Exercise 1. Let \( j : \mathbb{C}^\times \longrightarrow \mathbb{C} \) be the open immersion and \( j_! \mathbb{C}^\times \) be the constant sheaf on \( \mathbb{C}^\times \). Compute the stalks of \( j_! \mathbb{C}^\times \) and \( j_\ast \mathbb{C}^\times \) at 0. Deduce that the direct image of a local system is not always a local system.

Exercise 2. Using the proper base change theorem, show that \( R\pi_!(k_X)_x \cong R\Gamma_c(\pi^{-1}(x), k) \). From the previous exercise, show that \( R\pi_*(k_X)_x \) is not isomorphic to \( R\Gamma(\pi^{-1}(x), k) \) in general.

Exercise 3. Consider the following cartesian square, with \( f \) proper:

\[
\begin{array}{ccc}
Z \times_Y X & \overset{g'}{\longrightarrow} & X \\
\downarrow f' & & \downarrow f \\
Z & \underset{g}{\longrightarrow} & Y
\end{array}
\]

Use the proper base change theorem to show that \( g_! \circ Rf_* \cong Rf'_* \circ (g')^! \).

Exercise 4. Let \( X = U \sqcup F \) be a decomposition with \( U \) open and \( F \) closed. Let \( j : U \longrightarrow X \) and \( i : Z \longrightarrow X \) be the open and closed embeddings. Given a sheaf \( F \), show that there is an exact sequence

\[
0 \longrightarrow j_! j^* F \longrightarrow F \longrightarrow i_! i^* F \longrightarrow 0.
\]

Show that there is a distinguished triangle in \( D^b_c(X) \)

\[
(*) \quad Rj_! j^* F \longrightarrow F \longrightarrow Ri_! i^* F \stackrel{\sim}{\longrightarrow}.
\]

Deduce that there is a long exact sequence

\[
\cdots \longrightarrow H^i_c(F, k) \longrightarrow H^i_c(U, k) \longrightarrow H^i_c(X, k) \longrightarrow H^{i+1}_c(F, k) \longrightarrow \cdots
\]

What triangle does one obtain by dualizing \((*)\)?

Exercise 5. Let \( X = \bigsqcup_{i=0}^n X_i \) be a decomposition into locally closed subvarieties such that:

- each \( X_i \) is isomorphic to an affine space
- the closure of \( X_i \) is a union of some \( X_j \)'s for \( j \geq i \).

Using the previous exercise, compute the cohomology of \( X \). As an example, compute the cohomology of \( \mathbb{P}_1, \mathbb{P}_n \) and \( G/B \).

Exercise 6. Let \( \mathcal{L} \) be a local system on \( \mathbb{C}^\times \). Compute the stalks of \( Rj_! \mathcal{L} \) and \( Rj_* \mathcal{L} \) in terms of the monodromy of \( \mathcal{L} \).

Exercise 7. Let \( G \) be a finite group acting freely on \( X \) and \( \pi : X \longrightarrow X/G \) be the canonical projection. Compute \( \pi_! \mathbb{C}_X \) (as an example, one can start with \( z \mapsto z^2 \) in \( \mathbb{C}^\times \)). What is the relation between \( R\Gamma(X/G, k) \) and \( R\Gamma(X, k)^G \) (one should work with \( k = \mathbb{C} \) here!).
Exercise 8. Let \( \pi : \overline{X} \to X \) be a surjective proper map. We say that \( \pi \) is \textit{semi-small} if \( \overline{X} \times_X \overline{X} \subset \overline{X} \times \overline{X} \) has dimension \( d_X = \dim_{\mathbb{C}} X \).

(i) If \( \pi \) is semi-small, prove the following inequality for all \( i \geq 0 \):
\[
\dim_{\mathbb{C}} \{ x \in X \mid \dim_{\mathbb{C}} \pi^{-1}(x) \geq i \} \leq d_X - 2i
\]

(ii) Assuming that \( X \) is smooth, show that \( R\pi_* \underline{C}_X[d_X] \) is perverse.

(iii) Can one give a condition to guarantee that \( R\pi_* \underline{C}_X[d_X] \) is an intersection cohomology complex?

Exercise 9. Ask Daniel Juteau for fun (yes!) exercises on

- \( t \)-structures:
- the Springer correspondence;
- computations with constructible complexes and the 6 operations.