Exercise 1. Let $Q$ be a quiver of Dynkin type. Given representations $M$ and $X_1, \ldots, X_r$ of $Q$ over $\mathbb{F}_q$, we define $P_{X_1, \ldots, X_r}^M$ to be the number of series $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ such that $M_{i-1}/M_i \simeq X_i$. The Hall algebra of $Q$ is defined as the free $\mathbb{Z}$-module with basis the set of isoclasses of modules, and product

$$[X] \ast [Y] = \sum P_{X,Y}^M[M].$$

Show that this product is associative and compute the Hall algebra of $\bullet$.

Exercise 2. Let $Q = \bullet \rightarrow \bullet$ be the Dynkin quiver of type $A_2$. Let $L$ and $M$ be the two simple modules of the path algebra.

(i) Give the structure of the indecomposable modules of the path algebra. Which ones are projective? Injective?


(iii) Show that the quantum Serre relations hold.

Exercise 3. Let $Q$ be a finite quiver with no oriented cycles. Let $i$ and $j$ be two vertices in $Q$ such that there is no arrow between $i$ and $j$, and $n - 1$ arrows from $j$ to $i$. We want to show the quantum Serre relation

$$\sum_{t=0}^n (-1)^t \begin{pmatrix} n \\ t \end{pmatrix} q^{t(t-1)/2} [L_i]^t \ast [L_j] \ast [L_i]^{n-t} = 0$$

in the Hall algebra of $\mathbb{F}_q Q$, where $L_i$ (resp. $L_j$) denotes the simple 1-dimensional representation associated to $i$ (resp. $j$).

(i) Let $V$ be an $n$-dimensional $\mathbb{F}_q$-vector space. For $d \leq n$, show that the number of flags $0 \subset V_1 \subset V_2 \subset \cdots \subset V_d \subset V$ such that $\dim V_i = i$ is equal to

$$[n]_q[n-1]_q \cdots [n-d+1]_q = [n]_q^d/[n-d]_q^d.$$ 

Show that it is also the number of flags $V \supset V_1 \supset V_2 \supset \cdots \supset V_d \supset 0$ such that $\text{codim} V_i = i$.

(ii) Let $R$ be the full subquiver of $Q$ with vertices $\{i,j\}$. Let $M$ be a representation of $R$ such that $\dim M_i = n$ and $\dim M_j = 1$. Show that there is an indecomposable representation $N$ of $R$ such that $M \simeq N \oplus L^d_i$. Compute the socle and the head of $N$.

(iii) Let $M$ be a representation of $Q$ such that the coefficient of $[M]$ in $[L_i]^t \ast [L_j] \ast [L_i]^{n-t}$ is non-zero. Show that $M_k = 0$ for $k \notin \{i,j\}$ so that $M$ can be viewed as a representation of $R$ of the form $N \oplus L^d_i$ with $N$ indecomposable. Show that $1 \leq d$ and $t \leq d$.

(iv) The coefficient of $[M]$ in $[L_i]^t \ast [L_j] \ast [L_i]^{n-t}$ counts the number of composition series $M = M_0 \supset M_1 \supset \cdots \supset M_{n+1} = 0$ such that $M_{s-1}/M_s \simeq L_i$ for $s \neq t+1$ and $M_t/M_{t+1} \simeq L_j$. Show that such composition series are characterized by the fact that $N \subset M_t$ and $M_{t+1} \subset (\text{rad } N) \oplus L^d_i$. 

\[ 1 \]
(v) Deduce from (i) and (iv) that the coefficient of \([M]\) in \([L_i]^{*t} * [L_j] * [L_i]^{*(n-t)}\) equals
\[
\frac{[d]_q! [n-t]_q!}{[d-t]_q!}
\]

(vi) Conclude.

**Exercise 4.** Let \(Q\) be the following quiver

![Quiver](image)

subject to the relations \(bc = ca = 0\).

(i) Give the structure of the projective modules, and write a projective resolution for each simple module.

(ii) We consider the two indecomposable Soergel \((\mathbb{C}[x], \mathbb{C}[y])\)-bimodules for \(S_2\):

\[
B_1 = \mathbb{C}[x, y]/(x - y) \quad \text{and} \quad B_s = \mathbb{C}[x, y]/(x^2 - y^2).
\]

Let \(a : B_1 \to B_s\) be the multiplication by \(x + y\), \(b : B_s \to B_1\) be the canonical quotient map, and \(c : B_s \to B_s\) be the multiplication by \(x - y\). Show that \(\text{End}(B_1 \oplus B_s)\) is isomorphic to the path algebra of \(Q\).

**Exercise 5.** Let \(V\) be a finite set and \(\mathcal{P}(V)\) be the set of subsets of \(V\). For \(\psi\) a function on \(\mathcal{P}(V)\) and \(S \subset V\) we define

\[
(E \cdot \psi)(S) = \sum_{s \in V \setminus S} \psi(S \cup \{s\})
\]

\[
(F \cdot \psi)(S) = \sum_{s \in S} \psi(S \setminus \{s\})
\]

Show that \(E\) and \(F\) define an action of \(\mathfrak{sl}_2(\mathbb{C})\) on the set of functions \(\mathcal{P}(V) \to \mathbb{C}\) and compute the weight spaces.

**Exercise 6.** Let \(V\) be a \(\mathbb{F}_{q^2}\)-vector space of dimension \(n\) and \(\text{Gr}(V)\) be the set of subspaces of \(V\). For \(\psi\) a function on \(\text{Gr}(V)\) and \(S \in \text{Gr}(V)\) we define

\[
(E \cdot \psi)(S) = q^{1 - \dim S} \sum_{S' \supset S \atop \dim S'/S = 1} \psi(S')
\]

\[
(F \cdot \psi)(S) = q^{1 + \dim S - n} \sum_{S' \subset S \atop \dim S/S' = 1} \psi(S')
\]

Show that \(E\) and \(F\) define an action of \(U_q(\mathfrak{sl}_2(\mathbb{C}))\) on the set of functions on \(\text{Gr}(V)\) and compute the weight spaces.