The affine nil-Hecke algebra. The affine nil-Hecke algebra \( H_n \) has a natural polynomial representation \( \mathbb{Z}[x_1, \ldots, x_n] \), where the \( x_i \) act by multiplication and the \( \tau_i \) act on \( k[x_1, \ldots, x_n] \) by Demazure operators,

\[
\tau_i : f \mapsto \frac{f - s_i(f)}{x_{i+1} - x_i}.
\]

Exercise 1. Let \( w_0 \) be the longest element in \( S_n \). Define \( b_n := \tau_{w_0} x_2 x_3^2 \cdots x_{n-1} \).

(1) For \( n = 2 \), show explicitly that \( b_2^2 = b_2 \) and \( 0H_n b_n 0H_n = 0H_n \).

(2) Using the fact that \( Q[x_1, x_2] \) is a free \( Q[x_1, x_2]^{S_2} = Q[x + y, xy] \)-module of rank 2, with basis \( \{1, y-x\} \), construct an explicit isomorphism

\[
0H_2 \sim \text{Mat}_2 \left( Q[x_1, x_2]^{S_2} \right).
\]

What is the image of \( b_n \) under this isomorphism?

(3) Conclude that \( 0H_2 \simeq (0H_2 b_2)^{\oplus 2} \) as a left \( 0H_2 \)-module.

Diagrammatics. Let \( \Gamma \) be the graph with vertex set \( \{i, j\} \) and a single edge between \( i \) and \( j \) (so that \( i \cdot j = -1 \)). We will consider the algebra \( R(\nu) \), where \( \nu = i + j \). As explained in example (7) of [1], the algebra \( R(\nu) \) has six generators

\[
\alpha_1, \ldots, \alpha_6
\]

which we call \( \alpha_1, \ldots, \alpha_6 \). Recall that \( R(\nu) \) is graded. As in [1], we set \( R'(\nu) = R(\nu)/\langle Z(R(\nu)) \rangle \).

Exercise 2. (1) Using the alternative description of \( R(\nu) \) given in example (7) of [1], show that

\[
R'(\nu) \simeq Q : e_1 \quad a \\
\quad \quad b \\
\quad \quad \quad e_2
\]

with \( ab = ba = 0 \),

where the isomorphism sends the generator \( \alpha_1 \) to \( e_1 \), \( \alpha_2 \) to \( e_2 \), \( \alpha_3 \) to \( a \), \( \alpha_4 \) to \( b \) and \( \alpha_5, \alpha_6 \) to zero.

(2) Conclude that \( R(\nu) \) has only two simple, graded modules (up to shifts).

(3) What are the dimensions of these simple modules?

(4) Describe how the generators \( \alpha_1, \ldots, \alpha_6 \) act on them.
Exercise 3. In this exercise we check that the quantum Serre relations hold in $K_0(R)$. Again, we assume that $Q$ has vertices $i, j$ with $i \cdot j = -1$. Let

\[
\begin{align*}
    b_{0,1} &:= \begin{array}{c}
    \bullet \\
    i & j & i
    \end{array} \\
    b_{0,2} &:= \begin{array}{c}
    \bullet \\
    i & j & i
    \end{array} \\
    b_{1,1} &:= - \begin{array}{c}
    \bullet \\
    i & i & j
    \end{array} \\
    b_{1,2} &:= \begin{array}{c}
    \bullet \\
    i & i & j
    \end{array}
\end{align*}
\]

and

\[
B_0 = \begin{pmatrix} b_{0,1} \\ b_{0,2} \end{pmatrix}, \quad B_1 = \begin{pmatrix} b_{1,1} & b_{1,2} \end{pmatrix}.
\]

We also fix the idempotents

\[
\begin{align*}
e_{i}(2) &:= \begin{array}{c}
\bullet \\
 i & i & j
\end{array} \\
e_{j}(2) &:= \begin{array}{c}
\bullet \\
 j & i & i
\end{array} \\
e_{ij} &:= \begin{array}{c}
\bullet \\
 i & j & i
\end{array}
\end{align*}
\]

and define the projective right $R(2i+j)$-modules

\[
i^{(2)}jP = e_{i}(2)R(2i+j), \quad j^{(2)}iP = e_{j}(2)R(2i+j), \quad ijiP = e_{iji}R(2i+j).
\]

Using the the diagram relations, show that

\[
B_0 : ijiP \sim \rightarrow i^{(2)}jP \oplus j^{(2)}iP, \quad B_1 : i^{(2)}jP \oplus j^{(2)}iP \sim \rightarrow ijiP.
\]

Canonical basis. The exercises of this section are based on [2 Section 3.2]; it’s a good idea to read that section before starting. We study the canonical basis for $U_q(sl_3)$. Recall that we have the quantum numbers

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q[n-1]_q \cdots [1]_q.
\]

The Cartan matrix in this case is

\[
A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.
\]

The quantum enveloping algebra $U_q(b_+)$ of the upper part $b_+$ of $sl_3$ is generated by $K_1^{\pm 1}, K_2^{\pm 1}$ and $E_1, E_2$, satisfying the relations

\[
\begin{align*}
    K_1K_2 &= K_2K_1, \\
    K_1E_1K_1^{-1} &= q^2E_1, \\
    K_1E_2K_1^{-1} &= q^{-1}E_2, \\
    K_2E_1K_2^{-1} &= q^{-1}E_1, \\
    K_2E_3K_2^{-1} &= q^2E_2.
\end{align*}
\]
and the quantum Serre relations
\[ E_1^{(2)} E_2 - E_1 E_2 E_1 + E_2 E_1^{(2)} = 0, \]
\[ E_2^{(2)} E_1 - E_2 E_1 E_2 + E_1 E_2^{(2)} = 0, \]
where \( E_i^{(n)} = \frac{1}{m_i^q} E_i^n \) is the divided power of \( E_i \). If \( A = \mathbb{Z}[q, q^{-1}] \), then the Lusztig \( \mathbb{Z} \)-form \( U_q^Z(\mathfrak{b}_+) \) is the \( A \)-subalgebra of \( U_q(\mathfrak{b}_+) \) generated by all divided powers. We define a \( Q = \mathbb{Z} \epsilon_1 \oplus \mathbb{Z} \epsilon_2 \) grading on \( U_q(\mathfrak{b}_+) \) by assigning \( \deg(K_i) = 0 \), \( \deg(E_i) = \epsilon_i \). There exists a non-degenerate, \emph{homogeneous} symmetric Hopf pairing on \( U_q(\mathfrak{b}_+) \) uniquely defined by
\[ (K_i, 1) = 1, \quad (K_i, K_j) = q^{\epsilon_i \epsilon_j}, \quad (E_i, E_j) = \frac{1}{1 - q^{-2}}. \]
In particular, by Hopf pairing we mean that
\[ (a \cdot b, c) = (a \otimes b, \Delta(c)), \quad \forall a, b, c \in U_q(\mathfrak{b}_+). \]
The restriction of \((-,-)\) to \( U_q^Z(n_+) \) is non-degenerate. We define a \( \mathbb{Z} \)-linear ring involution \( x \mapsto \bar{x} \) on \( A \) by \( q \mapsto q^{-1} \). This extends uniquely to a ring involution on \( U_q^Z(n_+) \) by setting \( E_i^{(n)} = E_i^{(n)} \) for all \( i, n \).

**Definition** An element \( b \in U_q^Z(n_+) \) is said to be \emph{canonical} if \( \bar{b} = b \) and \( (b, b) \in 1 + q^{-1} \mathbb{N}[q^{-1}] \).

Lusztig’s main result can be thought of as saying: up to a choice of signs, there is a unique \( A \)-basis \( B \) of \( U_q^Z(n_+) \) such that
\[ \{ b \in U_q^Z(n_+) \mid b \text{ canonical} \} = B \cup -B. \]

The weight spaces \( U_q^Z(n_+)^{\epsilon_1+\epsilon_2}, U_q^Z(n_+)^{2\epsilon_1+\epsilon_2} \) and \( U_q^Z(n_+)^{\epsilon_1+2\epsilon_2} \) have basis \( \{ E_1 E_2, E_2 E_1 \}, \{ E_1 E_2 E_1, E_1^{(2)} E_2 \} \) and \( \{ E_2 E_1 E_2, E_1 E_2^{(2)} \} \) respectively. Recall that the multiplication in the tensor product \( U_q^Z(b_+) \otimes U_q^Z(b_+) \) is twisted:
\[ (x \otimes y)(z \otimes w) = q^{(\deg(y), \deg(z))} x z \otimes y w. \]

**Exercise 4.** (1) Calculate \( (E_1^{(2)}, E_1^{(2)}) \). What is \( (E_1^{(2)}, E_1 E_2) \)?

(2) Show that
\[ (E_1, K_1^n K_2^m E_1) = \frac{q^{2n-m}}{1 - q^{-2}}, \quad (E_2, K_1^n K_2^m E_2) = \frac{q^{2m-n}}{1 - q^{-2}}. \]

(3) By induction on \( n \), show that
\[ (E_i^{(n)}, E_i^{(n)}) = \frac{1}{(1 - q^{-2})^n} \]
(first show that \( (E_i^n, E_i^n) = \frac{|n|^2}{1 - q^{-2}} (E_1^{n-1}, E_1^{n-1}) \)). Conclude that \( E_i^{(n)} \) is canonical.

**Exercise 5.** Using \([\boxed{1}]\), show that
\[ (E_1 E_2, E_1 E_2) = (E_2 E_1, E_2 E_1) = \frac{1}{(1 - q^{-2})^2}, \quad (E_1 E_2, E_2 E_1) = (E_2 E_1, E_1 E_2) = \frac{q^{-2}}{(1 - q^{-2})^2}. \]
Hence conclude that \( \{ E_1 E_2, E_2 E_1 \} \) is a canonical basis of \( U_q^Z(n_+)^{\epsilon_1+\epsilon_2} \).
The usual action of $\mathfrak{sl}_3$ on the vectorial representation $V = \mathbb{C}^3 = \mathbb{C}\{e_1, e_2, e_3\}$ extends to $U_q(\mathfrak{sl}_3)$, where

\[ K_1 = \begin{pmatrix} q & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^{-1} \end{pmatrix}, \]

\[ E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \]

We think of $V$ as a lowest weight representation of $U_q(\mathfrak{sl}_3)$ with fixed lowest weight $e_3$. As in the undeformed case, this extends to $V \otimes V$ which decomposes as $S^2(V) \oplus \wedge^2 V$. Using perverse sheaves on the appropriate representation space, we will see tomorrow that the following $\mathbb{Z}[q, q^{-1}]$-modules have canonical basis

\[ U^\mathbb{Z}(n_+)(x_1+\epsilon_2) : \left\{ E_1^{(2)}E_2, \quad E_2E_1^{(2)} \right\} \]

\[ U^\mathbb{Z}(n_+)(\epsilon_1+2\epsilon_2) : \left\{ E_1E_2^{(2)}, \quad E_2^{(2)}E_1 \right\} \]

\[ U^\mathbb{Z}(n_+)(2\epsilon_1+2\epsilon_2) : \left\{ E_1^{(2)}E_2^{(2)}, \quad E_1E_2^{(2)}E_1, \quad E_2^{(2)}E_1^{(2)} \right\} \]

**Exercise 6.** Calculate the canonical basis of the lowest weight modules $V$, $\wedge^2 V$ and $S^2(V)$.

**Quiver Hecke algebras.** In this exercise we consider the polynomial representation of a quiver Hecke algebra. Let $Q$ be a quiver with no loops and, for each $i, j$ in the vertex set $I$, let $h_{i,j}$ denote the number of arrows from $i$ to $j$. Define $q_{i,j}(u,v) = (v-u)^{h_{i,j}}(u-v)^{h_{j,i}}$ if $i \neq j$ and $q_{i,i}(u,v) = 0$. For a dimension vector $\alpha$, let $I_\alpha$ denote the set of all words $i = (i_1, \ldots, i_n)$ in the alphabet $I$ such that $\sum_j i_j \epsilon_j = \alpha$. If $\alpha = \alpha_1 \epsilon_1 + \cdots + \alpha_\ell \epsilon_\ell$, then define the height of $\alpha$ to be $n = \sum_j \alpha_j$. The quiver Hecke algebra $R(\alpha)$ is the unital algebra generated by $\{1_i \mid i \in I_\alpha\} \cup \{x_1, \ldots, x_n\} \cup \{\tau_1, \ldots, \tau_{n-1}\}$ with relations

- The $1_i$ are orthogonal idempotents, whose sum is the identity of $R(\alpha)$;
- $1_i x_j = x_j 1_i$, and $1_i \tau_j = \tau_j 1_{s_j(i)}$;
- The $x_i$ pairwise commute;
- $(\tau_k x_l - x_{s_k(i)} \tau_k)1_i = \delta_{i,s_k^{-1}(l)}(\delta_{k+1,l} - \delta_{k,l})1_i$;
- $\tau_k^2 1_i = q_{i,i+k+1}(x_k, x_{k+1})1_i$;
- $\tau_k \tau_l = \tau_l \tau_k$ if $|k-l| > 1$;
- $\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k)1_i = \delta_{i,s_k^{-1}(l)}(x_{k,k+1} - q_{i,k+1}x_{k+1} - x_{k+1}x_{k+2})1_i$.

The above relations imply that there is a canonical morphism from the commutative algebra $P_{\alpha} := \bigoplus_{i \in I_\alpha} k[x_1, \ldots, x_n]1_i = \bigoplus_{i \in I_\alpha} P_{\alpha,i}$ to $R(\alpha)$.

**Exercise 7.** Clearly $P_{\alpha}$ has a natural structure of left $P_{\alpha}$-module. Check that the rule $\tau_k 1_i : P_{\alpha,i} \rightarrow P_{\alpha,s_k(i)}$,

\[ \tau_k 1_i(f 1_i) = \begin{cases} \frac{f-s_k(f)}{x_{k+1}}1_i & \text{if } s_k(i) = i, \\ (x_{k+1} - x_k)^{h_{i,s_k^{-1}(i)}}s_k(f)1_{s_k^{-1}(i)} & \text{otherwise} \end{cases} \]

makes $P_{\alpha}$ into a left $R(\alpha)$-module.
It is known that this representation is faithful - see [1]. Let $S_i$ denote the stabilizer of $i$ in $S_n$ and set $z_k = \sum_{\sigma \in S_n/S_i} x_{\sigma(k)}1_{\sigma(i)}$, where $S_n/S_i$ denotes the minimal length coset representatives of $S_i$ in $S_n$. The PBW theorem implies that $k[z_1, \ldots, z_n]$ is a polynomial subalgebra of $R(\nu)$.

Exercise 8. The aim of this exercise is to show that $Z(R(\alpha))$, the centre of $R(\alpha)$, equals $k[z_1, \ldots, z_n]^{S_i}$.

Let $Q$ be the quiver with one vertex and no arrows, so that $R(n) = NH_n$ is the affine nil-Hecke algebra. It is known that $Z(NH_n) = k[x_1, \ldots, x_n]^{S_n}$.

1. For $Q$ as above, using the fact that the polynomial representation is faithful, show that $k[x_1, \ldots, x_n]^{S_n} \subseteq Z(NH_n)$.

2. Now, assume that $Q$ is arbitrary. Under the decomposition $R(\alpha) = \bigoplus_{i \in \mathcal{I}} iR(\alpha)i$, show that $Z(R(\alpha)) \subseteq \bigoplus_{i \in \mathcal{I}} Z(iR(\alpha)i)$.

3. Let $z_i = 1_i z_1 1_i \in Z(iR(\alpha)i)$. By first showing that $\tau_k z_i = 1_{s_k(i)} s_k(z_i)1_{s_k(i)} \tau_k$, prove that $\sum_{\sigma \in S_n/S_i} 1_{\sigma(i)} \sigma(z_i) 1_{\sigma(i)}$ belong to the centre of $R(\alpha)$.

4. Conversely, if $z \in Z(R(\alpha))$, with $z = \sum_{i \in \mathcal{I}} z_i$, show that $z_j = \sigma(z_i)$, where $\sigma \in S_n/S_i$ such that $j = \sigma(i)$. Hint: assume otherwise, and consider the difference $z - \sum_{\sigma \in S_n/S_i} 1_{\sigma(i)} \sigma(z_i) 1_{\sigma(i)}$.

References
