subset M_c(triv)_{1+mh} \) with \( R \simeq V^* \), show that the Koszul resolution of \( L_c(triv) \) is a resolution

\[
0 \to M_c(\wedge^n V^*) \to \cdots \to M_c(V^*) \to M_c(triv) \to L_c(triv) \to 0
\]

of \( L_c(triv) \) as a \( H_c(G) \)-module, where \( n = \dim V \).

Hint: it is clear that the Koszul resolution is \( G \)-equivariant. Now use the fact that \( f \in \mathbb{C}[V] \) and \( y \in V \subset \mathbb{C}[V^*] \) implies that \([y, f] \in \mathbb{C}[V] \times G\).

**Solution 2.** Let \( \lambda \in \text{Irr}(G) \). The PBW theorem implies that \( M_c(\lambda) \simeq \mathbb{C}[V] \otimes \lambda \) as a \( \mathbb{C}[V] \times G \)-module, where \( \mathbb{C}[V] \) only acts on \( \mathbb{C}[V] \) and \( G \) acts diagonally i.e. \( w \cdot (f \otimes v) = (f^{w} \otimes w \cdot v) \). Under this identification, a Dunkl operator \( y \in V \subset \mathbb{C}[V^*] \) acts as follows:

\[
y \cdot (f \otimes v) = [y, f] \otimes v + f \otimes yv = [y, f] \otimes v
\]

where \( 1 \otimes yv = 0 \) follows from the definition of \( M_c(\lambda) \). So let's calculate the generalized eigen-spaces of \( \text{eu}_c \) on \( M_c(triv) \). If \( f \in \mathbb{C}[V] \) is homogeneous of degree \( r \) say, then \([\text{eu}_c, f] = rf\), which implies that \( \text{eu}_c \cdot (f \otimes v) = r(f \otimes v) + f \text{eu}_c \otimes v \). Since \( y \cdot v = 0 \),

\[
\text{eu}_c \cdot (1 \otimes v) = \sum_{s \in S} \frac{2c}{1-\lambda_s} 1 \otimes (1-s)v = 0
\]

because \( v \in \text{triv} \). Hence, the grading on \( M_c(triv) \) defined by the Euler element equals the usual positive grading on the polynomial ring \( \mathbb{C}[V] \) (for an arbitrary Verma module \( M_c(\lambda) \), the space \( 1 \otimes \lambda \) sits in some degree, dictated by the action of the element \( \sum_{s \in S} \frac{2c}{1-\lambda_s} (1-s) \in Z(G) \) on \( \lambda \), and then the Euler grading on \( M_c(\lambda) \) is just the naive polynomial grading shifted by the degree of \( 1 \otimes \lambda \). But the usual grading on \( \mathbb{C}[V] \) has Hilbert polynomial \( \frac{1}{(1-q)^n} \), as required.

Now we come to the interesting part. Recall that a regular sequence for \( \mathbb{C}[V] \) is a sequence \( f_1, \ldots, f_k \) of polynomials (where \( k \leq n \)) such that

- \( \bullet f_i \) is not a zero-divisor on \( \mathbb{C}[V]/(f_1, \ldots, f_{i-1}) \); and
- \( \mathbb{C}[V]/(f_1, \ldots, f_k) \neq 0 \).

When \( k = n \), the first condition implies that \( \text{Spec}(\mathbb{C}[V]/(f_1, \ldots, f_n)) \) is zero-dimensional, and the second implies that \( \text{Spec}(\mathbb{C}[V]/(f_1, \ldots, f_n)) \) is non-empty. Or, equivalently, that \( \mathbb{C}[V]/(f_1, \ldots, f_n) \neq 0 \). Now, it is a general result in commutative algebra that, given a regular sequence, the Koszul resolution

\[
0 \to \mathbb{C}[V] \otimes \wedge^n R \to \cdots \to \mathbb{C}[V] \otimes \wedge^1 R \to \mathbb{C}[V] \otimes \wedge^0 R \to \mathbb{C}[V]/(f_1, \ldots, f_n) \to 0
\]

is a free resolution of \( \mathbb{C}[V]/(f_1, \ldots, f_n) \) as a \( \mathbb{C}[V] \)-module, where \( R \) equals \( \text{Span}_C \{ f_1, \ldots, f_n \} \). The maps

\[
\phi_i : \mathbb{C}[V] \otimes \wedge^i R \to \mathbb{C}[V] \otimes \wedge^{i-1} R
\]

are given by \( \phi_i(\sum f_{j_1} \wedge \cdots \wedge f_{j_i}) = \sum_{s=1}^i f_{j_i} \otimes f_{j_1} \wedge \cdots \wedge \wedge f_{j_s} \). If \( G \) acts on \( \mathbb{C}[V] \) and the space \( R \) is a \( G \)-module, then \( \phi_i \) is actually a morphism of \( \mathbb{C}[V] \times G \)-modules, where \( G \) acts diagonally on \( \mathbb{C}[V] \otimes \wedge^i R \) (to give a rigorous proof of this fact, one has to first check that the map \( \phi \) is independent of the
graded, so for any homogeneous\( f \in \wedge^i R \) degree \( \sum_{j=1}^i \deg(f_j) \).

Hence, what we really need to show is that \( y \) is a \( \wedge \) morphism. Therefore, \( \ker \phi \) is the quotient map and hence obviously a \( \wedge \)-morphism of \( H_c(G) \)-modules. Since the sequence (2) is exact, it will be a "resolution" (it is not a projective resolution because the Verma modules are not projective \( H_c(G) \)-modules, but it is a resolution of a simple module by Verma modules, just like the Bernstein-Gelfand-Gelfand resolution of finite dimensional module in Lie theory) of \( H_c(G) \)-modules if we can show that each of morphisms \( \phi_i \) is actually a morphism of \( H_c(G) \)-modules.

Recall that the rational Cherednik algebra \( H_c(G) \) is generated by the linear functions \( V^* \subset \mathbb{C}[V] \) (the "\( x \)'s"), the group \( G \) and the Dunkl operators \( V \subset \mathbb{C}[V^*] \). So, to show that \( \phi_i \) is an \( H_c(G) \)-morphism, we must show that
\[
\phi_i(x \cdot f \otimes v) = x \phi_i(f \otimes v), \quad \phi_i(w \cdot f \otimes v) = w \phi_i(f \otimes v), \\
\phi_i(y \cdot f \otimes v) = y \phi_i(f \otimes v), \quad \forall x \in V^*, \; y \in V, \; w \in G.
\]
The first two equalities are clear because we already know that \( \phi_i \) is a morphism of \( \mathbb{C}[V] \times G \)-modules. So we just need to show that \( \phi_i(y \cdot f \otimes v) = y \phi_i(f \otimes v) \), for all \( y \in V \). At first sight, this just seems like an explicit calculation - we have an "explicit" definition of \( \phi_i \), so we just need to calculate how \( y \) interacts with \( \phi_i \). But, it is almost always the case with rational Cherednik algebras that explicit calculations are extremely difficult and, in the authors' opinion, should be avoided at all costs. So we need a cheap argument that will save us from having to make any explicit calculation. The key tool that allows us to do this is the grading provided by the Euler operator. We will prove inductively that each \( \phi_i \) is a morphism of \( H_c(G) \)-modules. The map \( \phi_0 : M_c(\text{triv}) \to L_c(\text{triv}) \) is the quotient map and hence obviously a \( H_c(G) \)-morphism. Notice that this implies that \( \ker \phi_0 \) is a \( H_c(G) \)-submodule of \( M_c(\text{triv}) \). So, let \( i > 0 \) and assume that \( \phi_{i-1} \) is a \( H_c(G) \)-morphism. Then, \( \text{Im} \phi_i = \ker \phi_{i-1} \) is a graded \( H_c(G) \)-submodule of \( M_c(\wedge^{i-1} R) \). Let's calculate,
\[
\phi_i(y \cdot f \otimes v) = \phi_i([y, f] \otimes v) + \phi_i(f \otimes yv) = \phi_i([y, f] \otimes v)
\]
because \( f \otimes yv = 0 \) by definition. Now, as noted in the exercise, the defining relations imply that \([y, f] \in \mathbb{C}[V] \times G \). But we know that \( \phi_i \) is a \( \mathbb{C}[V] \times G \)-morphism. Therefore, \( \phi_i(y \cdot f \otimes v) = [y, f] \phi_i(1 \otimes v) \). So we need to show that
\[
y \cdot \phi_i(f \otimes v) = [y, f] \phi_i(1 \otimes v) + f(\phi_i(1 \otimes v)).
\]
Hence, what we really need to show is that \( y \cdot \phi_i(1 \otimes v) = 0 \) for all \( y \in V \) and \( v \in \wedge R \). The fact that \([e u, y] = -y \) and that \( \phi_i \) is graded implies that \( \deg(y \cdot \phi_i(f \otimes v)) = \deg(\phi_i(f \otimes v)) - 1 \) for any homogeneous \( f \otimes v \). But, every homogeneous polynomial \( f \) is positively graded, so
\[
\deg \phi_i(f \otimes v) = \deg(f \phi_i(1 \otimes v)) = \deg f + \deg \phi_i(1 \otimes v) \geq \deg \phi_i(1 \otimes v).
\]
So \( \phi_i(1 \otimes \wedge^i R) \) sits in the lowest degree of the \( H_c(G) \)-module \( \text{Im}\phi_i \). On the other hand, \( \deg(y \cdot \phi_i(1 \otimes \wedge^i R)) = \deg \phi_i(1 \otimes \wedge^i R) - 1 \). Thus, we conclude that \( y \cdot \phi_i(1 \otimes \wedge^i R) = 0 \), as required.

**Exercise 3.** Again, fix \( c = (1 + mh)/h \). Using the fact that the regular sequence defining \( L_c(\text{triv}) \) has degree \( 1 + mh \), calculate the \( q \)-character of \( L_c(\text{triv}) \). More generally, for each \( w \in G \), show that

\[
(3) \quad \text{ch}(L_c(\text{triv}); w, q) = \frac{\det(1 - q^{1+mh}w)}{\det(1 - qw)}.
\]

**Solution 4.** We will use the resolution from problem 1 to show that the character of \( L_c(\text{triv}) \) at \( c = \frac{1+mh}{h} \), defined for \( w \in G \) as

\[
\text{char}(L_c, w, q) = \sum_i \text{tr}_{|L^i_c}(w)q^i,
\]

is equal to

\[
\frac{\det(1 - q^{1+mh}w)}{\det(1 - qw)}.
\]

**Claim 1.** For any \( w \in GL(V^*) \), acting on \( S^i V^* \) = homogeneous polynomials on \( V \) of degree \( i \),

\[
\text{char}(SV^*, w, q) = \sum_i \text{tr}_{|S^i V^*}(w)q^i = \frac{1}{\det(1 - qw)}.
\]

Both sides are invariant under conjugation, so we may assume \( w \) is upper triangular with \( \lambda_1, ..., \lambda_n \) on the diagonal. Then the right hand side is

\[
\frac{1}{\det(1 - qw)} = \frac{1}{(1 - q\lambda_1)(1 - q\lambda_2) \cdots (1 - q\lambda_n)} = \sum_i \left( \sum_{i_1 + \cdots + i_n = i} \lambda_1^{i_1} \cdots \lambda_n^{i_n} \right) q^i.
\]

On the left hand side, as \( w \) acts on \( V^* \) with eigenvalues \( \lambda_1, ..., \lambda_n \), it acts on \( S^i V^* \) with eigenvalues equal to all monomials in \( \lambda_1, ..., \lambda_n \) of total degree \( i \). The trace is then \( \text{tr}_{|S^i V^*}(w) = \sum_{i_1 + \cdots + i_n = i} \lambda_1^{i_1} \cdots \lambda_n^{i_n} \), and so

\[
\text{char}(SV^*, w, q) = \sum_i \text{tr}_{|S^i V^*}(w)q^i = \sum_i \left( \sum_{i_1 + \cdots + i_n = i} \lambda_1^{i_1} \cdots \lambda_n^{i_n} \right) q^i
\]

and the Claim 1 holds.

**Claim 2.** Similarly,

\[
\text{char}(AV^*, w, q) = \det(1 + qw).
\]

Again, assume \( w \) is upper triangular with \( \lambda_1, ..., \lambda_n \). Then both sides are equal to

\[
(1 + q\lambda_1)(1 + q\lambda_2) \cdots (1 + q\lambda_n).
\]
Claim 3. For any irreducible representation \( \tau \), the character of \( M_c(\tau) \) is equal to

\[
\text{char}(M_c(\tau), w, q) = q^{\text{eu}_c(\tau)\text{tr} \tau(w)} \frac{1}{\det(1 - qw)}
\]

As a graded representation of \( G \), \( M_c(\tau) \cong SV^* \otimes \tau \), and the grading by \( \text{eu}_c \) eigenvalues is compatible with the tensor product and with the natural grading on \( SV^* \). The character of \( M_c(\tau) \) is a product of the character of \( \tau \) as a graded \( G \) representation concentrated in one degree (which is equal to \( q^{\text{eu}_c(\tau)\text{tr} \tau(w)} \)) and the character of \( SV^* \) (which we calculated in Claim 1).

Claim 4. By the resolution of \( L_c(\text{triv}) \) at \( c = (1 + mh)/h \) given in Exercise 1,

\[
\text{char}(L_c(\text{triv}), w, q) = \sum_{i=0}^{n} (-1)^i \text{char}(M_c(\Lambda^i V^*), w, q).
\]

By Claim 3, this is equal to

\[
\sum_{i=0}^{n} \frac{(-1)^i q^{\text{eu}_c(\Lambda^i V^*)\text{tr} \Lambda^i V^*(w)}}{\det(1 - qw)} = \frac{1}{\det(1 - qw)} \text{char}(AV^*, w, -q^{mh+1}).
\]

(The reason for the exponent \( mh + 1 \) of \( q \) is that the grading by \( \text{eu}_c \) is, as seen in the Koszul resolution, \( (mh + 1) \) times the natural grading on \( AV^* \).) By Claim 2, this is now equal to

\[
\frac{\det(1 - q^{mh+1}w)}{\det(1 - qw)}.
\]

Exercise 5. In this exercise, we assume that \( G = \mathfrak{S}_n \), the symmetric group on \( n \) elements. Then, the Coxeter element \( h \) equals \( n \). Again, fix \( c = (1 + mn)/n \). We denote by \( Q \subset \mathfrak{h}^* \) the root lattice of \( \mathfrak{S}_n \).

1. Using formula (3), show that \( \text{ch}(L_c(\text{triv}); w) = (1 + nm)^{\text{dim} \ker h(1-w)} \).

2. Show that the character of \( w \in \mathfrak{S}_n \) acting on the permutation representation \( Q/(1 + nm)Q \) is \( (1 + nm)^{\text{#Cycles}(w) - 1} \). Hint: for this recall that \( Q \simeq \mathbb{Z}^n/\mathbb{Z} \) as a \( \mathfrak{S}_n \)-lattice and hence

\[
Q/(1 + nm)Q \simeq (\mathbb{Z}_{1+nm})^n/\mathbb{Z}_{1+nm}.
\]

3. Conclude\(^2\) that \( L_c(\text{triv}) \simeq \text{Span}_C(Q/(1 + nm)Q) \) as \( \mathfrak{S}_n \)-modules.

4. Using the fact that

\[
\langle \text{Ind}_{\mathfrak{S}_n}^\mathfrak{S}_\lambda \text{triv}, \text{sgn} \rangle = \begin{cases} 
1 & \text{if } \lambda \text{ equals } (1^n) \\
0 & \text{otherwise}
\end{cases}
\]

show that the number of copies of the sign representation in \( \text{Span}_C(Q/(1 + nm)Q) \) equals the \((m - 1)\)th Fuss-Catalan number

\[
\frac{1}{mn + 1} \left( \begin{array}{c} mn + 1 \\ n \end{array} \right).
\]

\(^2\)An analogous result, based on [4, Proposition 3.9], is valid for any finite Coxeter group; see [1] for details.
Solution 6. \( G = S_n, h = n, c = (mn + 1)/n \). The reflection representation \( V \) is the irreducible \( n - 1 \) dimensional representation which can be realized as \( \{(z_1, ... , z_n)|z_i \in \mathbb{C}, \sum z_i = 0\} \subseteq \mathbb{C}^n \), with elements of the symmetric group permuting the coordinates. It is sometimes more convenient to work with the irreducible \( V \) and sometimes with the full \( \mathbb{C}^n \).

**Part (1).** Assume \( w \in S_n \) has eigenvalues \( \lambda_1, ... , \lambda_n \) on \( V^* \), with \( \lambda_1 = ... = \lambda_l = 1 \) and \( \lambda_{l+1}, ... , \lambda_n \neq 1 \). Then \( \dim \ker V(w - 1) = \dim \ker V^*(w - 1) = l \). From Exercise 2 it follows that

\[
\text{char}(L_c(\text{triv}), w, q) = \frac{\det(1 - q^{mn + 1}w)}{\det(1 - qw)} = \frac{(1 - q^{mn + 1})^l \prod_{j=l+1}^n (1 - q^{mn + 1}\lambda_j)}{(1 - q)^l \prod_{j=l+1}^n (1 - q\lambda_j)},
\]

so

\[
\text{char}(L_c(\text{triv}), w, 1) = (1 + mn)^l \cdot \frac{\prod_{j=l+1}^n (1 - \lambda_j)}{\prod_{j=l+1}^n (1 - \lambda_j)} = (1 + mn)^{\dim \ker V(w - 1)}.
\]

**Part (2).** The root lattice \( Q \) of \( S_n \) can be thought of as the rank \( n - 1 \) free abelian group \( \{(z_1, ... , z_n)|z_i \in \mathbb{Z}, \sum z_i = 0\} \). Consequently,

\[
Q/(mn + 1)Q \cong \mathbb{Z}_{mn+1}^n / \mathbb{Z}_{mn+1} \cong \{(z_1, ... , z_n)|z_i \in \mathbb{Z}_{mn+1}, \sum z_i = 0\}.
\]

Consider the \( \mathbb{C} \) - vector space \( U \) with this set as a basis, and a representation of \( S_n \) acting on it by permuting the coordinates of \( (z_1, ... , z_n) \). Any element \( w \in S_n \) acts on \( U \) by a permutation matrix, so \( \text{char}_U(w) = \text{tr}U(w) \) is the number of 1s on the diagonal, i.e. the number of basis vectors \( (z_1, ... , z_n), z_i \in \mathbb{Z}_{mn+1}, \sum z_i = 0 \) that \( w \) preserves.

A cycle \( (m_1, ... m_l) \) preserves those vectors \( (z_1, ... , z_n) \) for which the coordinates corresponding to the cycle match, \( z_{m_1} = ... = z_{m_l} \). If \( w \in S_n \) is a product of \( j \) disjoint cycles (this count includes the cycles of length 1!), the number of basis vectors it preserves is \( (1 + mn)^{j-1} \) (a choice of a number in \( \mathbb{Z}_{mn+1} \) for each of \( j \) cycles, and a condition that the sum of coordinates is 0 which dictates the last choice).

**Part (3).** By part (1),

\[
\text{char}(L_c(\text{triv}), w) = (1 + mn)^{\dim \ker V(w - 1)},
\]

and by part (2),

\[
\text{char}(Q/(mn + 1)Q, w) = (1 + mn)^{\text{(number of cycles of } w\text{)} - 1}.
\]

To see that \( L_c(\text{triv}) \cong Q/(mn + 1)Q \), it is enough to see that for any \( w \in S_n \),

\[
\text{(number of cycles of } w\text{)} - 1 = \dim \ker V(w - 1)
\]

or equivalently that

\[
\text{number of cycles of } w = \dim \ker \mathbb{C}^n (w - 1).
\]

There is a bijection between the cycles in \( w \) and basis vectors in \( \ker \mathbb{C}^n (w - 1) \), associating to each cycle \( (m_1, ... m_l) \) in \( w \) a vector \( (z_1, ... , z_n) \in \mathbb{C}^n \) which has 1 in coordinates corresponding to the cycle and 0 everywhere else, i.e. \( z_i = 1 \) for \( i = m_1, ... , m_l \), \( z_i = 0 \) otherwise.
Part (4). To find the number of copies of \( \text{sgn} \) in \( \text{Span}_C Q/(1+nm)Q \), calculate the inner product of their \( \mathfrak{S}_n \) characters:

\[
\langle \text{charSpan}_C(Q/(1+nm)Q), \text{sgn} \rangle = \langle \text{char}(L_c(\text{triv})), \text{sgn} \rangle = \left\langle w \mapsto (1+mn)^{\dim \ker (w^{-1})}, w \mapsto \text{sgn}(w) \right\rangle = \frac{1}{n! mn + 1} \sum_{w \in \mathfrak{S}_n} (-1)^{l(w)} (1+mn)^{\text{number of cycles in } w}
\]

A permutation in \( \mathfrak{S}_n \) with \( k \) cycles has parity \((-1)^l(w) = (-1)^{n-k} \), and there are unsigned Stirling number of them. So, the above sum is equal to

\[
\frac{1}{n! mn + 1} \sum_k s(n,k)(1+mn)^k = \frac{1}{n! mn + 1} \frac{1}{mn + 1} (mn + 1)^n = \frac{1}{mn + 1} \binom{mn + 1}{n}
\]
of them.

Let \( e \) denote the trivial idempotent for \( \mathfrak{S}_n \) and \( e_- \) the sign idempotent. The spherical subalgebra of \( H_c(\mathfrak{S}_n) \) is \( e H_c(\mathfrak{S}_n) e \). Similarly, we have the anti-spherical subalgebra \( e_- H_c(\mathfrak{S}_n) e_- \). It is known that

\[
e_- H_{\frac{1}{n} + (m+1)}(\mathfrak{S}_n)e_- \simeq e H_{\frac{1}{n} + m} e, \quad \forall m \geq 0.
\]

The above isomorphism sends \( e \cdot \mathbf{eu} \) to \( e \mathbf{eu} + N \), where \( N = \frac{n(n+1)}{2} \) is the number of positive roots. For all \( c \in \mathbb{Q}_{\geq 0} \), multiplication by the idempotent \( e \) defines an equivalence \( e \cdot - : H_c(\mathfrak{S}_n) \mod e H_c(\mathfrak{S}_n) e \mod \) with quasi-inverse \( H_c(\mathfrak{S}_n) e \otimes e H_c(\mathfrak{S}_n) e \to - \). Similarly, for all \( c \in \mathbb{Q}_{\geq 1} \), multiplication by the idempotent \( e_- \) defines an equivalence \( e_- \cdot : H_c(\mathfrak{S}_n) \mod e_- H_c(\mathfrak{S}_n) e_- \mod \) with quasi-inverse \( H_c(\mathfrak{S}_n) e_- \otimes e_- H_c(\mathfrak{S}_n) e_- \). See section 3 of [3] for further details.

There is a natural "Bernstein" filtration on \( H_c(G) \), defined by putting

\[
\mathcal{F}_{-1} H_c(G) = 0, \quad \mathcal{F}_0 H_c(G) = \mathbb{C} G, \quad \mathcal{F}_1 H_c(G) = V + V^* + \mathbb{C} G
\]
and defining inductively \( \mathcal{F}_i H_c(G) = (\mathcal{F}_{i-1} H_c(G)) \cdot (\mathcal{F}_1 H_c(G)) \). Then, the Poincaré-Birkhoff-Witt theorem says that

\[
\text{gr}_\mathcal{F} H_c(G) \simeq \mathbb{C}[V \times V^*] \rtimes G.
\]

Exercise 7. We continue to consider the example \( G = \mathfrak{S}_n \). The idea is to show that one can filter \( L_c(\text{triv}) \) in such a way that the associated graded is a quotient of the ring of diagonal coinvariants.

1. Show, using the defining relations of the rational Cherednik algebra, that when \( c = 1/n \) we have \( L_c(\text{triv}) \simeq \mathbb{C} \).

2. Fix \( m = 1 \) so that \( c = (n+1)/n \). Conclude from the previous step that \( e_- H_c(\mathfrak{S}_n) e_- \) admits a one-dimensional representation \( \mathbb{C} \).

The next step is harder, and the reader should consult [1] for details:

\[
H_c(\mathfrak{S}_n) e_- \otimes e_- H_c(\mathfrak{S}_n) e_- \mathbb{C} \simeq L_c(\text{triv}).
\]

Here is an outline of the proof:
Using that $e_-\in\mathcal{H}_c(\mathfrak{g}_n)e_-\otimes\mathcal{H}_c(\mathfrak{g}_n)e_-\mathbb{C}$ is simple. Since it is finite dimensional, it belongs to category $\mathcal{O}_c$. Hence $L\cong L_c(\lambda)$ for some $\lambda\vdash n$.

- Show that, for $c = (n+1)/n$, the $\mathbf{eu}_c$-weights of $\mathcal{M}_c(\wedge^i\mathfrak{h})$ are all strictly greater than 0, except for $M_c(\operatorname{triv})$, where $M_c(\operatorname{triv})_0$ is spanned by $1\otimes v_0$.
- Let $\delta^c_\mathfrak{h}\in\mathbb{C}[V^*]$ be the product of all coweights. It is the Van-der-Monde determinant in the $y$'s and has degree $-N$. Key fact: $\delta^c_\mathfrak{h}e_-\otimes v_0$ is a non-zero element in $L$.
- For $c = (n+1)/n$, it is known that $\operatorname{Hom}_{\mathcal{H}_c(\mathfrak{g}_n)}(\mathcal{M}_c(\lambda),\mathcal{M}_c(\mu)) \neq 0$ only if $\lambda \simeq \wedge^i V$ and $\mu \simeq \wedge^j V$ for some $i$ and $j$. Conclude that $\mathcal{M}_c(\operatorname{triv}) \to L$ and hence $L \cong L_c(\operatorname{triv})$.

Now we can continue.

1. Define a filtration on $\mathcal{M}_c(\operatorname{triv}) = H_\mathfrak{c}(\mathfrak{g}_n)e_- \otimes e_- H_\mathfrak{c}(\mathfrak{g}_n)e_- \mathbb{C}$ by setting $F_i\mathcal{M}_c(\operatorname{triv}) = (F_i H_\mathfrak{c}(\mathfrak{g}_n)) \cdot (e_- \otimes v_0)$. Why does this define a surjective morphism $\mathbb{C}[V \times V^*]e_- \to \operatorname{gr}_F \mathcal{M}_c(\operatorname{triv})$?

2. Check that this makes $(\operatorname{gr}_F \mathcal{M}_c(\operatorname{triv})) \otimes \operatorname{sgn}$ into a bigraded, commutative algebra such that $(\operatorname{gr}_F \mathcal{M}_c(\operatorname{triv})) \otimes \operatorname{sgn} \cong \mathcal{M}_c(\operatorname{triv}) \otimes \operatorname{sgn}$ as $\mathfrak{g}_n$-modules.

3. Use exercise 5 to deduce that there is a unique copy of the trivial representation in $(\operatorname{gr}_F \mathcal{M}_c(\operatorname{triv})) \otimes \operatorname{sgn}$ and hence that $(\operatorname{gr}_F \mathcal{M}_c(\operatorname{triv})) \otimes \operatorname{sgn}$ is a graded quotient of the ring $\mathbb{C}[V \times V^*]^{\mathfrak{g}_n\mathfrak{g}_n}$ of diagonal coinvariants.

**Solution 8.** (1) It is always a bit tricky to work with the reflection representation for $\mathfrak{g}_n$ as opposed to the permutation representation. So we’ll consider both $V$ and $V^*$ as quotients of $\mathbb{C}^n$ and $(\mathbb{C}^n)^*$ by $y_1 + \ldots, y_n$ and $x_1 + \ldots + x_n$ respectively. To do this, we have to fix the pairing $\langle -, - \rangle : (\mathbb{C}^n)^* \times \mathbb{C}^n \to \mathbb{C}$ such that $\sum_i x_i$ and $\sum_j y_j$ span the kernel. Explicit,

$$\langle x, y \rangle = \sum_{i=1}^n y_i x_i - \frac{1}{n} \left( \sum_i y_i \right) \left( \sum_j x_j \right).$$

Our set of reflections $S$ equals $\{s_{i,j} | i < j\}$ with roots $\epsilon_i - \epsilon_j$ and coroots $\epsilon^\vee_i - \epsilon^\vee_j$. Now, if $M_c(\operatorname{triv})$ has a one dimensional quotient, then the fact that it is graded means that it has to be $\mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$. Thus, it is the unique module $\mathbb{C}v_0$ defined by $w \cdot v_0 = v_0$ for all $w \in \mathfrak{g}_n$ and $x_i \cdot v_0 = y_j \cdot v_0 = 0$. For this to be well-defined, we just have to check that $[y, x] \cdot v_0 = (yx - xy) \cdot v_0 = 0$ for all $x \in V^*$ and $y \in V$ i.e. we have to check that the main relation (1) holds when applied to $v_0$. Since $\langle x, \epsilon^\vee_i - \epsilon^\vee_j \rangle = x_i - x_j$ and $\langle \epsilon_i - \epsilon_j, y \rangle = y_i - y_j$, the relation we really need to check is

$$\langle x, y \rangle - 2c \sum_{i < j} \frac{\langle x, \epsilon^\vee_i - \epsilon^\vee_j \rangle \langle \epsilon_i - \epsilon_j, y \rangle}{\langle \epsilon_i - \epsilon_j, \epsilon^\vee_i - \epsilon^\vee_j \rangle} = 0,$$

---

3 This fact follows, via the Knizhnik-Zamolodchikov functor, from a corresponding fact for the Hecke algebra $H_q(\mathfrak{g}_n)$, where $q = e^{2\pi i N}$. 

\[\sum_{i < j} \frac{\langle x, \epsilon^\vee_i - \epsilon^\vee_j \rangle \langle \epsilon_i - \epsilon_j, y \rangle}{\langle \epsilon_i - \epsilon_j, \epsilon^\vee_i - \epsilon^\vee_j \rangle} = 0,\]
which becomes
\[ \sum_{i=1}^{n} x_i y_i - \frac{1}{n} \left( \sum_{i} x_i \right) \left( \sum_{j} y_j \right) - c \sum_{i<j} (x_i y_i + x_j y_j - x_i y_j - x_j y_i). \]

Now one can check that this is zero for all \( x, y \) precisely if \( c = \frac{1}{n} \).

(2) We’ve shown that \( L_{\frac{1}{n}}(\text{triv}) = 1 \otimes \text{triv} \). Hence \( eL_{\frac{1}{n}}(\text{triv}) \) is a one-dimensional representation of \( eH_{\frac{1}{n}}(\mathfrak{S}_n) \). As noted before the exercise, we have an isomorphism \( eH_{\frac{1}{n}}(\mathfrak{S}_n) \text{triv} \simeq e_{-}H_{\frac{n+1}{n}}(\mathfrak{S}_n) e_{-} \). So, the algebra \( e_{-}H_{\frac{n+1}{n}}(\mathfrak{S}_n) e_{-} \) admits a one-dimensional representation. We will fix this one-dimensional representation to be \( C v_0 \) for some spanning vector \( v_0 \).

(1) As noted before the exercise, the PBW property of \( H_c(\mathfrak{S}_n) \) implies that \( \text{gr}_\mathcal{F} H_c(\mathfrak{S}_n) \simeq \mathbb{C}[V \times V^*] \rtimes \mathfrak{S}_n \). Therefore, \( \text{gr}_\mathcal{F} L_c(\text{triv}) \) is a module for \( \mathbb{C}[V \times V^*] \rtimes \mathfrak{S}_n \). Since the image \( e_{-} \otimes v_0 \) of \( e_{-} \otimes v_0 \) in \( \text{gr}_\mathcal{F} L_c(\text{triv}) \) is a non-zero element that transforms by the sign representation (notice here that we have used the fact that each piece \( \mathcal{F}_i \text{L}_c(\text{triv}) \) of the filtration is \( \mathfrak{S}_n \)-stable), we can define a non-zero morphism \( \mathbb{C}[V \times V^*] e_{-} \rightarrow \text{gr}_\mathcal{F} L_c(\text{triv}) \) by sending \( e_{-} \) to \( e_{-} \otimes v_0 \). The hard part is to work out why this morphism is surjective. This actually follows from some general facts about filtrations, which I’ll try to explain. Firstly, we notice that we have a surjective map \( \eta : H_c(\mathfrak{S}_n) e_{-} \rightarrow L_c(\text{triv}) \) defined by sending \( e_{-} \) to \( e_{-} \otimes v_0 \) (it is surjective because \( L_c(\text{triv}) \) is irreducible). Now, the filtration on \( L_c(\text{triv}) \) can be defined by setting \( \mathcal{F}_i L_c(\text{triv}) = \eta((\mathcal{F}_i H_c(\mathfrak{S}_n)) e_{-}) \). It is a general (and easy) fact that, if we have a surjective morphism \( \eta : A \rightarrow B \) between filtered vector spaces such that \( \mathcal{F}_i B = \eta(\mathcal{F}_i A) \) (the morphism is ”strictly filtered”) then the associated graded map \( \text{gr} \eta : \text{gr}_\mathcal{F} A \rightarrow \text{gr}_\mathcal{F} B \) is also surjective. In particular, in our example we have
\[ \text{gr} \eta : \text{gr}_\mathcal{F} H_c(\mathfrak{S}_n) e_{-} = \mathbb{C}[V \times V^*] e_{-} \rightarrow \text{gr}_\mathcal{F} L_c(\text{triv}). \]

(2) Since the map \( \text{gr} \eta \) is \( \mathfrak{S}_n \)-equivariant, we can tensor both sides by \( \text{sgn} \) to get a surjective \( \mathbb{C}[V \times V^*] \rtimes \mathfrak{S}_n \)-module morphism \( \eta' : \mathbb{C}[V \times V^*] \rightarrow (\text{gr}_\mathcal{F} L_c(\text{triv})) \otimes \text{sgn} \). Any \( \mathbb{C}[V \times V^*] \)-module which is a quotient of \( \mathbb{C}[V \times V^*] \) is automatically a commutative ring. Since gradings are important, let me be a bit more specific about the bigrading on \( (\text{gr}_\mathcal{F} L_c(\text{triv})) \otimes \text{sgn} \). To begin with, the Euler element \( e_u \), makes \( H_c(\mathfrak{S}_n) \) \( \mathbb{Z} \)-graded by saying that \( f \in H_c(\mathfrak{S}_n) \) has degree \( k \) if \( [e_u, f] = kf \) (we say that the algebra \( H_c(\mathfrak{S}_n) \) has an ”internal” grading). Then, it is an easy fact (try to prove it!) that \( e_u \) acts locally finitely on any module in category \( \mathcal{O} \) and the generalized eigenspaces are finite dimensional. Therefore, any such module has a natural \( \mathbb{Z} \)-grading. In particular, \( L_c(\text{triv}) \) is \( \mathbb{Z} \)-graded. After taking \( \text{gr}_\mathcal{F} \), this will correspond to the \( \mathbb{Z} \)-grading on \( \mathbb{C}[V \times V^*] = \text{Sym}(V \oplus V^*) \) given by putting \( V^* \) in degree one and \( V \) in degree \(-1\).
Secondly, the associated graded of any filtered space has a natural $\mathbb{Z}$-grading, e.g.

$$(\text{gr}_F L_c(\text{triv}))_i := (\mathcal{F}_i L_c(\text{triv}))/\mathcal{F}_{i-1} L_c(\text{triv}).$$

This corresponds to the grading on $\mathbb{C}[V \times V^*]$ defined by putting $V$ and $V^*$ in sdegree one. Since we have shown that $(\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$ is a quotient of $\mathbb{C}[V \times V^*]$, this implies that the grading on $(\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$ coming from the filtration is actually an $\mathbb{N}$-grading. Thus, there is actually a $\mathbb{Z}^2$-grading on $(\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$.

(3) Recall that we have fixed $c = (n + 1)/n$, therefore exercise 5 says that there are

$$\frac{1}{n+1} \binom{n+1}{n} = 1$$

copies of sgn in $\text{gr}_F L_c(\text{triv})$. But we also know that $e_- \otimes v_0$ is at least one copy of sgn in $\text{gr}_F L_c(\text{triv})$. Hence, it is the only copy of sgn. Thus, $e_- \otimes v_0 \otimes \text{sgn}$ is the only copy of the trivial representation in $(\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$. So

$$\mathbb{C}[V \times V^*]^{S_n}_{\pm} \cdot (e_- \otimes v_0 \otimes \text{sgn}) \subset \mathbb{C} \cdot (e_- \otimes v_0 \otimes \text{sgn}).$$

On the other hand, recall that the surjection

$$\eta : \mathbb{C}[V \times V^*] \rightarrow (\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$$

is graded and $\mathbb{C}[V \times V^*]^{S_n}_{\pm}$ is strictly positively graded with respect to $\text{th}$ filtration grading. This implies that $\mathbb{C}[V \times V^*]^{S_n}_{\pm} \cdot e_- \otimes \mathbb{C} \otimes \text{sgn} = 0$, as required.

**Exercise 9.** The goal of this exercise is to try and describe $L_c(\text{triv})$ as a graded $S_n$-module when $n = 3$ and $c = (n + 1)/n$. Using what we already know about $L_c(\text{triv})$ from exercises 1 and 2, together with the fact that $(\text{gr}_F L(\text{triv})) \otimes \text{sgn}$ is a Gorenstein ring, describe the $q$-graded, $S_n$-structure of $L_c(\text{triv})$.

**Hint:** the fact that $(\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$ is Gorenstein implies that $[L_c(\text{triv}), : \lambda] = [L_c(\text{triv})_{-i+2N} : \lambda]$, $\forall i$,

the shift appearing because the identity element in $(\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$ is $\text{gr}_F(\delta_x \otimes e) \otimes \text{sgn}$.

Can you guess what the bigraded $S_3$-structure of $(\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$ is? Knowing that the 3rd $(q, t)$-Catalan polynomial is $q^3 + qt + q^2t + qt^2 + t^3$, which is the bigraded character of $(\mathbb{C}[V \times V^*]/\langle \mathbb{C}[V \times V^*]^{S_3}\rangle)^{S_3}$, is a big help here.

**Solution 10.** We know from the first exercise that $L_c(\text{triv})$ is the quotient of $M_c(\text{triv})$ by a regular sequence in degree 4 which is isomorphic as an $S_3$-module to $V$. Also, the hint using the fact that $(\text{gr}_F L_c(\text{triv})) \otimes \text{sgn}$ is Gorenstein implies that $L_c(\text{triv})_i = 0$ for $i > 6$. So let’s write out $\mathbb{C}[V]$ as a graded $S_3$-module up to degree 6. To do this we use the fact that $\mathbb{C}[V] = \mathbb{C}[V]^{\text{co}S_3} \otimes \mathbb{C}[V]^{S_3}$ and that $\mathbb{C}[V]^{\text{co}S_3} = T \oplus V[1] \oplus V[2] \oplus S[3]$, where $T$ is a copy of triv, $S$ a copy of
sgn and $V[2]$ means $V$ in degree 2 and so on. Hence,

$$
\begin{array}{c|c}
\text{M}_c(\text{triv}) & \\
\hline
0 & T \\
1 & V \\
2 & V T \\
3 & S V T \\
4 & V V T \\
5 & S V V T \\
6 & S V V T T \\
\end{array}
$$

Using the fact that we are moding out one copy of the $V$’s in degree four, together with the fact that we should remain symmetric, we get

$$
\begin{array}{c|c}
\text{L}_c(\text{triv}) & \\
\hline
0 & T \\
1 & V \\
2 & V T \\
3 & S V T \\
4 & V T \\
5 & V \\
6 & T \\
\end{array}
$$

Now to get the bigrading, we are told that we should filter the module by saying that the copy of $S$ in degree 3 lies in filtered degree zero. Applying just the $x$’s produces one copy of the coinvariant ring, to the right of $S$ (since $x$ has positive degree) and one copy to the left of the coinvariant ring by applying the $y$’s,

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
0 & S & \\
1 & V & V & \\
2 & V & V & \\
3 & T & T & \\
\end{array}
$$

This means that we are missing one copy of $V$ and 3 copies of $T$. The polynomial $q^3 + qt + q^2t + qt^2 + t^3$ tells us where all the $T$’s live, but it is with respect to the grading where $x$ has degree $q$ and $y$ degree $t$. So, in terms of the grading $s$ where $x$ has degree one and $y$ degree $-1$ (the Euler grading) and the grading $r$, where $x$ and $y$ both have degree one (the filtration degree), we get $s^3r^3 + s^2 + s^3r + s^3r^{-1} + s^3r^{-3}$. Thus, we must have

$$
\begin{array}{cccccccc}
\text{L}_c(\text{triv}) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \\
\hline
0 & S & \\
1 & V & V & \\
2 & V & V, T & V & \\
3 & T & T & T & T & \\
\end{array}
$$

Exercise 11. When the group $G$ is not well-generated it is more difficult to calculate the $m$th $q$-Fuss-Catalan number. Using the data in the table of section
3 of [2], calculate the $m$th $q$-Fuss-Catalan number for the exceptional complex reflection group $G_{13}$. What is the dimension of the $eH_{m+\frac{1}{13}}(G_{13})e$-module $eL_{c}(\text{triv})$?

Solution 12. This is just a case of plugging the numbers into the formula. We have $C_{G_{13}}^{(0)}(q) = 1$,

$$C_{G_{13}}^{(1)}(q) = \frac{[24]_q[32]_q}{[8]_q[12]_q} = q^{36} + q^{28} + q^{24} + q^{20} + q^{16} + q^{12} + q^{8} + 1,$$

$$C_{G_{13}}^{(2)}(q) = \frac{[44]_q[48]_q}{[8]_q[12]_q}, \quad \text{dim } L(\text{triv}) = 22,$$

$$C_{G_{13}}^{(3)}(q) = \frac{[56]_q[72]_q}{[8]_q[12]_q}, \quad \text{dim } L(\text{triv}) = 42.$$

References