1. The quantum group $U_q\mathfrak{g}$

In the first part we saw how a generalised Cartan matrix $A = [a_{ij}]$ contains the information that allows us to write a presentation of a Kac-Moody Lie algebra $\mathfrak{g}$. This presentation had generators $e_i, f_i, h = \alpha^\vee \in \mathfrak{h}$, and some relations. Notice that one could think of this presentation in two ways: either as a presentation of a Lie algebra $\mathfrak{g}$ (constructing a free Lie algebra in those generators, and modding out by the relations), or as a presentation of its universal enveloping algebra $U_\mathfrak{g}$ (constructing a free associative algebra in those generators, and modding out by the relations). The aim now is to write down $U_q\mathfrak{g}$, which is a deformation of $U_\mathfrak{g}$, using a similar presentation. Deformation means it is an algebra depending on some parameter $q$, such that its structure is similar to the structure of $U_\mathfrak{g}$, and such that setting $q = 1$ in an appropriate way recovers the original algebra $U_\mathfrak{g}$.

1.1. $U_q\mathfrak{g}$ for $q$ a formal parameter. Let $A = [a_{ij}]$ be a generalised $n \times n$ Cartan matrix as before. Let $q$ be an indeterminate, and let $\mathbb{K} = \mathbb{C}(q)$ be a field of rational functions in it.

We introduce the following notation:

$$d_i = \text{the symmetrising integers for the matrix } A, d_i a_{ij} = d_j a_{ji}$$
$$q_i = q^{d_i}$$
$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} = q^{m-1} + q^{m-3} + \ldots + q^{-m+1}$$
$$[m]_q! = [m]_q [m-1]_q \cdots [1]_q$$
$$\left[ \begin{array}{c} m \\ k \end{array} \right]_q = \frac{[m]_q!}{[k]_q! [m-k]_q!}$$

We define $U_q\mathfrak{g}$ as an associative algebra over $\mathbb{C}(q)$, with generators:

$$E_i, F_i, i = 1, \ldots, n$$

$$K_h^{\pm 1}, h \in Q^\vee \subset \mathfrak{h} = \text{coroot lattice}, \text{ with } K_{d_i h_i} = K_i$$
and relations:
\[
\begin{align*}
K_h K_{h'} &= K_{h+h'} \\
K_0 &= 1 \\
K_h E_i K_{h}^{-1} &= q^{\alpha_i(h)} E_i \\
K_h F_i K_{h}^{-1} &= q^{-\alpha_i(h)} F_i \\
[E_i, F_j] &= \delta_{ij} \frac{K_i - K_{i-1}}{q_i - q_{i-1}}
\end{align*}
\]

and the quantum Serre relations
\[
\begin{align*}
1 - a_{ij} \sum_{s=0}^{1-a_{ij}} (-1)^s \left[ 1 - a_{ij} \right]_q E_i^{1-a_{ij}} F_j E_i^s &= 0 \\
1 - a_{ij} \sum_{s=0}^{1-a_{ij}} (-1)^s \left[ 1 - a_{ij} \right]_q F_i^{1-a_{ij}} F_j F_i^s &= 0
\end{align*}
\]

**Remark 1.** Clearly \( K_h^{-1} = K_{-h} \). Notice that \( \alpha_i(h) \in \mathbb{Z} \) for \( h \in Q^\vee \), so \( q^{\alpha_i(h)} \in \mathbb{C}(q) \). Morally one should think of \( K_h \) as \( q^h \) and \( K_i = q_i^h \), and sometimes this notation is used.

Other choices are possible for the indices of \( K_h \), for example allowing \( h \in P^\vee \) the coweight lattice, and even \( h \in \mathfrak{h} \). The choice \( h \in Q^\vee \) is convenient because it means \( \lambda(h) \in \mathbb{Z} \) for integral weights \( \lambda \), so \( q^{\lambda(h)} \) makes sense in \( \mathbb{C}(q) \). If working with \( h \in P^\vee \), one might have to work over some extension of \( \mathbb{C}(q) \), e.g. \( \mathbb{C}(q^{1/2}) \).

1.2. **Specialising at** \( q = \epsilon \in \mathbb{C} \). Another possible meaning of a deformation is an algebra over \( \mathbb{C} \), depending on some parameter. To see how to specialise this, let \( \epsilon \in \mathbb{C} \) be a number, \( \epsilon \neq 0 \).

First, let us define an integral form of \( U_q \mathfrak{g} \). Let \( \mathcal{A} = \mathbb{Z}[q, q^{-1}] \), and notice that most relations are in fact over this ring. More precisely, define \( \mathcal{U}_{\mathcal{A}} \) to be an \( \mathcal{A} \) subalgebra of \( U_q \mathfrak{g} \) generated by
\[
E_i, F_i, K_i, \check{H}_i = \frac{K_i - K_{i-1}}{q_i - q_{i-1}}.
\]
(For some purposes, the divided powers \( E_i^{(m)} = \frac{E_i^m}{[m]_q} \) need to be included among the generators as well). It has a similar presentation as \( U_q \mathfrak{g} \), with the only difference that the relation
\[
[E_i, F_j] = \delta_{ij} \frac{K_i - K_{i-1}}{q_i - q_{i-1}}
\]
is replaced by
\[
[E_i, F_j] = \delta_{ij} \check{H}_i \\
(q_i - q_{i-1}) \check{H}_i = K_i - K_{i-1}
\]
and appropriate relations for commuting \( \check{H}_i \) and \( E_j, F_j \) are added.
Then $U_A$ is an integral form of $U_q\mathfrak{g}$, in the sense that $U_A \otimes_A \mathbb{C}(q) \cong U_q\mathfrak{g}$. Define a $\mathcal{A}$-module structure on $\mathbb{C}$ by $q.x = \epsilon x$. Define the specialisation of $U_q\mathfrak{g}$ at $q = \epsilon$ as $U_\epsilon \mathfrak{g} = U_A \otimes_A \mathbb{C}$.

This is an algebra over $\mathbb{C}$.

1.3. **Specialising at $\epsilon = 1$.** Choosing $\epsilon = 1$, we have that $U_1\mathfrak{g}/\langle K_i - 1 \rangle \cong U\mathfrak{g}$.

The isomorphism in question is

- $E_i \mapsto e_i$
- $F_i \mapsto f_i$
- $\tilde{H}_i \mapsto h_i$
- $K_i \mapsto 1$

We will not prove it is an isomorphism, but let us show how the relations for $U_1\mathfrak{g}$ transform into relations for $U\mathfrak{g}$.

\[
\begin{align*}
K_i E_j K_i^{-1} &= q_i^{a_{ij}} E_j & \mapsto & & 1 e_j 1 = e_j \\
K_i F_j K_i^{-1} &= q_i^{-a_{ij}} E_j & \mapsto & & 1 f_j 1 = f_j \\
[E_i, F_j] &= \delta_{ij} \tilde{H}_i & \mapsto & & [e_i, f_j] = \delta_{ij} \tilde{h}_i \\
(q_i - q_i^{-1}) \tilde{H}_i &= K_i - K_i^{-1} & \mapsto & & 0 = 0
\end{align*}
\]

\[
[\tilde{H}_i, E_j] = \frac{1}{q_i - q_i^{-1}} \left( (q_i a_{ij} - 1) E_j K_i - (1 - q_i^{-a_{ij}}) E_j K_i^{-1} \right)
\]

\[
= \frac{1}{q_i - q_i^{-1}} \left( (q_i a_{ij} - 1) E_j K_i + (q^{-a_{ij}} - 1) E_j \tilde{H}_i \right)
\]

\[
\mapsto [h_i, e_j] = a_{ij} e_j \cdot 1 + (1 - 1)e_j h_i = a_{ij} e_j.
\]

In this specialisation $q \to 1$, $[m]_q = q^{m-1} + \ldots + q^{-m+1}$ becomes $m$, so quantum Serre relations

\[
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ \begin{array}{c} 1 - a_{ij} \\ s \end{array} \right]_q E_i^{1-a_{ij}-s} E_j E_i^s = 0
\]

become the usual Serre relations $(R5)$

\[
\text{ad}(e_i)^{1-a_{ij}}(e_j) = \sum_{s=0}^{1-a_{ij}} (-1)^s \left( \begin{array}{c} 1 - a_{ij} \\ s \end{array} \right) e_i^{1-a_{ij}-s} e_j e_i^s = 0.
\]
Remark 2. Less formally, one can also write $q = e^h$, $K_h = q^{hh}$, and check that the relations for $U_q\mathfrak{g}$, up to order $h^2$, are exactly the relations for $U\mathfrak{g}$. For example,

$$K_h E_j K_h^{-1} = q^{\alpha_j(h)} E_j$$

becomes

$$(1 + hh + O(h^2)) \cdot E_j \cdot (1 - hh + O(h^2)) = (1 + h \alpha_j(h) + O(h^2)) \cdot E_j$$

from which it follows

$$E_j + h(h E_j - E_j h) + O(h^2) = E_j + h \alpha_j(h) E_j + O(h^2),$$

and reading off the coefficient of $h$ we get:

$$h E_j - E_j h = \alpha_j(h) E_j.$$

2. PBW

2.1. PBW for $U\mathfrak{g}$. The universal enveloping algebra $U\mathfrak{g}$ is the associative algebra with the generators $e_i, f_i, h \in \mathfrak{h}$, and the relations given in the first part of the talk. Alternatively, it is the universal associative algebra containing $\mathfrak{g}$, given as the quotient of the tensor algebra $T\mathfrak{g}$ on $\mathfrak{g}$ by the ideal generate by relations $xy - yx = [x, y]$ for $x, y \in \mathfrak{g}$. The tensor algebra $T\mathfrak{g}$ is graded by degree of $\mathfrak{g}$, and $U\mathfrak{g}$ inherits this as a filtration; consequently, it contains the Lie algebra $\mathfrak{g}$ as its degree 1 part. The Poincaré - Birkhoff -Witt theorem talks about a good choice of basis for $U\mathfrak{g}$.

Theorem 1 (PBW for $q = 1$). (1) The associated graded algebra to $U\mathfrak{g}$ is the symmetric algebra,

$$\text{gr}U\mathfrak{g} \cong S\mathfrak{g}.$$

(2) Let $x_1, \ldots, x_M$ be a basis for $\mathfrak{g}$. Then the set

$$\{x_1^{a_1} \cdots x_M^{a_M} | a_i \geq 0\}$$

is a basis for $U\mathfrak{g}$.

One can apply this to a particular basis: choose root vectors $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ for all positive roots, and choose a basis of $\mathfrak{h}$. The theorem than says that multiplication

$$U\mathfrak{n}^+ \otimes U\mathfrak{h} \otimes U\mathfrak{n}^- \rightarrow U\mathfrak{g}$$

is an isomorphism of vector spaces. Furthermore, if we choose an ordering on positive roots $\{\beta_1, \ldots, \beta_N\}$, it says that a basis of $U\mathfrak{n}^+$ is

$$\{e_{\beta_1}^{a_1} \cdots e_{\beta_N}^{a_N} | a_i \geq 0\},$$

and corresponding claims for $U\mathfrak{n}^-, U\mathfrak{h}$. 
2.2. The example of $U_q\mathfrak{sl}_2$. The Cartan matrix for $\mathfrak{sl}_2$ is the $1 \times 1$ matrix $A = [2]$. The coroot lattice is $Q^\vee = \mathbb{Z}h \subseteq \mathfrak{h}$, identified with the root lattice $Q = \mathbb{Z}\alpha \subseteq \mathfrak{h}^*$. Consequently, $U\mathfrak{sl}_2$ is an algebra over $\mathbb{C}$ with generators $e, f, h$ and relations
\[
[h, e] = 2e \quad [h, f] = -2f \quad [e, f] = h
\]

At $q = 1$, the PBW theorem says that a basis for $U\mathfrak{g}$ is the set
\[
\{e^ah^bf^c | a, b, c \in \mathbb{Z}_{\geq 0}\}.
\]
Analogously, $U_q\mathfrak{sl}_2$ is an algebra over $\mathbb{C}(q)$ with generators $E, F, K^{\pm 1} = K_{\pm h}$ and relations
\[
KK^{-1} = K^{-1}K = 1 \quad KEK^{-1} = q^2E \quad KFK^{-1} = q^{-2}F \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.
\]

There is an analogue of PBW for $U_q\mathfrak{g}$; in this case it says:

**Theorem 2** (PBW). The set $\{E^aK^bF^c | a, b, c \in \mathbb{Z}, a, c \geq 0\}$ is a basis for $U_q\mathfrak{sl}_2$.

2.3. PBW part 1 for $U_q\mathfrak{g}$. Define the following subalgebras of $U_q\mathfrak{g}$:
\[
U^+ = U_q\mathfrak{h}^+ = \text{the subalgebra generated by } E_i, i = 1, \ldots n
\]
\[
U^- = U_q\mathfrak{h}^- = \text{the subalgebra generated by } F_i, i = 1, \ldots n
\]
\[
U^0 = U_q\mathfrak{h}^0 = \text{the subalgebra generated by } K_h, h \in Q^\vee
\]

For example, for $U_q\mathfrak{sl}_2$, $U^+ = \mathbb{K}[E], U^- = \mathbb{K}[F], U^0 = \mathbb{K}[K^\pm]$. 

**Theorem 3** (PBW, Part 1). (1) Multiplication
\[
U^+ \otimes U^0 \otimes U^- \to U_q\mathfrak{g}
\]
is an isomorphism of vector spaces.

(2) The algebras $U^+, U^-, U^0$ have presentations:
\[
U^+ = \langle E_1, \ldots, E_n | \text{Serre relations} \rangle
\]
\[
U^- = \langle F_1, \ldots, F_n | \text{Serre relations} \rangle
\]
\[
U^0 = \mathbb{K}[Q^\vee].
\]

One would also like to have a good basis for $U^\pm$, in order to consider the PBW theorem complete. In $q = 1$ case, this was written down using the root vectors. In $q \neq 1$ case, this involves a choice.
2.4. The $\mathfrak{sl}_3$ example. Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ be the Cartan matrix of type $A_2$, i.e. $\mathfrak{g} = \mathfrak{sl}_3$.

The positive roots are the simple roots $\alpha_1, \alpha_2$, and the highest root $\theta = \alpha_1 + \alpha_2$. The corresponding root vectors are elementary upper triangular matrices,

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_\theta = [e_1, e_2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last one of these, $[e_1, e_2] = ad(e_1)(e_2) = -ad(e_2)(e_1)$ is, up to a multiplicative constant, the canonical choice of the root vector belonging to the root space of $\theta = \alpha_1 + \alpha_2$. The PBW theorem then said that the basis of $U^+$ was $\{e_1^a e_2^b e_\theta^c | a, b, c \geq 0\}$.

In the same construction for $U_q\mathfrak{sl}_3$, we no longer have the distinguished copy of the Lie algebra $\mathfrak{sl}_3$ inside $U_q\mathfrak{sl}_3$ as “degree 1” part of the filtered algebra. We still have $E_1$ and $E_2$ as generators, and one should try to choose an element of $U_q\mathfrak{sl}_3$ in the appropriate eigenspace. The two obvious choices are

$$ad(E_1)(E_2) = E_1 E_2 - q^{-1} E_2 E_1$$

$$ad(E_2)(E_1) = E_2 E_1 - q^{-1} E_1 E_2$$

which are no longer proportional.

In higher types, the challenge is to make these choices of “root vectors” consistently.

2.5. Lustig’s braid action. The set of roots is symmetric under the action of the corresponding Weyl group, and in the case of finite dimensional simple Lie algebras every positive root $\beta$ can be represented as $\beta = w(\alpha_i)$ for some choice of a simple root $\alpha_i$ and an element $w$ of the Weyl group. It is natural to ask whether this action extends to the algebra itself, i.e. whether we can find an action of the Weyl group $W$ on $\mathfrak{g}$ or $U\mathfrak{g}$ or $U_q\mathfrak{g}$. The answer is no, but there is an action of the braid group by Lustig’s automorphisms which serves the same purpose.

For every simple root $\alpha_i$, there is a simple reflection $s_i \in W$, and these reflections generate $W$. Associated to them, one can write elements $S_i$ in a certain completion of $U_q\mathfrak{g}$ (see e.g. Saito’s papers, or Lustig’s papers for references and an explicit formula). The map $x \mapsto S_i x S_i^{-1} = T_i(x)$ defines an automorphism $T_i : U_q\mathfrak{g} \to U_q\mathfrak{g}$, given on the generators by
the following formulas:

\[
T_i(E_i) = - E_i K_i \\
T_i(F_i) = - K_i^{-1} F_i \\
T_i(K_h) = K_{s_i(h)} \\
T_i(E_j) = \sum_{r=0}^{a_{ij}} (-1)^{-a_{ij} + r} q_i^{-r} \frac{E_i^{-a_{ij} - r}}{[-a_{ij} - r]_q!} E_j^r \frac{E_i^r}{[r]_q!} \\
T_i(F_j) = \sum_{r=0}^{a_{ij}} (-1)^{-a_{ij} + r} q_i^r \frac{F_j^r}{[r]_q!} F_i^{-a_{ij} - r} \frac{F_i^r}{[-a_{ij} - r]_q!} 
\]

(Notice that the last two equations have one term fewer that the corresponding Serre relations. They map \(E_j\), which is the lowest weight vector for the action of the copy of \(U_q\mathfrak{sl}_2\) corresponding to the simple root \(\alpha_i\), to \(T_i(E_j)\), which is the highest weight vector for it.)

These maps satisfy:

- \(T_i\) is an automorphism of \(U_q\mathfrak{g}\).
- \(T_i\) satisfy the braid relations. More precisely, \(W\) is a reflection group, generated by simple reflections \(s_i\), which satisfy the relations

\[
s_i^2 = 1 \quad s_i s_j s_i \ldots = s_j s_i s_j \ldots \quad \text{for some integers } m_{ij} \text{ which can be easily computed from } a_{ij}. \quad \text{These relations are called braid relations.} \quad \text{The group with generators } t_i, \text{ and relations } t_i t_j t_i \ldots = t_j t_i t_j \ldots \quad (\text{but not } t_i^2 = 1!) \text{ is called the braid group } BW \text{ of } W. \quad \text{It is infinite, the Weyl group } W \text{ is its quotient (by } t_i^2 = 1), \text{ and the Hecke algebra of } W \text{ is the quotient of its group algebra (by a deformation of the relation } t_i^2 = 1). \quad \text{Because } T_i \text{ satisfy the braid relations, it is possible to define } T_w = T_{i_1} T_{i_2} \ldots T_{i_k} \text{ for every } w \in W, \text{ and this decomposition does not depend on the choice of a reduced decomposition } w = s_{i_1} \ldots s_{i_k}.

2.6. PBW, part 2. From now on, assume that \(A\) is a Cartan matrix of finite type. In that case, for every positive root \(\beta\) we can choose \(w \in W\) and a simple root \(\alpha_i\) such that \(\beta = \alpha_i\). Then define

\[
E_\beta = T_w(E_i), \\
F_\alpha = T_w(F_i).
\]

These will be our replacements for the root vectors \(e_\beta, f_\beta\). Also choose an ordering on the set of all positive roots, labelling them \(\beta_1, \ldots, \beta_N\).
Theorem 4 (PBW, Part 2). With the above choices, $U^+$ and $U^-$ have bases
\[ E_{\beta_1}^{a_1} \cdots E_{\beta_N}^{a_N} | a_i \in \mathbb{Z}_{\geq 0} \],
\[ F_{\beta_1}^{b_1} \cdots F_{\beta_N}^{b_N} | a_i \in \mathbb{Z}_{\geq 0} \].

Remark 3. A consistent way to make these choices, of both the presentation $\beta = w(\alpha_i)$ and the ordering on the set of positive root vectors, is to choose a reduced decomposition of the longest element $w_0 \in W$, $w_0 = s_{i_1} \ldots s_{i_N}$. Then the set
\[ \beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \ldots, \quad \beta_j = s_{i_1} \ldots s_{i_{j-1}}(\alpha_{i_j}), \quad \ldots, \quad \beta_N = s_{i_1} \ldots s_{i_{N-1}}(\alpha_{i_N}) \]
is the set of all positive roots.

Notice that the basis obtained really depends on the choices made. For example, for $U_q\mathfrak{sl}_3$ as above, the choice $w_0 = s_{1}s_{2}s_1$ gives the ordering
\[ \beta_1 = \alpha_1, \beta_2 = \alpha_1 + \alpha_2, \beta_3 = \alpha_2 \]
and the corresponding vectors
\[ E_1, T_1(E_2) = -E_1 E_2 + q^{-1} E_2 E_1, E_2, \]
while the choice $w_0 = s_2s_1s_2$ gives the ordering
\[ \beta_1 = \alpha_2, \beta_2 = \alpha_1 + \alpha_2, \beta_3 = \alpha_1 \]
and the corresponding vectors
\[ E_2, T_2(E_1) = -E_2 E_1 + q^{-1} E_1 E_2, E_1. \]

3. Hopf algebra structure

There is also a Hopf algebra structure on $U_q\mathfrak{g}$, giving a tensor structure on its representation category. The coproduct $\Delta : U_q\mathfrak{g} \to U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ (allowing us to define a tensor product of represent ions) is given by
\[ \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} \otimes 1 + \otimes F_i, \quad \Delta(K_h) = K_h \otimes K_h \]
the count $\Delta : U_q\mathfrak{g} \to \mathbb{C}(q)$ (allowing us to define a trivial representation) is given by
\[ \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0, \quad \varepsilon(K_h) = 1, \]
and the antipode $S : U_q\mathfrak{g} \to U_q\mathfrak{g}$ (allowing us to define a dual representation) is given by
\[ S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i, \quad S(K_h) = K_h^{-1}. \]
Notice that \( \text{flip} \circ \Delta \neq \Delta \), i.e. the coproduct is not cocommutative. This means that the obvious map \( \text{flip} : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1 \) is not an isomorphism of representations; the search for this isomorphism produces interesting operators.

**Example 1.** Let us use the coproduct to justify the adjoint action formulas for \( U_q \mathfrak{sl}_3 \) which we wrote earlier. In general, the adjoint action is given by
\[
\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \quad \text{ad}(x)(y) = \sum x_{(1)} y S(x_{(2)}).
\]
So,
\[
\text{ad}(E_i)(E_j) = E_i E_j S(1) + K_i E_j S(E_i) = E_i E_j - q^{a_{ij}} E_j E_i.
\]