SYMPLECTIC SINGULARITIES AND THEIRQUANTIZATIONS

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Abstract. These lecture notes are based on a mini-course given by the author at Università degli Studi di Padova. The aim is to explain the basic theory of symplectic singularities and their quantizations - this is an area of geometric representation theory commonly referred to as “symplectic representation theory”.

We start by recalling the definition of a symplectic singularity, and the notion of symplectic resolutions. In the second and third lectures, I will describe Namikawa’s results on the existence of a universal Poisson deformation of a symplectic singularity, and its implication for the existence of symplectic resolutions. Then, in the fourth lecture, we consider quantizations of symplectic singularities. This recovers many important algebras such as primitive quotients of enveloping algebras and symplectic reflection algebras. It allows us to use geometry to study their representation theory. Finally, we briefly discuss the basics of geometric category $\mathcal{O}$.

A heavy emphasis is placed throughout on examples to illustrate the applicability of the theory. In particular, we explain how the representation theory and geometry of semi-simple complex Lie algebras can be viewed as a particular instance of symplectic representation theory.

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Introduction

The purpose of these lectures is to introduce the reader to symplectic singularities, their quantizations and associated representation theory.

Originating in Sophus Lie’s study of continuous symmetries of solutions of differential equations, Lie theory is an important part of mathematics, providing a bridge between group theory, representation theory, geometry, integrable systems... In recent years it has become apparent that many of the key structures that underlie Lie theory exist in several other important examples occurring in representation theory. As the heart of all these examples is a conic symplectic singularity, which in the case of Lie theory corresponds to the nilpotent cone inside a semi-simple Lie algebra. The

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notion of symplectic singularities was introduced by Beauville in [21], and is a fruitful generalization of symplectic manifold to the category of singular algebraic varieties. Namikawa then focused on conic symplectic singularities [25] which are affine symplectic singularities with a contracting $\mathbb{C}^\times$-action compatible with the symplectic structure. This important property allows one to lift formal constructions (deformations or quantizations, for instance) to the global setting. In particular, conic symplectic singularities admit universal

- flat Poisson deformations
- filtered quantizations

which are, moreover, compatible with symplectic resolutions of singularities. It is these two key properties that allows one to generalize much of Lie theory to arbitrary conic symplectic singularities. This has lead to the development of the new field of “symplectic representation theory” (a phrase coined by Andrei Okounkov).

Of course, such a generalization is pointless unless new examples can be studied in this framework. Thankfully, there is no shortage of such examples. These include finite quotient singularities (which leads to the study of representations of symplectic reflection algebras, and in particular rational Cherednik algebras), quiver varieties and their quantizations, hypertoric varieties, and more recently Coulomb branches associated to reductive group representations.

The goal of these notes is to give a very brief overview of the basics of the theory. For a much more detailed account we recommend the reader consult [10].
1. Symplectic singularities and symplectic resolutions

Throughout, a variety will mean a reduced, separated scheme of finite type over \( \mathbb{C} \).

1.1. Poisson varieties. If \( A \) is a commutative \( \mathbb{C} \)-algebra, then we say that a bilinear form \( \{-,-\} : A \times A \to A \) makes \( A \) into a Poisson algebra if \( (A, \{-,-\}) \) is a Lie algebra and

\[
\{ab, c\} = \{a, c\}b + a\{b, c\}, \quad \forall \ a, b, c \in A.
\]

Note that the above derivation rule implies that the Poisson structure is uniquely defined by the bracket of generators of the algebras.

Example 1.1. If \( A = \mathbb{C}[u, v, w]/(uw + v^3) \), then there is a unique Poisson structure on \( A \) satisfying

\[
\{v, u\} = 3u, \quad \{v, w\} = -3w, \quad \{u, w\} = 6v^2.
\]

We say that an affine variety \( X \) is a Poisson variety if \( \mathbb{C}[X] \) is a Poisson algebra.

1.2. Symplectic manifolds. Dual to the notion of a Poisson variety is the notion of a symplectic manifold. Abusing terminology somewhat, for us a symplectic manifold will mean a smooth irreducible variety \( X \) with a globally defined algebraic closed non-degenerate 2-form \( \omega \) i.e.

\[
\omega_x : T_x X \times T_x X \to \mathbb{C}
\]

is a symplectic form on \( T_x X \) for every closed point \( x \in X \).

Example 1.2. If \( X = \mathbb{C}^{2n} \), then it has a standard symplectic structure given by

\[
\omega = dy_1 \wedge dx_1 + \cdots + dy_n \wedge dx_n.
\]

Example 1.3. If \( X \) is the hypersurface in \( \mathbb{C}^3 \) defined by the equation \( uw - v^2 = 1 \), then it is symplectic, with symplectic form \( \omega = w^{-1} dv \wedge dw \).

Though Poisson structure and symplectic structures are dual, in the sense that a Poisson structure is a choice of a global bivector, and a symplectic structure is a choice of a two form, the fact that the symplectic structure is required to be non-degenerate means that the symplectic structure is equivalent to a non-degenerate Poisson structure.

1.3. Symplectic singularities. We wish to consider symplectic structures in algebraic geometry. Therefore we have to deal with singular spaces. Here, the definition of symplectic structure does not make sense. A good generalization was proposed by Beauville [2].

Definition 1.4. A normal variety \( X \) is said to have symplectic singularities if

(a) there exists a symplectic form \( \omega \) defined on the smooth locus \( X_{\text{sm}} \) of \( X \); and

(b) if \( \rho : Y \to X \) is a resolution of singularities (i.e. a birational proper map from a smooth variety) then \( \rho^* \omega \) is a regular 2-form on \( Y \).

The following result, which is a special case of [2, Proposition 2.4], provides us with several immediate examples of symplectic singularities.
Proposition 1.5. If $V$ is a symplectic vector space and $G \subset \text{Sp}(V)$ is a finite group, then $V/G := \text{Spec} \mathbb{C}[V]^G$ has symplectic singularities.

Example 1.6. Take $\mathbb{C}^2$ with the standard symplectic structure. Then $\text{Sp}(\mathbb{C}^2) = \text{SL}(2, \mathbb{C})$ and hence each finite group $\Gamma \subset \text{SL}(2, \mathbb{C})$ gives a symplectic singularity $\mathbb{C}^2/\Gamma$. These groups are classified, up to conjugation, by the simply laced Dynkin diagrams.

<table>
<thead>
<tr>
<th>Diagram</th>
<th>Group</th>
<th>Singularity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\mathbb{Z}_{n+1}$</td>
<td>$xy = z^{n+1}; n \geq 1$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\mathbb{B}D_{n-2}$</td>
<td>$x^2 + y^2 + z^{n-1} = 0; n \geq 3$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$T$</td>
<td>$x^2 + y^3 + z^4 = 0$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$O$</td>
<td>$x^2 + y^3 + yz^3 = 0$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$I$</td>
<td>$x^2 + y^3 + z^5 = 0$</td>
</tr>
</tbody>
</table>

where $\mathbb{B}D_n$ is the binary dihedral group of order $4n$, $T$ the binary tetrahedral group of order 24, $O$ the binary octahedral group of order 48 and $I$ the binary icosahedral group of order 120.

Example 1.7. The other main source of examples of symplectic singularities are the coadjoint nilpotent orbit closures. Let $G$ be a simple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra and $N \subset \mathfrak{g}^*$ the nilpotent cone. Take any coadjoint orbit $O \subset \mathfrak{g}^*$. It has a canonical symplectic structure, the Kirillov-Kostant-Souriau form, given by

$$\omega(x, y) = -\epsilon([x, y]), \quad \forall \ x, y \in T_eO = \mathfrak{g}/\mathfrak{g}_e,$$  

where $\mathfrak{g}_e = \{ x \in \mathfrak{g} | x \cdot e := \epsilon([x, -]) = 0 \}$ is the centralizer of $e \in O$ in $\mathfrak{g}$. In what follows, it is easier to work with adjoint orbits. Therefore, using the Killing form $\kappa$ to make a $G$-equivariant identification $\mathfrak{g}^* \simeq \mathfrak{g}$, we consider $O \subset \mathfrak{g}$. Then the symplectic form (1) becomes

$$\omega(x, y) = -\kappa(e, [x, y]), \quad \forall \ x, y \in T_eO = \mathfrak{g}/\mathfrak{g}_e,$$  

It was shown by Panyushev [27] that:

Proposition 1.8. The normalization $\tilde{O}$ of the closure of a coadjoint orbit $O$ has symplectic singularities.

Proof. Let $e \in O$. Then by the Jacobson-Morozov Theorem, we can extend it to an $\mathfrak{sl}_2$-triple $\{e, f, h\}$. The semi-simple element $h$ defines a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ on $\mathfrak{g}$ with $\mathfrak{g}_i = \{ x \in \mathfrak{g} | [h, x] = ix \}$. Let $\mathfrak{n}_2 = \bigoplus_{i \geq 2} \mathfrak{g}_i$, and $P \subset G$ a connected subgroup with Lie algebra $\bigoplus_{i \geq 0} \mathfrak{g}_i$. The element $e$ belongs to $\mathfrak{n}_2$ and the $P$-orbit $P \cdot e$ is dense in $\mathfrak{n}_2$ by [11, Proposition 5.7.3]. Let $Y = G \times_P \mathfrak{n}_2$. Since $P$ is parabolic, the map $\tau : G \times_P \mathfrak{n}_2 \to \mathfrak{g}$ given by $(g, x) \mapsto g(x)$ is proper. Since it contains $O$, it follows that it contains $\overline{O}$. On the other hand, the fact that the orbit $P \cdot e$ is dense in $\mathfrak{n}_2$ implies that the image of $\tau$ is contained in $\overline{O}$. The variety $G \times_P \mathfrak{n}_2$ is smooth, therefore the map $\tau$ factors through a unique map $\rho : G \times_P \mathfrak{n}_2 \to \overline{O}$. As noted in [15, Proposition 4], the map $\pi$ is a projective resolution of singularities of $\overline{O}$; the non-trivial part of this is the statement that $\pi$ is an isomorphism over $\overline{O}$.
Let \((g, x) \in G \times \rho \mathfrak{n}_2\). Then \(T_{(g, x)}Y = (\mathfrak{g} \oplus \mathfrak{n}_2)/p\), where \(p \to \mathfrak{g} \oplus \mathfrak{n}_2\) is given by \(z \mapsto (-z, [x, z])\).

We define a 2-form \(\omega'\) on \(Y\) by

\[
\omega'((A_1, y_1), (A_2, y_2)) = \kappa(y_2, A_1) - \kappa(y_1, A_2) + \kappa(x, [A_1, A_2]) \quad \forall (A, y), (B, y_2) \in T_{(g, x)}Y.
\]

Note that if \(x = p \cdot e \in O\) then the fact that \(P \cdot e \subset \mathfrak{n}_2\) is dense implies that each \(y \in \mathfrak{n}_2\) can be written as \([x, z]\) for some \(z \in \mathfrak{p}\). If \((g, x) = (1, p \cdot e)\) then

\[
\omega(d\pi(A, x), d\pi(B, y)) = \omega(g \cdot y_1 + g \cdot [A_1, x], g \cdot y_2 + g \cdot [A_2, x])
\]

since \(d_{(g, x)}\pi(A, y) = g \cdot y + g \cdot [A, x]\). In the last but one line we have used the fact that \(\mathfrak{n}_2\) is perpendicular to \([\mathfrak{p}, \mathfrak{n}_2]\) under \(\kappa\). Be careful! In the fourth line, we have used \([\mathfrak{g}, \mathfrak{g}] = 0\), but you have to notice that one realizes \(T_xO = \mathfrak{g}/\mathfrak{g}_x\) as a subspace of \(T_x\mathfrak{g} = \mathfrak{g}\) via \(\pi \mapsto [x, z]\).

This implies that \(\omega'|_{x^{-1}(O)} = \pi^*\omega\). Since \(\omega'\) is regular, we deduce that \(O\) has symplectic singularities. \(\square\)

In particular the nilpotent cone \(\mathcal{N}\) has symplectic singularities.

Remark 1.9. If \(X\) is an affine variety with symplectic singularities then \(\mathbb{C}[X]\) is Poisson algebra.

1.4. **Symplectic resolutions.** If \(\rho : Y \to X\) is a resolution of singularities then the regular 2-form \(\rho^*\omega\) need not be non-degenerate in general.

**Definition 1.10.** A resolution of singularities \(\rho : Y \to X\) of a symplectic singularity \(X\) is said to be a **symplectic resolution** if \(\rho^*\omega\) is a non-degenerate 2-form on \(Y\).

Since \(\rho^*\omega\) is automatically closed, this implies in particular that \((Y, \rho^*\omega)\) is a symplectic manifold.

**Remark 1.11.** There are many examples of symplectic singularities, but symplectic resolutions are rare. It is an interesting problem to try and classify when symplectic singularities admit symplectic resolutions.

**Remark 1.12.** If \(K_X\) denotes the canonical divisor on \(X\), then recall that a resolution of singularities is **crepant** if \(\rho^*K_X = K_Y\). A symplectic resolution is the same as a crepant resolution.

**Proof.** First, note that if \(\omega\) is the symplectic form on \(X_{\text{sm}}\) and \(\dim X = 2n\) then \(\wedge^n\omega\) is a regular, nowhere vanishing section of \(K_X\). In particular, it trivializes \(K_X\).

If the resolution is symplectic then \(\rho^*(\wedge^n\omega) = \wedge^n(\rho^*\omega)\) is regular and nowhere vanishing hence \(K_Y = 0\) too and \(\rho\) is crepant.

Conversely, assume that \(\rho\) is crepant. Then \(K_Y = \rho^*K_X = 0\). The form \(s = \rho^*(\wedge^n\omega)\) trivializes \(K_Y\) since \(\wedge^n\omega\) trivializes \(K_X\). In particular, it has no zeros. Thus, \(\rho^*\omega\) is non-degenerate. \(\square\)
We recall that a morphism \( \rho \) is *semi-small* if for every closed subvariety \( Z \subset Y \),
\[
2\text{codim}_Y Z \geq \text{codim}_X \rho(Z).
\]
This is a very strong property of the morphism \( \rho \). A key property of symplectic resolutions is that:

**Theorem 1.13** (Kaledin). *Symplectic resolutions are semi-small.*

The example illustrated in figure (2.2) is a symplectic resolution.

1.5. **Examples.** We give some examples of symplectic resolutions.

**Example 1.14.** If \( \Gamma \subset \text{SL}(2, \mathbb{C}) \) is any finite group and \( \rho : Y \to X = \mathbb{C}^2/\Gamma \) is the minimal resolution then it is a symplectic resolution. This follows from the fact that the minimal resolution is crepant i.e. \( \rho^* K_X = K_Y \). See [28]. In practice, the most simple way to construct this resolution is to iteratively blow-up
\[
X_i \to X_{i-1} \to \cdots \to X_1 \to X = \mathbb{C}^2/\Gamma
\]
the reduced singular locus. In this way one sees that the singularities of \( X_i \) are still Kleinian (but the rank drops each time) and the pull-back \( \rho_i^* \omega \) of the symplectic form on \( X_{\text{sm}} \) extends to a symplectic form on the smooth locus of \( X_i \). We illustrate this by considering the \( E_7 \)-singularity \( x^2 + y^3 + yz^3 = 0 \). First, we note that if \( X = (f = 0) \subset \mathbb{C}^3 \) has isolated singularities then
\[
\omega = \frac{dx \wedge dy}{(\partial f/\partial z)} - \frac{dx \wedge dz}{(\partial f/\partial y)} = \frac{dy \wedge dz}{(\partial f/\partial x)}
\]
is regular and symplectic on \( X_{\text{sm}} \).

Blowing up the reduced singular locus of the \( E_7 \)-singularity \( x^2 + y^3 + yz^3 = 0 \) produces a quasi-projective variety \( Y \) with three affine open charts \( U_0, U_1 \) and \( U_2 \) with
\[
\mathbb{C}[U_0] = \mathbb{C}[u_1, u_2, u_3]/(f_1 = 1 + u_1 u_2^3 + u_1^2 u_2 u_3^3),
\]
\[
\mathbb{C}[U_1] = \mathbb{C}[v_1, v_2, v_3]/(f_2 = v_1^2 + v_2 + v_2^3 v_3^3),
\]
\[
\mathbb{C}[U_2] = \mathbb{C}[x_1, x_2, x_3]/(f_3 = x_1^2 + x_2 x_3^3 + x_2^2 x_3).
\]
The first two charts are smooth and \( U_3 \) is a \( D_6 \)-Kleinian singularity. Pulling back the form \( \omega \) gives
\[
U_0: \quad \rho^* \omega = \frac{1}{3u_1^2 u_2 u_3^3} du_1 \wedge du_2 = \frac{du_1 \wedge du_2}{(\partial f_1/\partial u_3)},
\]
\[
U_1: \quad \rho^* \omega = \frac{1}{2v_1} dv_2 \wedge dv_3 = \frac{dv_2 \wedge dv_3}{(\partial f_2/\partial v_1)},
\]
\[
U_2: \quad \rho^* \omega = \frac{1}{2x_1} dx_2 \wedge dx_3 = \frac{dx_2 \wedge dx_3}{(\partial f_3/\partial x_1)}.
\]

**Proof.** The blowup of \( X = \text{Spec} \mathbb{C}[x, y, z]/(x^2 + y^3 + yz^3) \) at the singular point is the irreducible component \( Y \) of
\[
\{(x, y, z, [u : v : w]) \mid x^2 + y^3 + yz^3 = 0, uy = xv, uz = wx, vz = wy\} \subset \mathbb{A}^3 \times \mathbb{P}^2
\]
mapping birationally onto \( X \). When \( u \neq 0 \), this gives
\[
y = x \frac{v}{u}, \quad z = x \frac{w}{u}, \quad 1 + x \left( \frac{v}{u} \right)^3 + x^2 \left( \frac{v}{u} \right) \left( \frac{w}{u} \right)^3 = 0.
\]
We take \( x = u_1, \frac{w}{u} = u_2 \) and \( \frac{w}{w} = u_3 \). Then the hypersurface \( 1 + u_1 u_2^2 + u_1^2 u_2 u_3^3 = 0 \) is smooth. When \( v \neq 0 \), this gives
\[
x = y \frac{u}{v}, \quad z = y \frac{w}{v}, \quad \left( \frac{u}{v} \right)^2 + y + y^2 \left( \frac{w}{v} \right)^3 = 0.
\]
We take \( v_1 = \frac{u}{v}, v_2 = y \) and \( v_3 = \frac{w}{v} \). Then the hypersurface \( v_1^2 + v_2 + v_3^2 = 0 \) is smooth. When \( w \neq 0 \), this gives
\[
x = z \frac{u}{w}, \quad y = z \frac{v}{w}, \quad \left( \frac{u}{w} \right)^2 + z \left( \frac{v}{w} \right)^3 + z^2 \left( \frac{v}{w} \right)^3 = 0.
\]
We take \( x_1 = \frac{u}{w}, x_2 = z \) and \( x_3 = \frac{v}{w} \). Then the hypersurface \( x_1^2 + x_2 x_3^2 + x_2^2 x_3 = 0 \) has an isolated singularity at 0. It follows from Proposition 2 of [29, §8.3] that \( x_1^2 + x_2 x_3^2 + x_2^2 x_3 = 0 \) is isomorphic to the Kleinian singularity of type \( D_6 \).

On \( U_0 \), the map \( \rho \) is given by \( x = u_1, y = u_1 u_2 \) and \( z = u_1 u_3 \). Then,
\[
\rho^* \left( \frac{1}{3}y^2 dx \wedge dy \right) = \frac{1}{3u_1^2 u_2^2 u_3} du_1 \wedge d(u_1 u_2) = \frac{1}{3u_1^2 u_2^2 u_3} du_1 \wedge du_2 = \frac{du_1 \wedge du_2}{(\partial f_1/\partial u_3)}.
\]
On \( U_1 \), the map \( \rho \) is given by \( x = v_1 v_2, y = v_2 \) and \( z = v_2 v_3 \). Then,
\[
\rho^* \left( \frac{1}{2} dy \wedge dz \right) = \frac{1}{2v_1 v_2} dv_2 \wedge d(v_2 v_3) = \frac{1}{2v_1} dv_2 \wedge dv_3 = \frac{dv_2 \wedge dv_3}{(\partial f_2/\partial v_1)}.
\]
On \( U_2 \), the map \( \rho \) is given by \( x = x_1 x_2, y = x_2 x_3 \) and \( z = x_2 \). Then,
\[
\rho^* \left( \frac{1}{2} dy \wedge dz \right) = \frac{1}{2x_1 x_2} dx_2 \wedge dx_3 = -\frac{1}{2x_1} dx_2 \wedge dx_3.
\]

\[ \square \]

Example 1.15. If we take \( G \) a simple algebraic group with Lie algebra \( \mathfrak{g} \) and \( B_+ \subset G \) a fixed Borel subgroup with Lie algebra \( \mathfrak{b}_+ \) then the Springer resolution is the map
\[
\pi : T^* G / B_+ \simeq \{ (b, x) \in G / B_+ \times \mathfrak{g}_+ \mid x \in \mathfrak{b}_+ \} \simeq G \times_{B_+} \mathfrak{b}_+ \to \mathcal{N},
\]
given by projection onto \( \mathfrak{g}_+ \). Here \( G / B_+ \) is identified with the flag manifold of all Borel subalgebras of \( \mathfrak{g} \). This is a symplectic resolution of the nilpotent cone. This follows from the proof of Proposition 1.8 since the extension of \( \pi^* \omega \) defined there is non-degenerate in this particular case.

Example 1.16. The Hilbert-Chow morphism. If we take the symmetric group \( \Sigma_n \) acting by permutation of coordinates on \( \mathbb{C}^n \) then the induced action on \( T^* \mathbb{C}^n = \mathbb{C}^{2n} \) is symplectic and the quotient \( \mathbb{C}^{2n} / \Sigma_n \) is a symplectic singularity by Beauville’s result. A well-known resolution of singularities of \( \mathbb{C}^{2n} / \Sigma_n \) is given by the Hilbert-Chow morphism
\[
\pi : \text{Hilb}^n \mathbb{C}^2 \to \mathbb{C}^{2n} / \Sigma_n
\]
where \( \text{Hilb}^n \mathbb{C}^2 \) is the Hilbert scheme of \( n \) points in the plane. In fact this turns out to be the unique resolution of singularities, as shown by Beauville [1]. See [23] for details.

Example 1.17. A large class of examples are given by hypertoric varieties. Let \( G = (\mathbb{C}^*)^k \) be a torus, acting on a representation \( \mathbb{C}^n \). This action is encoded by the matrix of weights \( A = (a_{i,j}) \)
where \((t_1, \ldots, t_k) \in G\) acts on \(x_j\) by
\[
(t_1, \ldots, t_k) \cdot x_j = t_1^{a_1,j} \cdots t_k^{a_k,j} x_j.
\]
Then the action of \(G\) on \(T^* \mathbb{C}^n\) is Hamiltonian with moment map \(\mu : T^* \mathbb{C}^n \to \mathbb{C}^k\) given by
\[
\mu(x_1, \ldots, x_n, y_1, \ldots, y_n) = \left( \sum_{i=1}^{n} a_{i,j} x_i y_i \right)^{k}_{j=1}.
\]
The affine hypertoric variety is defined to be \(\mu^{-1}(0)/G\). This is a symplectic singularity. When \(A\) is unimodular i.e. all \(k \times k\)-minors are in \(-1, 0, 1\) then variation of GIT gives a symplectic resolution.

Example 1.18. Quiver varieties provide a large, and extremely interesting, source of symplectic singularities. See [22] and [13] for the definition. The fact that they have symplectic singularities is shown in [7].

As we will see in the next section, all the examples listed above are conic symplectic singularities. There are many other examples of symplectic singularities that are not conic. For instance, the moduli spaces of sheaves on K3 surfaces have symplectic singularities [16], as do the character varieties of closed Riemann surfaces [7].

1.6. Conic symplectic singularities. Assume now that \(X\) is an affine symplectic singularity. Then an action of \(\mathbb{C}^\times\) on \(X\) is the same as a \(\mathbb{Z}\)-grading on the coordinate ring \(\mathbb{C}[X]\).

Definition 1.19. The variety \(X\) is said to be a conic symplectic singularity if
\begin{enumerate}
\item \(\mathbb{C}[X] = \bigoplus_{n \geq 0} \mathbb{C}[X]_n\) with \(\mathbb{C}[X]_0 = \mathbb{C}\); and
\item the Poisson bracket is homogeneous of degree \(-\ell\), for some \(\ell > 0\).
\end{enumerate}

Geometrically, part (a) of Definition 1.19 means that there is a cone point \(o \in X\) such that \(\lim_{t \to \infty} t \cdot x = o\) for all \(x \in X\). Part (b) of Definition 1.19 means that if \(a \in \mathbb{C}[X]_n\) and \(b \in \mathbb{C}[X]_m\) then \(\{a, b\} \in \mathbb{C}[X]_{n+m-\ell}\) is homogeneous too.

As noted above, nilpotent orbit closures, quotient singularities, hypertoric varieties and quiver varieties are all conic symplectic singularities. We note, as shown in [14, Lemma 5.3]:

Proposition 1.20. If \(X\) is a conic symplectic singularity and \(\rho : Y \to X\) a symplectic resolution then the \(\mathbb{C}^\times\)-action lift to an action of \(\mathbb{C}^\times\) on \(Y\) making \(\rho\) equivariant.

1.7. \(\mathbb{Q}\)-factorial terminalizations. As explained earlier, in general symplectic resolutions need not exist. Therefore one can ask if there is a “next best thing”. Either one could ask for a resolution that is as close to symplectic as possible, or one could ask for a partial resolution that is still crepant but as smooth as possible. It seems that the latter is the best approach.

A normal variety is said to be locally factorial if every Weil divisor is Cartier. Equivalently, if every local ring \(O_{Y,y}\) is a unique factorization domain, for \(y \in Y\) a closed point. We say that \(Y\) is \(\mathbb{Q}\)-factorial if for each Weil divisor \(D\) there exists some \(n > 0\) such that \(nD\) is Cartier. Equivalently, if the divisor class group \(\text{Cl}(O_{Y,y})\) of every local ring, with \(y \in Y\) closed, is torsion.
The variety $Y$ is said to have *terminal singularities* or is terminal, if for each resolution of singularities, $f : Z \to Y$,

$$K_X = f^*K_Y + \sum_i a_i E_i$$

with all $a_i > 0$. Here the sum is over all irreducible exceptional divisors $E_i$ of $f$.

The minimal model programme \cite{9} Corollary 1.4.3 implies that every symplectic singularity admits a $\mathbb{Q}$-factorial terminalization i.e. a projective birational map $\rho : Y \to X$ such that

(a) $Y$ is $\mathbb{Q}$-factorial and terminal; and

(b) $\rho^*K_X = K_Y$.

**Lemma 1.21.** If $\rho : Y \to X$ is a $\mathbb{Q}$-factorial terminalization then $Y$ has symplectic singularities and the morphism $\rho$ is Poisson.

**Proof.** First, we claim that $\rho^*\omega$ defines a symplectic form on the smooth locus of $Y$. Since $X$ is symplectic, $\rho^*\omega$ defines a regular closed 2-form on $Y_{\text{sm}}$. Its locus where it is degenerate is precisely the zero locus of $\wedge^n\omega$ (where $2n = \dim X$). But since $\rho^*K_X = K_Y = 0$, this section is non-where vanishing. Thus, $\omega' = \rho^*\omega$ is symplectic on $Y_{\text{sm}}$. Next, if $f : Z \to Y$ is a resolution of singularities, then $\rho \circ f$ is a resolution of singularities for $X$. Hence $f^*\omega' = (\rho \circ f)^*\omega$ is a regular form on $Z$. To see that $\rho^* : \rho^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ is Poisson, consider local sections $a, b$ of $\rho^{-1}\mathcal{O}_X$. Then we know that $\rho^*$ is Poisson when restricted to $\rho^{-1}(X_{\text{sm}})$ since it is an isomorphism on this locus. This means that $\{\rho^*a, \rho^*b\} - \rho^*\{a, b\}$ vanishes on the dense open set $\rho^{-1}(X_{\text{sm}})$. Hence it is zero. \qed

We have the following remarkable result by Namikawa \cite{26}:

**Theorem 1.22.** If $X$ is a conic symplectic singularity then $X$ admits only finitely many $\mathbb{Q}$-factorial terminalizations.

It is an interesting problem, in all of the examples of conic symplectic singularities mentioned above, to explicitly construct every $\mathbb{Q}$-factorial terminalization.

## 2. Deformations of symplectic singularities

Remarkably one can understand a great deal about the symplectic resolutions of a symplectic singularity by studying instead its Poisson deformations. This extremely fruitful idea proposed by Ginzburg-Kaledin and Namikawa.

### 2.1. Poisson deformations

Let $X$ be an affine Poisson variety and $S$ a (possibly formal) affine scheme over $\mathbb{C}$. Then a flat Poisson deformation of $X$ over $S$ is an affine $S$-scheme $\nu : \tilde{X} \to S$ such that:

(a) $\mathbb{C}[\tilde{X}]$ has a $\mathbb{C}[S]$-linear Poisson bracket; and

(b) there is an isomorphism of Poisson varieties $\nu^{-1}(0) \simeq X$, where $0 \in S$ is a distinguished closed point.

**Remark 2.1.** Property (a) implies that every fibre of $\nu$ is a Poisson subvariety of $\tilde{X}$. 

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It is natural to ask if one can classify the Poisson deformations of $X$? We note that if one drops the Poisson condition and only considers flat deformations of $X$ then when $\dim X > 2$, the space of infinitesimal deformations (those deformations where $S = \text{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$) is infinite dimensional and there are obstructions to lifting those infinitesimal deformations to formal deformations. However, the theory is very well-behaved for flat Poisson deformations.

**Theorem 2.2.** If $X$ is an affine symplectic singularity then there exists a smooth formal scheme $\tilde{T}_X$ and flat Poisson deformation $\tilde{\nu} : \tilde{X} \to \tilde{T}_X$ such that for any $\nu : \tilde{X} \to S$ a flat Poisson deformation with $S$ local Artinian, then there is a unique morphism $S \to \tilde{T}_X$ such that $\tilde{X} \simeq \tilde{X} \times_{\tilde{T}_X} S$.

Here, a local Artinian scheme just means Spec $A$ for some finite dimensional local $\mathbb{C}$-algebra $A$. Namikawa also showed that when $X$ is conic, one can lift this statement to a statement about graded Poisson deformations of $X$. We say that a flat Poisson deformation $\nu : \tilde{X} \to S$ is a graded flat Poisson deformation if, in addition:

(c) $\mathbb{C}$ acts on $S$ and $\tilde{X}$ making $\nu$ equivariant,

(d) the action on $S$ contracts all points to 0 i.e. $\lim_{t \to \infty} t \cdot s = 0$ for all $s \in S$; and

(d) the bracket on $\tilde{X}$ has degree $-\ell$.

**Theorem 2.3.** Let $X$ be a conic symplectic singularity. Then there exists a universal graded flat Poisson deformation $X \to T_X$, where $T_X \simeq \mathbb{A}^d$, for some $d$.

Just as in Theorem 2.2 this means that if $\nu : \tilde{X} \to S$ is a graded Poisson deformation then there exists a unique $\mathbb{C}^\times$-equivariant map $S \to T_X$, mapping $0 \in S$ to $0 \in T_X$ such that $\tilde{X} \simeq X \times_{T_X} S$.

This has the advantage in that $S$ (resp. $T_X$) need not be local Artinian (resp. formal). One can recover the formal scheme $\tilde{T}_X$ by completing $T_X$ at zero.

### 2.2. Deformations of the resolution

Assume now that we are given a symplectic resolution $\rho : Y \to X$ of our conic symplectic singularity. By Proposition 1.20 the $\mathbb{C}^\times$-action lifts to $Y$. The variety $Y$ also admits a universal graded flat Poisson deformation $Y \to T_Y$. Namikawa has also shown in [25, 24] that:

**Theorem 2.4.** There exists a finite group $W$, acting linearly on $T_Y$, and a morphism $\rho : Y \to X$ of Poisson varieties such that:

(a) The diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & T_Y \\
\rho \downarrow & & \downarrow q \\
X & \longrightarrow & T_X
\end{array}
$$

commutes and is $\mathbb{C}^\times$-equivariant.

(b) There is an isomorphism $T_Y/W \simeq T_X$ such that $q : T_Y \to T_X$ is the quotient map.

(c) $T_Y \simeq H^2(Y, \mathbb{C})$ and $W$ acts on $H^2(Y, \mathbb{Q})$ as a finite Coxeter group.

Since $W$ is a finite group acting faithfully, this means that $q$ is a finite map.
Figure 1. The resolution and the deformation of the $\mathbb{Z}_2$ quotient singularity.

**Example 2.5.** If we take $X = (xy - z^4) \subset \mathbb{C}^3$ to be the Kleinian singularity of type $A_3$, then

$$x = \text{Spec} \frac{\mathbb{C}[x, y, z, s_2, s_3, s_4]}{(xy - z^4 - s_2 z^2 - s_3 z - s_4)} \to S = \text{Spec} \mathbb{C}[s_2, s_3, s_4]$$

is the universal graded Poisson deformation. Here the grading is given by

$$\deg x = \deg y = 4, \quad \deg z = 2, \quad \deg s_i = 2i.$$ 

The group $W$ is the symmetric group $\mathfrak{S}_4$ acting on $h = \{(a_1, a_2, a_3, a_4) \in \mathbb{C}^4 | a_1 + a_2 + a_3 + a_4 = 0\}$ by permuting coordinates and $s_i$ is the $i$th symmetric polynomial in the $a_j$.

Replace $X$ by $X' = X \times_{TX} T_Y$. Then the above diagram becomes

$$\begin{array}{c}
\mathcal{Y} \\
\downarrow \rho \\
T_Y
\end{array} \xymatrix{ \mathcal{Y} \ar[r]^-{\rho} \ar[d]_{\mathcal{Y}_c} & X' \ar[d]_{X'_c} \\
T_Y & X'_c}
$$

and for each $c \in T_Y$, we can specialize $\rho$ to get a map $\rho_c : \mathcal{Y}_c \to X'_c$.

**Theorem 2.6.** There exist finitely many hyperplanes $\{H_i | i = 1, \ldots, k\}$ in $T_Y = H^2(Y, \mathbb{Q})$ such that the map $\rho_c : \mathcal{Y}_c \to X'_c$ is an isomorphism if and only if $c$ does not lie on any of the complexified hyperplanes $H_i \otimes_{\mathbb{Q}} \mathbb{C} \subset H^2(Y, \mathbb{C}) = T_Y$.

In particular, this implies that the fiber $\mathcal{Y}_c$ is actually affine for generic $c$. In fact, the complex hyperplanes $H_i \otimes_{\mathbb{Q}} \mathbb{C}$ are also characterized by the fact that $\mathcal{Y}_c$ is affine if and only if $c$ does not lie on any of these hyperplanes.

**Corollary 2.7.** The symplectic singularity $X$ admits a projective symplectic resolution $Y$ if and only if the generic fibre $X_c$ of its universal graded flat Poisson deformation is a smooth variety.
number of different symplectic resolutions admitted by $X$. Namely, this number equals

$$\frac{1}{|W|} \dim_{\mathbb{C}} H^*(U, \mathbb{C}).$$

(3)

Notice that $U$ is the complement to a hyperplane arrangement, therefore $H^*(U, \mathbb{C})$ is an Orlik-Solomon algebra. See [5] for details.

2.3. $\mathbb{Q}$-factorial terminalizations. The full statement of Corollary 2.7 does not directly follow from Theorem 2.6. For this, one requires a more precise version of Theorem 2.6 where one begins with an arbitrary $\mathbb{Q}$-factorial terminalization $\rho : Y \to X$. Given such a morphism, let $Y_{\text{sm}}$ denote the smooth locus of $Y$. Again, $Y$ admits a universal graded flat Poisson deformation $\mathcal{Y} \to T_Y$. Theorems 2.4 and 2.6 still hold in this generality, except that $T_Y = H^2(Y_{\text{sm}}, \mathbb{C})$. Moreover, we have the following key result:

**Theorem 2.8.** Forgetting the Poisson structure on $Y$, the deformation $\mathcal{Y} \to T_Y$ is locally trivial in the analytic topology.

More precisely, this means that each point in $Y \subset \mathcal{Y}$ has an analytic open neighbourhood $U \subset \mathcal{Y}$ isomorphic to $(U \cap Y) \times B$. In particular, the singularities of $Y$ do not change as one deforms along $T_Y$.

Let us now explain how the proof of Corollary 2.7 follows from Theorems 2.4 and 2.8. Assume first that $\rho : Y \to X$ is a projective symplectic resolution. In particular $Y$ is smooth. Theorem 2.8 implies that every fibre $\mathcal{Y}_c$ is smooth. By Theorem 2.6 we have an isomorphism $\mathcal{Y}_c \simeq X_c$ for generic $c \in T_Y$. Thus, $X'_c$ is smooth for generic $c$. Conversely, assume that $X'_c$ is never smooth. Fix a $\mathbb{Q}$-factorial terminalization $\rho : Y \to X$. Then, again by Theorem 2.6 we see that the generic fibre $\mathcal{Y}_c$ is also not smooth. By Theorem 2.8 this implies that $Y$ itself is also not smooth and hence cannot be a symplectic resolution. Notice that this argument shows in particular that if $X$ admits a projective symplectic resolution then every $\mathbb{Q}$-factorial terminalization of $X$ is a symplectic resolution.

In the next subsections, we will try to describe explicitly the above set-up in key examples.

2.4. The nilpotent cone. Let $G$ be a connected simple algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra and $\mathcal{N} \subset \mathfrak{g}^*$ the nilpotent cone. Recall that a projective symplectic resolution of $\mathcal{N}$ is given by the Springer resolution $\rho : T^* B \to \mathcal{N}$, where $B = G/B_+$ is the flag manifold. We can write

$$T^* B = \{(b, x) \mid b \in B, x \in (\mathfrak{g}/b)^*\} = G \times_{B_+} (\mathfrak{g}/b_+)^*,$$

with $T^* B \to \mathfrak{g}^*$ just projection onto $x$. Let

$$\tilde{\mathfrak{g}} = \{(b, x) \mid b \in B, x \in (\mathfrak{g}/[b, b])^*\} = G \times_{B_+} (\mathfrak{g}/[b_+, b_+])^*,$$
mapping to \( \mathfrak{g} \) by projection onto \( x \). This is called Grothendieck’s simultaneous resolution. We can form the commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{h}^* \\
\downarrow & & \downarrow \\
\mathfrak{g}^* & \longrightarrow & \mathfrak{h}^*/W
\end{array}
\]

where \( \tilde{\mathfrak{g}} \rightarrow \mathfrak{h}^* \) is given by \( (b, x) \mapsto x|_{b/\mathfrak{b},b} \) using \( \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \simeq \mathfrak{h} \) the abstract Cartan. The map \( \mathfrak{g}^* \rightarrow \mathfrak{h}^*/W \) is the quotient map \( \mathfrak{g}^*/G \rightarrow \mathfrak{h}^*/G \simeq \mathfrak{h}^*/W \). Here \( \mathbb{C}^\times \) acts on \( \mathfrak{g}^* \) via the usual scaling action on \( \mathfrak{g} \) and on \( \tilde{\mathfrak{g}} \) by \( t \cdot (b, x) = (b, t \cdot x) \).

**Theorem 2.9.** Grothendieck’s simultaneous resolution is the universal graded flat Poisson deformation of \( T^*\mathcal{B} \) and \( \mathfrak{g}^* \rightarrow \mathfrak{h}^*/W \) is the universal graded flat Poisson deformation of \( \mathcal{N} \).

**Proof.** The first part of the statement follows from a result of Yamada [30] and the second is a special case of a general result of Lehn-Namikawa-Sorger [17, Theorem 1.2]. \( \square \)

**Remark 2.10.** The above result has been generalized to many other nilpotent orbit closures by Lehn-Namikawa-Sorger [17, Theorem 1.2].

In this example, the hyperplane arrangement of Theorem 2.6 are precisely the reflecting hyperplanes for the action of \( W \) on \( \mathfrak{h} \). It is known that \( \dim H^*(U, \mathbb{C}) = |W| \) in this case. Hence, we see that the Springer resolution is the unique projective symplectic resolution of the nilpotent cone \( \mathcal{N} \).

2.5. **Kleinian singularities.** Recall that Slodowy [29] showed that the Kleinian singularities can be realised as slices to the subregular orbit in the nilpotent cone. Namely, to each \( \Gamma \subset \text{Sl}(2, \mathbb{C}) \) we associate a simply laced Dynkin diagram. Let \( \mathfrak{g} \) be the corresponding simple Lie algebra. If \( e \in O_{\text{sub}} \) is in the subregular nilpotent orbit and \( S_e \subset \mathfrak{g}^* \) is the Slodowy slice then \( S_e \cap \mathfrak{N} \) is isomorphic to \( \mathbb{C}^2/\Gamma \).

**Proposition 2.11.** The universal graded flat Poisson deformation of \( \mathbb{C}^2/\Gamma \) is given by the composite map \( S_e \hookrightarrow \mathfrak{g}^* \rightarrow \mathfrak{h}^*/W \). Moreover, if \( \tilde{S}_e \) is the pre-image of \( S_e \) in \( \tilde{\mathfrak{g}} \) then \( \tilde{S}_e \rightarrow \mathfrak{h} \) is the universal graded flat Poisson deformation of the minimal resolution \( \tilde{S}_e \cap T^*\mathcal{B} \) of \( \mathbb{C}^2/\Gamma \).

**Proof.** It was conjectured by Grothendieck, and proven by Brieskorn that \( S_e \rightarrow \mathfrak{h}^*/W \) is the universal deformation of the Kleinian singularity. This is explained in [29]. But it is only very recently that a complete proof by Lehn-Namikawa-Sorger [17] of the above result has appeared. \( \square \)

Once again, in this example, the hyperplane arrangement of Theorem 2.6 are precisely the reflecting hyperplanes for the action of \( W \) on \( \mathfrak{h} \). Hence, we see that formula (3) says that the minimal resolution is the unique projective symplectic resolution, as we already know.

2.6. **Quotient singularities.** To any finite group \( G \subset \text{Sp}(V) \), Etingof and Ginzburg associated a family of non-commutative algebras \( H_{t,c}(G) \), called symplectic reflection algebras. Here \( t \in \{0, 1\} \) and \( c \in \mathfrak{c} \), a certain finite dimensional vector space. Let \( e = \frac{1}{|\mathfrak{c}|} \sum_{g \in G} g \) be the trivial idempotent; it is an element in \( H_{t,c}(G) \). At special parameter \( t = 0 \), the spherical subalgebra \( eH_{t,c}(G)e \) is commutative, and gives a flat graded Poisson deformation \( \tilde{X}(G) \rightarrow \mathfrak{c} \) of \( V^*/G \). The space \( \tilde{X}(G) \) is
called the *Calogero-Moser space* associated to $G$. This is almost the universal Poisson deformation of $V^*/G$. More specifically, if $W$ is the Namikawa Weyl group of $V^*/G$ and $Y \to V^*/G$ any $\mathbb{Q}$-factorial terminalization, then we show in [5] that:

**Theorem 2.12.** There is a $W$-equivariant isomorphism $c \simeq H^2(Y_{\text{sm}}, \mathbb{C})$ such that

$$\tilde{X}(G) \simeq \mathfrak{X} \times_{\mathcal{T}_X} c.$$

In the notation of section 2.2 this says that $\tilde{X}(G) = \mathfrak{X}'$. If we consider the special case of $S_n$ acting on $\mathbb{C}^{2n}$, as in example 1.16, then $c \simeq \mathbb{C}$ acted upon by $W = \mathfrak{S}_2$. Thus, $U = \mathbb{C}^\times$ and hence $\dim H^1(\mathbb{C}, \mathbb{C}) = 2$. Then equation (3) implies that the Hilbert-Chow morphism is the unique projective symplectic resolution of $\mathbb{C}^{2n}/\mathfrak{S}_n$.

**Example 2.13.** Let $\Gamma \subset SL(2, \mathbb{C})$ be a finite group. Then $\Gamma^n$ acts on $\mathbb{C}^{2n}$ in the obvious way. The symmetric group $S_n$ also acts by permuting pairs of coordinates. This gives an action of the wreath product $G := \mathfrak{S}_n \ltimes \Gamma^n \rtimes \mathfrak{S}_n$ on $\mathbb{C}^{2n}$, which preserves the natural symplectic structure. Let $\mathfrak{h}$ be the reflection representation of the Weyl group $W_{\Gamma}$ associated to $\Gamma$ via the McKay correspondence and $\Phi \subset \mathfrak{h}^*$ the root system. The symplectic singularity $\mathbb{C}^{2n}/G$ admits a projective symplectic resolution $Y$ and there is a natural isomorphism $H^2(Y, \mathbb{C}) = \mathfrak{h} \oplus \mathbb{C}$. In this case, the Namikawa Weyl group is $W = W_{\Gamma} \rtimes \mathfrak{S}_2$, where $W_{\Gamma}$ acts trivially on the second factor and $\mathfrak{S}_2$ acts only on the second factor. If $t$ is the coordinate defining the hyperplane $\mathfrak{h}$ in $\mathfrak{h} \oplus \mathbb{C}$, then the hyperplane arrangement is

$$H_\infty = \{t = 0\}, \quad H_{\alpha,m} = (\alpha + mt)^\perp, \quad \alpha \in \Phi, \quad -n < m < n.$$  

This is precisely the cone over the $n$-Catalan arrangement.

**3. Quantizations of symplectic singularities**

The fact that symplectic singularities have a natural Poisson structure suggests that they are shadows of a non-commutative algebra i.e. that they can be quantized. In this section, we make this precise by describing the filtered quantizations of symplectic singularities.

**3.1. Filtered quantizations.** Assume that $X$ is a conic symplectic singularity, where the bracket $\{\cdot, \cdot\}$ has degree $-\ell$.

**Definition 3.1.** A filtered quantization of $X$ is a $\mathbb{C}$-algebra $A$ equipped with a filtration $A = \bigcup_{i \in \mathbb{Z}} A_i$ such that:

(a) if $a \in A_i, b \in A_j$ then $[a, b] \in A_{i+j-\ell}$,

(b) there is an isomorphism of graded algebras $\text{gr} A \simeq \mathbb{C}[X]$; and

(c) the isomorphism $\text{gr} A \simeq \mathbb{C}[X]$ is Poisson, where

$$\{\bar{a}, \bar{b}\} := [a, b] \mod A_{i+j-\ell-1} \quad (4)$$

if $\bar{a} \in (\text{gr} A)_i$ and $\bar{b} \in (\text{gr} A)_j$ is the Poisson bracket on $\text{gr} A$.

In (a), the fact that $\ell > 0$ means that $\text{gr} A$ is a commutative algebra. In (b), the fact that $A$ is filtrated means that $\text{gr} A$ is graded with $(\text{gr} A)_i := A_i/A_{i-1}$. The fact that the Poisson structure on
X is non-zero (indeed, it is generically non-degenerate) implies that A must be a non-commutative algebra.

**Example 3.2.** Let $A = \mathbb{C}\langle x, \partial_x \rangle$ be the first Weyl algebra, where $\partial_x x - x \partial_x = 1$. Then A is a quantization of $\mathbb{C}^2$ with its standard symplectic structure. Here the filtration on $A$ is the Bernstein filtration, where $x$ and $\partial_x$ are in $A_1$.

**Example 3.3.** If $\Gamma \subset \text{SL}(2, \mathbb{C})$ is finite then $\Gamma$ acts on $A = \mathbb{C}\langle x, \partial_x \rangle$ by algebra automorphisms and $A^\Gamma$ is a quantization of $\mathbb{C}^2/\Gamma$.

**Example 3.4.** Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{m} \subset Z(U(\mathfrak{g}))$ a maximal ideal of the centre. Let $J$ be the primitive central ideal of $U(\mathfrak{g})$ generated by $\mathfrak{m}$. Then $U(\mathfrak{g})/J$ inherits a filtration from the standard PBW-filtration on $U(\mathfrak{g})$ such that $\text{gr}(U(\mathfrak{g})/J) \simeq \mathbb{C}[\mathcal{N}]$. Hence $U(\mathfrak{g})/J$ is a filtered quantization of the nilpotent cone $\mathcal{N}$.

### 3.2. Symplectic reflection algebras

Let $V$ be a symplectic vector space and $G \subset \text{Sp}(V)$ a finite group. Then $\text{Weyl}(V)^G$ is a quantization of $V^*/G$, where

$$\text{Weyl}(V) = TV/(v \otimes w - w \otimes v - \omega(v, w))$$

is the Weyl algebra for $V$. Here $TV = \mathbb{C} \oplus V \oplus V^2 \oplus V^3 \oplus \cdots$ is the tensor algebra on $V$. As noted in section 2.6, Etingof and Ginzburg introduced a family of algebras called the symplectic reflection algebras $H_{t, c}(G)$. At $t = 1$, the spherical subalgebras $eH_{1, c}(G)e$ provide a flat family of filtered quantizations of $V^*/G$. For more details, see [6].

### 3.3. Classifying quantizations

Remarkably, just as Namikawa has shown the existence of a universal graded flat Poisson deformation of $X$, Losev [20] has shown the existence of a universal filtered quantization of $X$. We fix a $\mathbb{Q}$-factorial terminalization $\rho : Y \to X$ of $X$. Since $X$ need not admit a projective symplectic resolution, we cannot assume that $Y$ is smooth. Let $Y_{\text{sm}}$ denote the smooth locus of $Y$. Recall that $T_Y = H^2(Y_{\text{sm}}, \mathbb{C})$ is the base of universal graded flat Poisson deformation of $Y$.

We will first state the intuitive version of Losev’s classification, and then give the more precise version.

**Theorem 3.5.** Let $X$ be a conic symplectic singularity.

(a) For each $\lambda \in T_Y$ there exists a filtered quantization $A_{\lambda}$ of $X$.

(b) If $A$ is a filtered quantization of $X$ then there exists some $\lambda \in T_Y$ and filtered isomorphism $A \simeq A_\lambda$.

(c) If $\lambda, \mu \in T_Y$ then $A_\lambda \simeq A_\mu$ as filtered algebras if and only if $\mu \in W\lambda$.

Thus, we see that the filtrated quantizations of $X$ are parametrized by $H^2(Y_{\text{sm}}, \mathbb{C})/W \simeq T_X$, just as for graded Poisson deformations.

In order to state the more general version of Losev’s result, we need to generalize Definition 3.1. Let $B = \bigcup_i B_i$ be a filtered commutative algebra and $A = \bigcup_i A_i$ a filtered $B$-algebra; this means that there is a unital embedding $B \to Z(A)$ such that $B_i = A_i \cap B$. Assume that:

---

*Notes*
(a') \([a, b] \in A_{i+j-\ell} \) if\( a \in A_i \) and \( b \in A_j \). Since \( \ell > 0 \), (a') implies that \( \mathbb{C}[[X]] := \text{gr } A \) is a commutative graded algebra over the graded algebra \( \mathbb{C}[S] := \text{gr } B \). Moreover, (4) implies that \( \mathbb{C}[[X]] \) has a \( \mathbb{C}[S] \)-linear Poisson bracket. Then we say that \( A \) is a filtered family of quantizations of \( X \) if, in addition to (a'), we have

(b') \( \nu : \tilde{X} \to S \) is a graded flat Poisson deformation of \( X \).

In particular, this implies that \( A \) is flat over \( B \). In [20, Proposition 3.5] it is shown that:

**Theorem 3.6.** Let \( X \) be a conic symplectic singularity. Then \( X \) admits a universal family \( A_X \) of filtered quantizations of \( X \) with base \( B_X := \mathbb{C}[H^2(Y_{\text{sm}}, \mathbb{C})]^W \).

This means that if the \( B \)-algebra \( A \) is a family of filtered quantizations of \( X \) then there is a unique filtered morphism \( B_X \to B \) such that

(i) \( A \simeq A_X \otimes_{B_X} B \) as filtered algebras; and

(ii) \( \text{gr } A \simeq \mathbb{C}[X] \otimes_{\mathbb{C}[H^2(Y_{\text{sm}}, \mathbb{C})]^W} \mathbb{C}[S] \) as graded Poisson deformations of \( X \).

Here (ii) is saying that the associated graded of the universal family of filtered quantizations is the universal graded Poisson deformation.

**Example 3.7.** In the case of the nilpotent cone \( \mathcal{N} \), we have \( A_X = U(\mathfrak{g}) \), considered as a \( Z(U(\mathfrak{g})) \simeq \mathbb{C}[[\mathfrak{h}^*]^W \)-algebra. Notice that the Harish-Chandra isomorphism \( Z(U(\mathfrak{g})) \simeq \mathbb{C}[[\mathfrak{h}^*]^W \) is as filtered algebras. The fact that \( A_X = U(\mathfrak{g}) \) can be deduced from Section 2 of [4], in particular the isomorphism (2.1), once one knows that all filtered quantizations of \( T^*B \) are given by twisted differential operators. The latter fact is explained in [10].

**Example 3.8.** In the case of \( X = V^*/G \), for \( G \subset \text{Sp}(V) \) a finite group, \( A_X \) is isomorphic to the universal spherical subalgebra \( eH_1(G)e \) of the symplectic reflection algebra at \( t = 1 \). This is shown in [20, Proposition 3.13].

### 3.4. Quantization of the resolution.

How does one go about constructing these quantizations? For simplicity, we assume that \( X \) admits a symplectic resolution \( \rho : Y \to X \); see remark 3.17 for the general situation. Since \( X \) is normal, \( H^0(Y, \mathcal{O}) = \mathbb{C}[X] \) and the Grauert-Riemenschneider vanishing theorem implies that \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \). Thus, the global sections of a quantization of \( Y \) give a quantization of \( X \).

**Example 3.9.** As a particular example, \( H^0(T^*B, \mathcal{O}) = \mathbb{C}[\mathcal{N}] \). The structure sheaf on \( T^*B \) has a natural quantization. Namely, we can take \( \mathcal{D}_B \), the sheaf of differential operators on \( B \) (strictly speaking this is a quantization of \( \pi_*\mathcal{O}_{T^*B} \) where \( \pi : T^*B \to B \) is projection). This is a sheaf of filtered algebras, where we take the filtration by order of differential operator, and there is an isomorphism of filtered algebras \( H^0(B, \mathcal{D}_B) \simeq U(\mathfrak{g})/J_0 \) where is the primitive central ideal annihilating the unique one-dimensional representation of \( \mathfrak{g} \).

In general, the above approach cannot work for two reasons. First, as noted above, \( \mathcal{D}_B \) is a sheaf on \( B \) and not \( T^*B \) (one can actually overcome this issue using microlocalization). Secondly, and more crucially, very few symplectic resolutions are of the form \( T^*Z \) for some \( Z \). Instead, we must use formal quantization and then kill the formal parameter using the \( \mathbb{C}^\times \) action of Proposition
First, we will consider sheaves of deformation-quantization algebras on \( Y \). Let \( \hbar \) be a formal parameter.

**Definition 3.10.** A sheaf of deformation-quantization algebras \( B \) is a sheaf of \( \mathbb{C}[\hbar] \)-algebras \( B \) that are \( \hbar \)-adically complete, and flat over \( \mathbb{C}[\hbar] \), such that there is an isomorphism of Poisson algebras \( B/\hbar B \cong \mathcal{O}_Y \).

In a formal neighbourhood of any point \( y \in Y \), any deformation-quantization algebra isomorphic to \( \mathbb{C}[x_1, \ldots, x_n; y_1, \ldots, y_n][\hbar] \), with the usual Moyal-Weyl product:

\[
    f \star g = \sum_{i,j \in \mathbb{N}^n} \hbar^{|i|} \frac{\partial^i f}{\partial y \partial^i g},
\]

where \( i! = i_1! \cdots i_n! \), \( \partial^i / \partial x = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n} \) and \( |i| = i_1 + \cdots + i_n \). Bezrukavnikov-Kaledin [8] showed that:

**Theorem 3.11.** The sheaves of deformation quantization algebras on \( Y \) are classified by \( H^2(Y, \mathbb{C})[\hbar] \).

Recall that \( \mathbb{C}^\times \) acts on \( Y \) and the form \( \omega \) has weight \( \ell \). We make \( \mathbb{C}^\times \) act on \( \mathbb{C}[\hbar] \) by giving \( \hbar \) weight \( \ell \) too. Then we can consider \( \mathbb{C}^\times \)-equivariant deformation-quantization algebras on \( Y \) too. Using Theorem 3.11, Losev showed in [18] that:

**Theorem 3.12.** The sheaves of equivariant deformation-quantization algebras are classified by \( H^2(Y, \mathbb{C}) \).

Finally, we once again use the \( \mathbb{C}^\times \) action on \( Y \) to kill the formal parameter \( \hbar \). To do this, we must consider the conic topology on \( Y \) i.e. a subset \( U \) of \( Y \) is open in this topology if and only if it is Zariski open and \( \mathbb{C}^\times \)-stable.

**Definition 3.13.** A filtered quantization of \( \mathcal{O}_Y \) is a sheaf (in the conic topology) of sheaves of filtered algebras \( A = \bigcup_{i \in \mathbb{Z}} A_i \) such that

1. the filtration is separating \( \bigcap_{i \in \mathbb{Z}} A_i = 0 \), and complete \( A = \lim_{\rightarrow -\infty} A/\mathcal{A}_i \),
2. \([a, b] \in A_{i+j-\ell} \) for \( a \in A_i \) and \( b \in A_j \); and
3. there is an isomorphism of graded Poisson algebras \( \text{gr} A \cong \mathcal{O}_Y \).

In the case of a filtered quantization \( A \) of a conic symplectic singularity \( X \), the analogue of condition (i) of Definition 3.13 is automatic since \( A_{-1} = 0 \). Let \( B \) be an equivariant deformation-quantization algebra on \( Y \). For each \( \mathbb{C}^\times \)-stable affine open subset \( U \) (which form a base for the conic topology on \( Y \)), let

\[
    A_i(U) = \Gamma(U, Bh^{-i})^{\mathbb{C}^\times}, \quad \forall i \in \mathbb{Z}.
\]

Then \( A(U) = \bigcup_{i \in \mathbb{Z}} A_i(U) \) is a filtered algebra, and they glue to form a filtered quantization of \( \mathcal{O}_Y \). This is an equivalence between equivariant deformation-quantization algebras on \( Y \) and filtered quantization of \( \mathcal{O}_Y \). We deduce:

**Corollary 3.14.** The filtered quantizations of \( \mathcal{O}_Y \) are parametrized by \( H^2(Y, \mathbb{C}) \).
In particular filtered quantizations exist! Moreover, just as in Theorem 3.6, there is a universal filtered quantization sheaf $\mathcal{A}_Y$ on $Y$, which is a sheaf of $B_Y = \mathbb{C}[H^2(Y, \mathbb{C})]$-algebras. Taking global sections, $\Gamma(Y, \mathcal{A}_Y)$ is a family of filtered quantization of $X$. The group $W$ acts on both $B_Y$ and $\Gamma(Y, \mathcal{A}_Y)$ by filtered algebra automorphisms. Then, in [20], Losev proves the following result, of which Theorem 3.6 is an immediate corollary.

**Theorem 3.15.** The $B^W$-algebra $\Gamma(Y, \mathcal{A}_Y)^W$ is the universal filtered quantization of $X$.

**Remark 3.16.** Many notions from the theory of $\mathcal{D}$-modules make sense for filtered quantizations. One can talk about holonomic $\mathcal{A}$-modules, characteristic cycles, Verdier duality etc.

**Remark 3.17.** If $X$ does not admit a projective symplectic resolution, then we take instead $\rho : Y \to X$ to be a $\mathbb{Q}$-factorial terminalization, as in section 1.7. Let $i : Y_{\text{sm}} \to Y$ be the inclusion. Then we can apply Corollary 3.14 since it is known that $H^0(Y_{\text{sm}}, \mathcal{O}) = \mathbb{C}[X]$ and $H^i(Y_{\text{sm}}, \mathcal{O}) = 0$ for all $i > 0$. If $\mathcal{A}$ is a filtered quantisation of $Y_{\text{sm}}$, then $i_\ast \mathcal{A}$ is a filtered quantization of $Y$ and this establishes a bijection between quantizations of the two spaces. See [10, Proposition 3.4] for details.

### 3.5. Localization

Given a filtered quantization $\mathcal{A}$ of $X$, we have seen above that there exists a filtered quantization $\mathcal{A}_Y$ such that $\Gamma(Y, \mathcal{A}) \simeq \mathcal{A}$ as filtered algebras.

**Definition 3.18.** We say that a $\mathcal{A}$-module $M$ is coherent if it has a filtration $M = \bigcup_{i \in \mathbb{Z}} M_i$, compatible with the filtration on $\mathcal{A}$, such that

1. the filtration is separating $\bigcap_{i \in \mathbb{Z}} M_i = 0$ and complete $M = \lim_{\to} M/M_i$; and
2. the associated graded $\text{gr} M$ is a coherent $\mathcal{O}_Y$-module.

The category of coherent $\mathcal{A}$-modules is denoted $\text{Coh}(\mathcal{A})$ and the category of finitely generated $\mathcal{A}$-modules is denoted $\mathcal{A}$-$\text{mod}$. We have a pair of adjoint functors

$$\text{Loc} : \mathcal{A}$-mod \to \text{Coh}(\mathcal{A}), \quad \text{Loc}(M) = \mathcal{A} \otimes_{A} M,$$

$$\Gamma : \text{Coh}(\mathcal{A}) \to \mathcal{A}$-mod, \quad \Gamma(M) = \Gamma(Y, M).$$

As a vast generalization of a result of Beilinson-Bernstein [3], McGerty and Nevins have shown in [21]:

**Theorem 3.19.** If $\mathcal{A}$ has finite global dimension, then $\mathbb{L}\text{Loc} : D^b(\mathcal{A}$-mod) $\to D^b(\text{Coh}(\mathcal{A}))$ is an equivalence with quasi-inverse $\mathbb{R}\Gamma$.

We have $\mathcal{A} = \mathcal{A}_\lambda$ for some $\lambda \in H^2(Y, \mathbb{C})$, and write $\mathcal{A}_\lambda := \Gamma(Y, \mathcal{A}_\lambda)$.

**Problem 3.20.** Describe precise the locus in $H^2(Y, \mathbb{C})$ for which $\mathcal{A}_\lambda$ has finite global dimension.

It was shown by Braden-Licata-Proudfoot-Webster in [10] that:

**Theorem 3.21.** For any $\lambda \in H^2(Y, \mathbb{C})$ and $\eta = c_2(\mathcal{L})$ for $\mathcal{L}$ an ample line bundle on $Y$,

$$\text{Loc} : \mathcal{A}_{\lambda + k\eta}$-mod $\to \text{Coh}(\mathcal{A}_{\lambda + k\eta})$$

is an equivalence for all but finitely many $k \in \mathbb{Z}$. 

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Thus, in almost all cases, studying the representation theory of $A_\lambda$ is equivalent to studying shaves of modules for $A_\lambda$ on $Y$. One can ask for the precise values in $H^2(Y, \mathbb{C})$ where localization holds. There does not seem to be a general answer at present.

4. Category $\mathcal{O}$

One of the iconic results in geometric representation theory is Beilinson-Bernstein’s Localization Theorem, and their consequent proof of Jantzen’s conjecture. A key step here was to realise category $\mathcal{O}$ for a simple Lie algebra as a certain category of holonomic $\mathcal{D}$-modules on the flag manifold. Localization allows us to do the same for filtered quantizations of conic symplectic singularities. However, in order to be able to define category $\mathcal{O}$ in the first place, we need a Hamiltonian $\mathbb{C}^\times$-action on $X$. Such actions do not always exist; for instance this is the case for Kleinian singularities not of type $A$. A Hamiltonian $\mathbb{C}^\times$-action on $X$ is an action of a torus $T \simeq \mathbb{C}^\times$ that fixes the symplectic form. Equivalently, it is one for which the Poisson bracket is homogeneous of degree zero. We will always assume that our Hamiltonian actions are faithful, and commute with the contracting $\mathbb{C}^\times$ action. We assume throughout that there is a fixed projective symplectic resolution $\rho : Y \to X$. The action of $T$ lifts to $Y$. We assume that the set $Y^T$ is finite. Examples satisfying these conditions include:

1. The Springer resolution $\rho : T^*B \to N$.
2. The Hilbert-Chow morphism $\rho : \text{Hilb}^n \mathbb{C}^2 \to \mathbb{C}^{2n}/\mathfrak{S}_n$.
3. More generally, the projective symplectic resolutions of $\mathbb{C}^{2n}/\mathfrak{S}_n \wr \mathbb{Z}_\ell$.
4. Hypertoric varieties.

4.1. Algebraic category $\mathcal{O}$. Choose a filtered quantization $A$ of $X$. Then the Hamiltonian torus $T$ will act on $A$, defining a grading $A = \bigoplus_{i \in \mathbb{Z}} A(i)$ (not to be confused with the filtration $A = \bigcup_{j \in \mathbb{Z}} A_j$). Let

$$A(\geq) = \bigoplus_{i \geq 0} A(i),$$

a subalgebra of $A$. Notice that $A$ is Noetherian since $\mathbb{C}[X]$ is Noetherian. Hence the category of finitely generated $A$-modules is abelian.

**Definition 4.1.** Algebraic category $\mathcal{O}_a$ is defined to be the category of finitely generated $A$-modules that are locally finite for $A(\geq)$.

Here a module $N$ is locally finite for $A(\geq)$ if, for each $n \in N$, the space $A(\geq) \cdot n$ is finite dimensional. In the examples listed above, algebraically we get:

1. The principal block of category $\mathcal{O}$ for a simple Lie algebra.
2. Category $\mathcal{O}$ for the rational Cherednik algebra associated to the symmetric group.
3. Category $\mathcal{O}$ for the rational Cherednik algebra associated to $\mathfrak{S}_n \wr \mathbb{Z}_\ell$.
4. Hypertoric category $\mathcal{O}$.

4.2. Geometric category $\mathcal{O}$. Let $Y^+ = \{ y \in Y \mid \lim_{t \to \infty} t \cdot y \text{ exists} \}$ be the attracting locus for $T$. It is a Lagrangian subvariety of $Y$. We choose a filtered quantization $A$ of $Y$ such that
\( \Gamma(Y, \mathcal{A}) = A \). The derivative \( \zeta \) of the \( \mathbb{T} \)-action on \( Y \) can be lifted to an element of \( \mathcal{A} \) (also denoted \( \zeta \)), meaning that for \( a \in \mathcal{A} \), a local section, \( t \cdot a = t^i a \) for all \( t \in \mathbb{T} \) if and only if \( [\zeta, a] = ia \).

**Definition 4.2.** Geometric category \( \mathcal{O}_g \) is defined to be the full subcategory of \( \text{Coh}(\mathcal{A}) \) consisting of sheaves \( \mathcal{M} = \bigcup_{i \in \mathbb{Z}} \mathcal{M}_i \) supported on \( Y^+ \) such that \( \zeta(\mathcal{M}_0) \subset \mathcal{M}_0 \).

We recall next the important notion of highest weight category, as exemplified by category \( \mathcal{O} \) of a semi-simple Lie algebra \( g \) over \( \mathbb{C} \). The abstract notion of a highest weight category was introduced in [12].

**Definition 4.3.** Let \( \mathcal{C} \) be an abelian, \( \mathbb{C} \)-linear and finite length category, and \( \Lambda \) a poset. We say that \( (\mathcal{C}, \Lambda) \) is a **highest weight category** if

1. There is a complete set \( \{ L(p) \mid p \in \Lambda \} \) of non-isomorphic simple objects labelled by \( \Lambda \).
2. There is a collection of *standard* objects \( \{ \Delta(p) \mid p \in \Lambda \} \) of \( \mathcal{C} \), with surjections \( \Delta(p) \twoheadrightarrow L(p) \) such that all composition factors \( L(q) \) of the kernel satisfy \( q < p \).
3. Each \( L(p) \) has a projective cover \( P(p) \) in \( \mathcal{C} \) and the projective cover \( P(p) \) admits a \( \Delta \)-filtration \( 0 = F_0 P(p) \supset F_1 P(p) \supset \cdots \supset F_m P(p) = P(p) \) such that
   - \( F_m P(p)/F_{m-1} P(p) \simeq \Delta(p) \).
   - For \( 0 < i < m \), \( F_i P(p)/F_{i-1} P(p) \simeq \Delta(q) \) for some \( q > p \).

For geometric category \( \mathcal{O} \), one takes \( \Lambda = Y^\mathbb{T} \). Clearly the fixed points lie inside \( Y^+ \). For each \( p \in Y^\mathbb{T} \), define \( Y^+_p = \{ y \in Y \mid \lim_{t \to \infty} t \cdot y = p \} \). Then each \( Y^+_p \) is a smooth locally closed subset of \( Y \) and

\[
Y^+ = \bigsqcup_{p \in Y^\mathbb{T}} Y^+_p.
\]

We define a partial ordering on \( Y^\mathbb{T} \) (called the geometric ordering) by saying that

\[
p > q \quad \text{if} \quad Y^+_p \cap Y^+_q \neq \emptyset.
\]

Braden-Licata-Proudfoot-Webster [10] showed that:

**Theorem 4.4.** The category \( \mathcal{O}_g \) is a highest weight category with poset \( Y^\mathbb{T} \).

Thus, to each \( p \in Y^\mathbb{T} \), one can associate a Verma module \( \Delta(p) \), with simple quotient \( L(p) \) etc. In particular, \( \mathcal{O}_g \) has enough projectives, and hence is equivalent to \( D\text{-mod} \) for some finite dimensional algebra \( D \). Recall that there must exist some \( \lambda \in H^2(Y, \mathbb{C}) \) such that \( \mathcal{A} = \mathcal{A}_\lambda \). By [10, Corollary 3.19], we have:

**Proposition 4.5.** Assume that the parameter \( \lambda \) is chosen so that \( \text{Loc}_\lambda : \mathcal{A}_\lambda\text{-mod} \to \mathcal{A}_\lambda\text{-mod} \) is an equivalence. Then it restricted to an equivalence \( \text{Loc}_\lambda : \mathcal{O}_a \to \mathcal{O}_g \).

We deduce from Theorem 4.4 that:

**Corollary 4.6.** For all parameters \( \lambda \) for which localization holds, \( \mathcal{O}_a \) is a highest weight category.

**Problem 4.7.** Find a geometric interpretation of the multiplicities \( \left[ \Delta(\lambda) : L(\mu) \right] \).
Remark 4.8. Notice that both $O_a$ and $O_g$ depend on the choice of a Hamiltonian torus $T$. Thus, in reality, one gets families of category $O$s depending on this choice of torus.

Remark 4.9. The papers [10] and [19] develop the theory of geometric category $O$ much further. We recommend the reader consult these articles to learn more about this beautiful theory.

References


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