Semisimple and Kac-Moody Lie algebras

§1. Review of semisimple Lie algebras

Def. A Lie algebra \( g \) is a vector space over a field \( K \) equipped with a bilinear map \( [\cdot,\cdot]:g \times g \to g \) which is alternating and satisfies the Jacobi identity

\[
[lx,[y,z]] + [l[x,y],z] + [x,[l,y,z]] = 0 \quad \forall x,y,z \in g.
\]

Ex. \( gl_n(K) \), \( sl_n \), ...

Def. A representation of a Lie algebra \( g \) is a vector space \( V \) together with a Lie algebra map \( g \to gl(V) = \text{End}(V) \).

Ex. (Key) The adjoint representation of \( g \) is \( \text{ad} : g \to gl(g) \), where

\[
\text{ad} x : g \to g, \quad \text{ad} x(y) = [x,y].
\]

\( \phi \) Def. - An ideal \( \mathfrak{I} \) of \( g \) is a vector space \( \mathfrak{I} \subset g \) such that \( [\mathfrak{I},g] \subset \mathfrak{I} \);

- \( g \) is simple if it has no non-trivial ideals and \( 2 \leq \dim(g) < \infty \);
- \( g \) is semisimple if it is direct sum of finitely many simple Lie algebras.

Ex. \( sl_n \) is simple.

Prop. Let \( g \) be a finite dimensional Lie algebra. Then, the following are equivalent:

(i) \( g \) is semisimple;
(ii) \( g \) has no non-trivial abelian ideals;
(iii) The Killing form, defined as the bilinear symmetric form \( \kappa : g \times g \to K \),

\[
(\text{Killing form}) \quad x, y \mapsto \text{tr}(\text{ad}(x) \text{ad}(y))
\]

is non-degenerate.

Cartan If \( g \) is semisimple, there is the following decomposition of \( g \) in weight spaces with respect to the adjoint action of \( h \) on \( g \):

\[
g = h \oplus \bigoplus_{\lambda \in \Lambda^0} g_{\lambda},
\]
where \( H \) is a Cartan subalgebra of \( g \), i.e., an abelian subalgebra such that \( N_g(H) = H \) and \( g_x = \{ x \in g \mid [H, x] = x(h) \}, \) the \( H^\vee \).

Def. If \( g_x \neq 0 \), \( x \) is called \text{root}. The set of roots is denoted by \( \Delta \) via the Killing form we have an isomorphism \( h \cong h^\vee \) (\( g \) is semisimple), so we get a symmetric bilinear non-degenerate form on \( h^\vee \), denoted by \( (\cdot, \cdot) \).

Def. - For \( x \in \Delta \), we denote by \( s_x \) the reflection defined by \( s_x : h^\vee \to h^\vee \):

\[
s_x(u) = u - 2 \frac{(u, x)}{(x, x)} x.
\]

- The group generated by \( \{s_x \mid x \in \Delta \} \) is called \text{Weyl group} and denoted by \( W \).

**Rule:** \( W \) is finite.

**Then:**

(i) \( \Delta \) spans \( h^\vee \), \( \Delta \cong \mathbf{1} \Delta \);

(ii) If \( x, c \in \Delta \) for \( c \in K \), then \( c = \pm 1 \);

(iii) \( W(\Delta) = \Delta \);

(iv) \( \langle x, \beta \rangle := 2 \frac{(x, \beta)}{(x, x)} \in \mathbb{Z} \), \( \forall x, \beta \in \Delta \) (Cartan integers)

**Rule:** the previous conditions are the axioms for a root system on a Euclidean space (\( \mathbf{1} \) in our case \( h^\vee \)).

**Def.** \( x \in \Delta \) is a \text{simple root} if it cannot be written as the sum of two or more roots. The set of simple roots is denoted by \( \Pi \) and is called a \text{basis of} \( \Delta \).

**Then:** - \( W \) is generated by \( \{s_\alpha \mid \alpha \in \Pi \} \);

- \( W \) acts simply transitively on the connected components of \( h^\vee \setminus \cup H_x \), where \( H_x \) is the hyperplane orthogonal to \( x \) (called the \text{Weyl chambers}) and \( \Pi \) the \text{bases of} \( \Delta \).

**Def.** - For \( x \in \Delta \), we define the \text{concord} \( x^\vee \in h \) by \( \langle x^\vee, \beta \rangle = \langle x, \beta \rangle \), \( \forall \beta \in \Delta \);

- The \text{Cartan matrix} associated to a basis of roots \( \Pi \) is the matrix \( \alpha \).
\[ A = (\langle x_i^*, x_j \rangle)_{i,j=1}^{n,m}, \text{ where } \Pi = \vec{x}_1, \ldots, \vec{x}_n. \]

**Remark:** The Cartan matrix only depends on \( g \) (up to reordering of rows and columns) and it determines \( \Delta \) up to isomorphism.

**Thus:** Let \( A = (a_{ij})_{i,j=1}^{n,m} \) be a Cartan matrix. Then, we have:

(i) \( a_{ii} = 2 \) \( \forall i = 1, \ldots, m \);

(ii) \( a_{ij} \leq 0 \) \( \forall i \neq j \);

(iii) \( a_{ij} = 0 \iff a_{ji} = 0 \).

(iv) \( \det A > 0 \).

However, any matrix with these properties is a Cartan matrix for some semisimple Lie algebra \( g \).

**Remark** the Cartan matrices are the main ingredients for the classification of simple Lie algebras.

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§ 2. **Kac-Moody Lie algebras**

**Def.** A realisation of a matrix \( A \in \text{Mat}_{n,m}(k) \) is a triple \( (\mathfrak{h}, \Pi, \Pi^*) \), where \( \mathfrak{h} \) is a vector space, \( \Pi = \vec{x}_1, \ldots, \vec{x}_m \subset \mathfrak{h}^* \) and \( \Pi^* = \vec{x}_1^*, \ldots, \vec{x}_n^* \), satisfying:

1. \( \Pi \) and \( \Pi^* \) are linearly independent in \( \mathfrak{h}^* \) and \( \mathfrak{h} \) respectively;
2. \( \langle x_i^*, x_j \rangle = a_{ij} \) \( \forall i, j = 1, \ldots, m \);
3. \( \dim \mathfrak{h} = m + \text{corank}(A) \).

**Prop:** There exists a unique realisation of a matrix \( A \) (up to isomorphism).

**Def.** (i) \( A \) is decomposable if \( A = (A_1, A_2) \), up to reordering of rows and columns.

(ii) \( Q \subset \mathfrak{h}^* \) is the \( \mathbb{Z} \)-module generated by \( \Pi \). It is called the root lattice;

\[ Q^+ = \bigoplus_{i=1}^{m} \mathbb{Z}_{\geq 0} \vec{x}_i. \]
Def. Let \( A = (a_{ij})_{i,j=1}^m \) and \( (R, \Pi, D) \) a realisation of \( A \). We define \( \tilde{g}(A) \) to be the Lie algebra generated by \( (e_i)_{i=1}^m , (f_i)_{i=1}^m \) and \( h \) and subject to the following relations:

\[
\begin{align*}
(\mathrm{R}_1) \quad [e_i, f_j] &= \delta_{ij} h, \\
(\mathrm{R}_2) \quad [h, h] &= 0, \\
(\mathrm{R}_3) \quad [h, e_i] &= a_{ii} h, \\
(\mathrm{R}_4) \quad [h, f_i] &= a_{ii} h.
\end{align*}
\]

Let \( \tilde{\mathfrak{h}}_+ \) and \( \tilde{\mathfrak{n}}_+ \) be the Lie subalgebras of \( \tilde{g}(A) \) generated by \( (h)_{i=1}^m \) and \( (e_i)_{i=1}^m \) respectively.

Then:

\[ \tilde{g}(A) = \tilde{\mathfrak{h}}_+ \oplus \tilde{\mathfrak{n}}_+ \oplus \tilde{\mathfrak{h}}. \]

- There exists a unique ideal \( \mathfrak{r} \) of \( \tilde{g}(A) \) which is maximal among the ideals intersecting \( \tilde{\mathfrak{n}}_+ \) trivially.

Set \( g(A) := \tilde{g}(A) / \mathfrak{r} \). We have the following decomposition:

\[ g(A) = \left( \bigoplus_{\alpha \in \mathbb{Q}} g_{\alpha} \right) \oplus \mathfrak{h}_+ \oplus \left( \bigoplus_{\alpha \in \mathbb{Q}^+} g_{\alpha} \right) \]

Def. A Kac-Moody Lie algebra is the Lie algebra \( g(A) \) associated to a generalised Cartan matrix \( A \), i.e. a matrix satisfying conditions (i), (ii), (iii) of Theorem (\( x \)).

Thm: Any two Kac-Moody Lie algebras are isomorphic if and only if their generalised Cartan matrices are obtained from each other by a reordering of rows and columns.

Prop. Let \( A \) be an indecomposable generalised Cartan matrix.

Then, \( g(A) \) is simple if and only if \( \det(A) \neq 0 \).
Prop: The Kac-Moody Lie algebra $\mathfrak{g}(n)$ is generated by $(e_i)_{i=1}^n$, $(h_i)_{i=1}^n$, and $\mathfrak{h}$, and satisfies the relations:

(R5) $(\text{ad } e_i)^{n-1} e_i = 0$ for $i \neq j$;
(R6) $(\text{ad } h_i)^{n-1} h_i = 0$ for $i \neq j$.

(in addition to (R1), (R2), (R3) and (R4)).