Examples \( U(s_L) \) and \( U_q(s_L) \)

1) char = 0 / generic case:

\[ g = \mathfrak{sl}_2 \] with basis \( \{ h, e, f \} \) s.t. \( [h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h \)

\[ g = \mathfrak{sl}_2 \] \( \triangleright \) Identify \( \lambda \rightarrow \mathfrak{u}(\mathfrak{h}) \)

\( \rightarrow \) the root lattice \( \Lambda = 2 \mathfrak{h}^\vee \) is identified w/ \( 2 \mathfrak{h} \)

\( \rightarrow \) the integral weight lattice \( \Lambda = \{ \lambda \in \mathfrak{h}^\vee | <\lambda, \mathfrak{h}> \in \mathbb{Z} \} \) is identified w/ \( 2 \mathfrak{h} \).

\( \rightarrow \) the dominant integral weights are identified w/ \( \mathbb{N} \).

Recall: \( M(\Lambda) = U(\mathfrak{g}) \otimes \mathbb{C} \rho \) as \( U(\mathfrak{g}) \)-module

By PBW for \( \mathfrak{sl}_2 \), \( U(\mathfrak{g}) \otimes \mathbb{C} \rho \approx \mathbb{C}[\mathfrak{h}] \) and \( \rho \) is of weight \(-2\).

Let \( v_0 \in M(\lambda)_\lambda \) be a highest weight vector.

Set \( v_i = \frac{f(v_0)}{i!} \in M(\Lambda, \lambda_{i-2}; \mathbb{C}) \) for \( i \geq 1 \).

\( v_1, \ldots, v_i \) is a basis of \( M(\Lambda) \) consisting of weight vectors (linearly independent, eigenvectors of different eigenvalues).

\( M(\Lambda) \) has the following weight space decomposition:

\[ M(\Lambda) = \bigoplus_{i \geq 0} M(\Lambda, \lambda_{i-2}; \mathbb{C}) \]

where space is one-dimensional.

In general:

\[ h \cdot v_i = (\lambda - 2; i) v_i \]

\[ e \cdot v_i = (i+2) v_{i+1} \]

\[ f \cdot v_i = (\lambda + 1; i-2) v_{i-1} \]

Q: When does \( M(\Lambda) \) contain a non-trivial submodule?

\[ (i - 2 < 0 \Rightarrow i \geq 1) \Rightarrow v_i \in M(\Lambda, \lambda_{i-2}; \mathbb{C}) \] generates a simple highest weight module of \( M(\Lambda) \) of weight \( \lambda_{i-2} \).

(\( M(\Lambda) \) is simple for \( \lambda < 0 \) and has \( M(\lambda - 2; \mathbb{C}) \) as unique submodule for \( \lambda > 0 \).)

(See general form.\)

For \( \lambda > 0 \), \( M(\Lambda) = \frac{M(\Lambda)}{M(\lambda - 2; \mathbb{C})} \) is of degree \( \lambda + 1 \) modulus of the same weight are isomorphic.)
\[ M(\lambda) \]
\[ \text{LW} \]
\[ L(-\lambda - 2) = M(-\lambda - 2) \]
\[ \text{in a composition series for } M(\lambda) \]
\[ 0 \]

Note: \( L(0) \) is the trivial module.
\( L(1) \) is the natural 2-dim rep. of \( \mathfrak{sl}_2 \).
\( L(2) \) is the adjoint representation.

Fact 1) \( \exists \) finite dim. simple \( \mathfrak{sl}_2(\mathbb{C}) \)-modules \( \xrightarrow{\phi} M(\lambda) \rightarrow \lambda + 1 \)

Fact 2) (Weyl's complete reducibility theorem)
All finite-dim. \( \mathfrak{sl}_2(\mathbb{C}) \)-modules are semisimple.

b) Fix ground field \( k \), \( q \in k \setminus \{0,1\} \) with \( q^2 + 1 \)

Recall: \( U_q(\mathfrak{sl}_2) \) is unital, cocommutative \( k \)-algebra with generators \( E, F, K \) and \( K^{-1} \) subject to the relations:

\[
\begin{align*}
K^{-1} &= 1 = K^{-2} K \\
KEK^{-1} &= q^2 E \\
KFK^{-1} &= q^{-2} F \\
EF &= q^{-1} FE \quad \text{with } q = K + K^{-1}
\end{align*}
\]

By the PBW-Theorem, \( \sum E^{j_1} F^{j_2} K^{j_3} | j_1, j_2, j_3 \geq 0, j_1 + j_2 + j_3 \text{ are } \lambda \text{-weights} \) are a basis of \( U_q(\mathfrak{sl}_2) \).

For \( \lambda \in k \setminus \{0,1\} \) consider the \( U_q(\mathfrak{sl}_2) \)-module \( M(\lambda) \) with basis \( m_0, m_1, m_2, \ldots \).

\[
\begin{align*}
K m_i &= q^{-i} m_i \\
F m_i &= m_{i+1} \\
E m_i &= \begin{cases} 0 & \text{if } i = 0 \\
\frac{q^{-i} - q^i}{q - q^{-1}} m_{i+1} & \text{if } i > 0
\end{cases}
\end{align*}
\]

Assume from now on: \( q \) is not a root of unity \( \Rightarrow [\lambda] + 0 \forall \lambda \geq 1 \)

\[ M(\lambda) = \bigoplus_{i=0}^{\lambda} M(\lambda)_{q^{-i}} \lambda \quad \text{where } M(\lambda)_{q^{-i}} \lambda = \text{span} \{ m_i : \lambda = \text{weight} \}
\]
Q: When does $M(q)$ contain a non-trivial submodule?

Note: $(E_{i}=0 \Rightarrow x_{i}^{-1} - \frac{1}{q} y_{i}^{-1} = 0) \Rightarrow x=q^{-i/4} \Rightarrow x=q^{i/4}$

\[ \begin{cases} 
\text{If } x=q^{n} \text{ for all integers } n \geq 0, \text{ then } M(q) \text{ is simple} \\
\text{If } x=q^{n} \text{ for some } n>0, \text{ then } M(q) \text{ has a unique non-trivial submodule. It is spanned by the } \nu^{i} \text{ for } i \equiv n \pmod{2} \text{ and zero. So } M(q^{-2^{n+1}q})
\end{cases} \]

For $n>0$ define $L(n)=\frac{M(q^{n})}{<m_{i}|i>n}$

$L(n)$ is a simple $U_{q}(sl_{1})$ module of dim $n!$ and basis given by the images of $\sum_{\lambda_{i}=0}^{n} m_{i}$ in the quotient $\nu^{n}$ in which the action looks like:

\[
\begin{align*}
K_{m_{i}} &= q^{-n} m_{i} \\
E_{m_{i}} &= (n-i+1) m_{i+1} \\
F_{m_{i}} &= (i+1) m_{i+1}
\end{align*}
\]

For $\text{char}(k)<2$, we have $L(n) \cong L(n+1)$

But they are not isomorphic as:

- $m_{i}$ is the $i$-th power of $m_{1}$
- $(\pm q^{i} n)$ is the eigenvalue of $K_{m_{i}}$

End: 1) Define dim simple $U_{q}(sl_{1})$ module $Z_{q}$

\[
\begin{align*}
1:1 \quad &Z_{q} \xrightarrow{L(n)} \{L(n+1) \} \quad n \geq 0 \\
L(n) &\xrightarrow{1:1} n \rightarrow 0
\end{align*}
\]

For $\text{char}(k)=2$, the additional assumption that the module is the dual space of its weight space is necessary.

1) Any finite dim $U_{q}(sl_{1})$ module is semisimple.

2) How to build a $U_{q}(sl_{1})$ module of type 1 if $k$ is odd?

Say that a $U_{q}(sl_{1})$-module $V$ is of type 1 if $V$ is a $U_{q}(sl_{1})$-module of type 1.

The set of all $U_{q}(sl_{1})$-modules of type 1 form a monoidal subcategory that looks a lot like the set of all $U_{q}(sl_{1})$-modules.

Tensoring with $L(n)$ gives $U_{q}(sl_{1})$-modules of type 1 and -1.

2) char $=p$ / odd of Weyl case?

Concentrate on $U_{q}(sl_{2})$ for now.

Assume $q$ is primitive with $n$ odd for $L$ odd (for simplicity, of course replace $L$ by $L^{*}$ odd where $L^{*}$ is dual).

Q: What changes?
\[ m(\lambda) = \bigoplus_{n \geq 0} K_{m+n} \text{ with } 0 \leq n \leq r \]

Thus exist f.d. \( U_q(\mathfrak{sl}_2) \)-modules that are not semisimple.

Consider certain baby Verma.

This case resembles a lot the representation theory of \( \mathfrak{sl}_2 \) over an algebraically closed field of characteristic 0 if \( r \) is prime.
What is the reason for this?

1) The center of \( U_q(\mathfrak{sl}_2) \) for \( q \) a primitive \( l \)-th root of unity

is a finitely gen. \( k \)-alg. and \( U_q(\mathfrak{sl}_2) \) is a finitely gen. module

over its center.

(\( \Rightarrow \) any irreducible rep of \( U_q(\mathfrak{sl}_2) \) is finite dim.)

Write \( U_q(\mathfrak{sl}_2) = \bigoplus_{i=1}^{\infty} \mathfrak{g}(U_q(\mathfrak{sl}_2)) \mathfrak{u}_i \).

Let \( V \) be a simple \( U_q(\mathfrak{sl}_2) \)-module.

For \( v \in \mathfrak{V} \), we have \( V = U_q(\mathfrak{g}) v = \bigoplus_{i=1}^{\infty} \mathfrak{g}(U_q(\mathfrak{sl}_2)) \mathfrak{u}_i v \).

\( V \) is finitely gen. over \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \).

Since \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \) is finitely gen. \( k \)-alg., \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \) is Noetherian.

and there exists a maximal \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \)-module \( V \leq V \).

\( V^i = \mathfrak{g}(U_q(\mathfrak{sl}_2)) \) is a \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \)-module for some \( i \).

Eventually, \( V / V^i = \mathfrak{g}(U_q(\mathfrak{sl}_2)) \).

\( m V \neq V \) is a \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \)-module.

\( m V = 0 \) if the action of \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \) on \( V \) factors over \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) / m = k \).

Prop: \( q \) is not a root of unity.

\( \Rightarrow \) \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \cong k[C] \) where \( C = \frac{K_2 + K^{-1} q^{-1}}{(q - q^{-1})^2} = EF \frac{K_2 + K^{-1} q^{-1}}{(q - q^{-1})^2} \).

11) \( q \) is a primitive \( l \)-th root of unity with \( l \) odd, \( l > 3 \).

\( \Rightarrow \) \( \mathfrak{g}(U_q(\mathfrak{sl}_2)) \) is gen. by \( \mathfrak{g}(E^0, F^0, K^0, K^{-1}) \) and \( C \) in the ring generated over this field.
The eigenvalue decompositions of $U_q(gl_n)$ under conjugation with $K$ is not fine enough!

Note $U_q(gl_n)$ is a $2$-graded algebra via $q$

$\deg(E_i) = 1$, $\deg(F_i) = -1$ and $\deg(k) = 0 = \deg(k^{-1})$

(Relations are homogeneous)

A monomial $P^q k^n E_i$ is homog. of degree $i - s$.

Check: $k^n k^{-i} = q^i n$ for $n \in U_q(gl_n)$ homogenous of degree $i$.

a) $q$ is not a root of unity $\Rightarrow$ eigenvalues of $v \mapsto k v k^{-1}$ are exactly the graded pieces.

b) $q$ is a root of unity $\Rightarrow$ the grading is finer than the eigenvalue decomposition.

Recall: Weight space decompose is the shining point of the proof of semisimplicity of a module.