ON SINGULAR CALOGERO-MOSER SPACES

GWYN BELLAMY

Abstract. Using combinatorial properties of complex reflection groups we show that if the group $W$ is different from the wreath product $S_n \wr \mathbb{Z}/m\mathbb{Z}$ and the binary tetrahedral group (labelled $G(m,1,n)$ and $G_4$ respectively in the Shephard-Todd classification), then the generalised Calogero-Moser space $X_c$ associated to the centre of the rational Cherednik algebra $H_{t,c}(W)$ is singular for all values of the parameter $c$. This result and a theorem of Ginzburg and Kaledin imply that there does not exist a symplectic resolution of the singular symplectic variety $h \times h^*/W$ when $W$ is a complex reflection group different from $S_n \wr \mathbb{Z}/m\mathbb{Z}$ and the binary tetrahedral group. Conversely it has been shown by Etingof and Ginzburg that $X_c$ is smooth for generic values of $c$ when $W \cong S_n \wr \mathbb{Z}/m\mathbb{Z}$. We show that this is also the case when $W$ is the binary tetrahedral group. A conjecture by Namikawa, if true, would imply the existence of a symplectic resolution in this case. Finally, we note that the above results together with work of Chlouveraki are consistent with a conjecture of Gordon and Martino on block partitions in the restricted rational Cherednik algebra.

1. Introduction

Let $W$ be an irreducible complex reflection group and $\mathfrak{h}$ its reflection representation. Etingof and Ginzburg [EG] associated to $W$ a family of algebras, the rational Cherednik algebras $H_{t,c}(W)$, depending on parameters $t$ and $c$. The definition is given in Section 2. When $t = 0$, these algebras have large centres and the geometry of the centre strongly influences the representation theory of the algebra. The affine variety $X_c$ corresponding to the centre of the rational Cherednik algebra was called the generalised Calogero-Moser space at $c$ by Etingof and Ginzburg. They showed [EG, Corollary 1.14], that for generic values of the parameter $c$, $X_c$ is smooth when $W \cong G(m,1,n)$. However, Gordon [Go, Proposition 7.3] showed that, for many Weyl groups $W$ not of type $A$ or $B(= C)$, $X_c$ is a singular variety for all choices of the parameter $c$. Using similar methods we extend this result to all irreducible complex reflection groups.

Theorem 1.1. Let $W$ be an irreducible complex reflection group, not isomorphic to $G(m,1,n)$ or $G_4$, and $X_c$ the generalised Calogero-Moser space associated to $W$. Then $X_c$ is a singular variety for all choices of the parameter $c$. Conversely for $W \cong G_4$, $X_c$ is a smooth variety for generic values of $c$.

This completes the classification of rational Cherednik algebras for which $X_c$ is smooth for generic $c$.

In [GK, Corollary 1.21], Ginzburg and Kaledin show that the existence of a symplectic resolution of the symplectic singularity $\mathfrak{h} \times \mathfrak{h}^*/W$ implies that $X_c$ is smooth for generic $c$. This result, together with Theorem 1.1 above implies the following corollary.
Corollary 1.2. Let $W$ be an irreducible complex reflection group with reflection representation $\mathfrak{h}$. Then there does not exist a symplectic resolution of $\mathfrak{h} \times \mathfrak{h}^*/W$ when $W \not\cong G(m,1,n)$ or $G_4$.

A related conjecture by Namikawa [Na, page 2] states that a symplectic resolution of $\mathfrak{h} \times \mathfrak{h}^*/W$ exists if and only if it has a smoothing by a Poisson deformation. If true, this conjecture and [GK, Theorem 1.18] imply that $\mathfrak{h} \times \mathfrak{h}^*/W$ has a symplectic resolution if and only if $X_c$ is smooth for generic values of $c$. Thus we conjecture

Conjecture 1.3. There exists a symplectic resolution of the singular symplectic variety $\mathfrak{h} \times \mathfrak{h}^*/G_4$.

In order to prove Theorem 1.1 we show that the restricted rational Cherednik algebra $\overline{H}_{0,e}(W)$ has irreducible representations of dimension $< |W|$ for all values of $c$ when $W$ is different from $G(m,1,n)$ and $G_4$. This implies that there exist blocks in $\overline{H}_{0,e}(W)$ with nonisomorphic irreducible modules. Therefore the corresponding block partition of $\text{Irr}(W)$, as described in [GM], is trivial for generic values of $c$ if and only if $W$ is $G(m,1,n)$ or $G_4$. A conjecture of Gordon and Martino [GM] then implies that the partitioning of $\text{Irr}(W)$ induced by the Rouquier families of the Hecke algebra $\mathcal{H}_q(W)$ should also be trivial for generic choices of $c$ if and only if $W$ is $G(m,1,n)$ or $G_4$. Work of Chlouveraki [Ch] on the cyclotomic Hecke algebras of exceptional complex reflection groups shows that this is indeed the case.

2. The rational Cherednik algebra at $t = 0$

2.1. Definitions and notation. Let $W$ be a complex reflection group, $\mathfrak{h}$ its reflection representation over $\mathbb{C}$ with $\dim(\mathfrak{h}) = n$, and $S$ the set of all complex reflections in $W$. Let $\omega : \mathfrak{h} \oplus \mathfrak{h}^* \to \mathbb{C}$ be the symplectic form on $\mathfrak{h} \oplus \mathfrak{h}^*$ given by $\omega((f_1,f_2),(g_1,g_2)) = f_2(g_1) - g_2(f_1)$ and $c : S \to \mathbb{C}$ a $W$-invariant function. For $s \in S$, define $\omega_s : \mathfrak{h} \oplus \mathfrak{h}^* \to \mathbb{C}$ to be the restriction of $\omega$ on $\text{Im}(1-s)$ and the zero form on $\text{Ker}(1-s)$. The rational Cherednik algebra at parameter $t = 0$, as introduced by Etingof and Ginzburg [EG, page 250], is the quotient of the skew group algebra of the tensor algebra $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ with $W$, $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$, by the ideal generated by the relations

$$[x, y] = \sum_{s \in S} c(s)\omega_s(x, y)s$$

$\forall x, y \in \mathfrak{h} \oplus \mathfrak{h}^*$

Let $Z_c$ denote the centre of $H_{0,e}$ and $X_c = \text{maxspec}(Z_c)$ the affine variety corresponding to $Z_c$. The space $X_c$ is called the generalised Calogero-Moser space associated to the complex reflection group $W$ at parameter $c$. By [EG, Proposition 4.5], we have an inclusion $A = \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \subset Z_c$ and correspondingly a surjective morphism $\Upsilon_c : Z_c \to \mathfrak{h}/W \times \mathfrak{h}^*/W$. This allows us to define the restricted rational Cherednik algebra $\overline{H}_{0,e}(W)$ as

$$\overline{H}_{0,e}(W) = \frac{H_{0,e}(W)}{\langle A_+ \rangle}$$

where $A_+$ denotes the ideal in $A$ of elements with zero constant term. From the defining relations (1) we see that putting $\mathfrak{h}^*$ in degree 1, $\mathfrak{h}$ in degree $-1$ and $\mathbb{C}W$ in degree 0 defines a $\mathbb{Z}$-grading on the rational Cherednik algebra $H_{t,e}(W)$. The ideal $\langle A_+ \rangle$ is generated by elements that are homogeneous with respect
to this grading, therefore $H_{0,e}$ is also a $\mathbb{Z}$-graded algebra.

We denote by $\mathbb{C}[h]^{\omega W}$ the coinvariant ring $\mathbb{C}[h]/\mathbb{C}[h]_+^W$, where $\mathbb{C}[h]_+^W$ is the ideal in $\mathbb{C}[h]$ generated by the elements in $\mathbb{C}[h]^W$ with zero constant term. We follow the notation introduced in [Ste] with regards to complex reflection groups, and set $M := H_{0,e}$-module associated to the irreducible $W$-module $\lambda$. This module is a graded $H_{0,e}$-module with $M(\lambda)_i = 0$ for $i < 0$. By [Go, Proposition 4.3], $M(\lambda)$ has a simple head which we denote $L(\lambda)$.

We follow the notation of [Ste] with regards to complex reflection groups, and set $d = m/p$ when considering the group $G(m, p, n)$. For an arbitrary $\mathbb{Z}$-graded vector space $M = \bigoplus_{i \in \mathbb{Z}} M_i$, the Poincaré polynomial of $M$ will be denoted $P(M, t)$. We denote by $f_\lambda(t)$ the fake polynomial of the irreducible representation $\lambda$ of $W$. This is defined as

$$f_\lambda(t) := \sum_{i \in \mathbb{Z}_{\geq 0}} (\mathbb{C}[h]^{\omega W}_i : \lambda)t^i$$

where $(\mathbb{C}[h]^{\omega W}_i : \lambda)$ is the multiplicity of $\lambda$ in $i^{th}$ degree of the coinvariant ring $\mathbb{C}[h]^\omega W$ (thought of here as a graded $W$-module).

Let $\text{Irr}(W)$ be a complete set of non-isomorphic irreducible representation of $W$. We will also require the surjective map $\Theta : \text{Irr}(W) \to \mathcal{Y}^{-1}(0)$, taking $\lambda$ to the annihilator of $L(\lambda)$ in $\mathbb{Z}_e$, as defined in [Go, paragraph 5.4]. This map has the property that a fiber $\Theta^{-1}(m)$ is a singleton set if and only if $m$ is a smooth closed point in $X_e$ ([Go, Theorem 5.6]).

2.2. General results. Let $\{s_1, \ldots, s_k\}$ be a conjugacy class consisting of complex reflections in $W$ and $\zeta$ the eigenvalue of $s_1$ (and hence all $s_i$) not equal to 1 when thinking of $W$ as a subgroup of $GL(\mathfrak{h})$. For $1 \leq i \leq k$, let $\omega_{s_i}$ be the restricted symplectic form on $\mathfrak{h} \oplus \mathfrak{h}^*$ as defined above. Let $\pi_{s_i} : \mathfrak{h} \oplus \mathfrak{h}^* \to \text{Im}(1-s_i)$ be the projection map along $\text{Ker}(1-s_i)$, so that $\omega_{s_i} = \omega \circ \pi_{s_i}$, and define $\Omega = \sum_{i=1}^k \omega_{s_i}$.

Lemma 2.1. Let $W$, $\omega$ and $\Omega$ be as above. Then $\Omega = \frac{b}{n}(1-\zeta)^{-1}(1-\zeta^{-1})^{-1}(2-\zeta-\zeta^{-1})\omega$.

Proof. Since each $\omega_{s_i}$ is alternating and $\mathbb{C}$-linear, $\Omega \in \Lambda^2(\mathfrak{h} \oplus \mathfrak{h}^*)$. Let $x \in \mathfrak{h} \oplus \mathfrak{h}^*$. Then $x$ decomposes uniquely as $x_1 + x_2$, with $x_1 \in \text{Im}(1-s_i)$ and $x_2 \in \text{Ker}(1-s_i)$. By definition, there exists $y \in \mathfrak{h} \oplus \mathfrak{h}^*$ such that $(1-s_i)y = x_1$. Then $(1-gs_1g^{-1})(gy) = g(1-s_i)g^{-1}gy = g(1-s_i)y = gx_1$ implying that $gx_1 \in \text{Im}(1-gs_1g^{-1})$. Also $(1-s_i)x_2 = 0$ implies that $(1-gs_1g^{-1})gx_2 = 0$ hence $gx$ decomposes as $gx_1 + gx_2$ with $gx_1 \in \text{Im}(1-gs_1g^{-1})$ and $gx_2 \in \text{Ker}(1-gs_1g^{-1})$. Therefore $\pi_{gs_1g^{-1}} = g\pi_{s_i}g^{-1}$ and $\omega_{s_i}(g^{-1}x, g^{-1}y) = \omega_{gs_1g^{-1}}(x, y)$. Hence $\Omega \in (\Lambda^2(\mathfrak{h}^* \oplus \mathfrak{h}))^W$. By [EG, Lemma 2.23] $\dim(\Lambda^2(\mathfrak{h}^* \oplus \mathfrak{h}))^W = 1$, therefore there exists $\lambda \in \mathbb{C}$ such that $\Omega = \lambda \omega$. Consider $\Omega'(x, y) = \Omega((x, 0), (0, y))$, where $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^*$. Recall that $\zeta$ is the eigenvalue of $s_1$ not equal to 1, then $\pi_{s_i}(x) = (1-\zeta)^{-1}(1-s_i)x$ and
\[ \pi_{s_i}(y) = (1 - \zeta^{-1})^{-1}(1 - s_i)y. \] Expanding \( \Omega'(x, y) \)

\[
\Omega'(x, y) = \sum_{i=1}^{k} \omega((1 - \zeta)^{-1}(1 - s_i)x, (1 - \zeta^{-1})^{-1}(1 - s_i)y)
\]

\[
= (1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1}\sum_{i=1}^{k} [\omega(x, y) - \omega(s_i x, y) - \omega(x, s_i y) + \omega(s_i x, s_i y)]
\]

\[
= (1 - \zeta)^{-1}(1 - \zeta^{-1})^{-1}\omega(x, (\sum_{i=1}^{k} 2 - s_i - s_i^{-1})y)
\]

Define \( \phi = (\sum_{i=1}^{k} 2 - s_i - s_i^{-1}) : \mathfrak{h}^* \to \mathfrak{h}^* \), a \( W \)-homomorphism. The trace of \( \phi \) is \( 2nk - (n - 1)k - k\zeta - (n - 1)k - k\zeta^{-1} = k(2 - \zeta - \zeta^{-1}) \). Since \( \mathfrak{h}^* \) is irreducible, Schur’s lemma says that \( \phi(y) = \frac{k}{n} (2 - \zeta - \zeta^{-1})y \) and therefore \( \lambda = \frac{k}{n} (1 - \zeta^{-1})^{-1}(2 - \zeta - \zeta^{-1}) \).

We also require the notion of a generalised baby Verma module, which are baby Verma modules above points other than the origin in \( \mathfrak{h}/W \times \mathfrak{h}^*/W \).

**Definition 2.2.** Let \( (p, q) \in \mathfrak{h}/W \times \mathfrak{h}^* \), \( W_q \) the stabiliser subgroup of \( q \in W \) and \( E \) an irreducible \( W_q \)-module. Then we define the **generalised baby Verma module**

\[
\Delta_c(E; p, q) := H_{0,c}(W) \otimes_{C[\mathfrak{h}]^W \otimes C[\mathfrak{h}^*]^W} W_q E
\]

where the action of \( C[\mathfrak{h}]^W \otimes C[\mathfrak{h}^*]^W \) on \( E \) is given by \((f \otimes g \otimes w) \cdot e = f(p)g(q)w \cdot e \) for all \( f \in C[\mathfrak{h}]^W \), \( g \in C[\mathfrak{h}^*] \), \( w \in W_q \).

Since \( C[\mathfrak{h}]^W \otimes C[\mathfrak{h}^*]^W \subseteq Z_c \), Schur’s lemma implies that, for every irreducible \( H_{0,c} \)-module \( L \), there exists \((p, r) \in \mathfrak{h}/W \times \mathfrak{h}^*/W \) such that \((f \otimes g) \cdot l = f(p)g(r)l \) for all \( l \in L \), \( f, g \in C[\mathfrak{h}]^W \otimes C[\mathfrak{h}^*]^W \). Choosing a point \( q \) in the orbit represented by \( r \) we write \((p, r) = (p, W_q) \) and say that the irreducible \( H_{0,c} \)-module \( L \) lies above \((p, W_q) \).

**Lemma 2.3.** Let \( L \) be an irreducible \( H_{0,c} \)-module lying above \((p, W_q) \). Then there exist \( E \in \text{Irr}(W_q) \) and a surjective \( H_{0,c} \)-homomorphism \( \phi : \Delta_c(E; p, q) \to L \).

**Proof.** The action on \( L \) of the commutative ring \( C[\mathfrak{h}]^* \) gives a decomposition \( L = \bigoplus_{q' \in \mathfrak{h}^*} L_{q'}^{gen} \) of \( L \) into generalised eigenspaces. That is, for each \( l \in L_{q'}^{gen} \) and \( f \in C[\mathfrak{h}]^* \), there exists an \( N \in \mathbb{N} \) such that \((f - f(q'))^N \cdot l = 0 \) (since \( L \) is finite dimensional, we can choose \( N \) to be independent of \( f \) and \( l \)).

Choose \( q' \) such that \( L_{q'}^{gen} \neq 0 \), so that \((f - f(q'))^N \) acts as zero on \( L_{q'}^{gen} \) for all \( f \in C[\mathfrak{h}]^* \). As \( L \) lies over \((p, W_q) \) we see that \((f - f(q')) \) also acts nilpotently on \( L_{q'}^{gen} \) and \( f(q) = f(q') \). Since \( W \) is a finite group, each orbit in \( \mathfrak{h}^* \) is closed, therefore \( q' \in W_q \) and we can find \( w \in W \) such that \( w \cdot q = q' \). Now let \( 0 \neq L_q \subseteq L_{q'}^{gen} \) be the space of elements \( l \) in \( L_{q'}^{gen} \) such that \((f - f(q')) \cdot l = 0 \), for all \( f \in C[\mathfrak{h}]^* \). Then \( w^{-1}(L_q) \neq 0 \) and \( f \cdot (w^{-1}l) = (fw^{-1}) \cdot l = w^{-1} \cdot (w'f)(q')l = w^{-1} \cdot (fw^{-1}q')l = f(q)w^{-1}l \) implies that \( w^{-1}(L_q) \subseteq L_q \). Thus \( L_q \) is a nonzero \( W_q \)-submodule of \( L \) because \( f \cdot (v \cdot l) = (fv) \cdot l = v \cdot (v^{-1}f) \cdot l = v \cdot f(q)l = v \cdot f(q)l = f(q)l = f(q)(v \cdot l) \) for all \( f \in C[\mathfrak{h}]^* \), \( v \in W_q \) and \( l \in L_q \). Choose an irreducible \( W_q \)-submodule \( E \) of \( L_q \). The inclusion \( E \hookrightarrow L \) induces a \( H_{0,c} \)-homomorphism \( \phi : \Delta_c(E; p, q) \to L \). The fact that \( L \) is irreducible implies that this is a surjection. \( \square \)
3. Singular generalised Calogero-Moser Spaces

3.1. The main result.

**Theorem 3.1.** For all $W$ not isomorphic to $G(m, 1, n)$ or $G_4$ and for all parameters $c$, the variety $X_c$ is singular.

By [EG, Proposition 3.8] the statement of Theorem 3.1 is equivalent to the statement: for $W$ not isomorphic to $G(m, 1, n)$ or $G_4$ and for all parameters $c$ there exists an irreducible $H_{0,c}(W)$-module $L$ with $\dim L < |W|$. Therefore Theorem 3.1 follows from

**Proposition 3.2.** For each $W$ not isomorphic to $G(m, 1, n)$ or $G_4$, there exists an irreducible $W$-module $\lambda$ such that for all parameters $c$, the irreducible $H_{0,c}(W)$-module $L(\lambda)$ has dimension $< |W|$.

The proof of Proposition 3.2 will occupy the remainder of Section 3. The irreducible complex reflection groups were classified by Shephard and Todd [ST] and either belong to an infinite family labelled $G(m, p, n)$, where $m, p, n \in \mathbb{N}$ and $p|m$, or to one of 34 exceptional groups $G_4, \ldots G_{37}$.

**Lemma 3.3.** Let $W$ be a complex reflection group. Let $\lambda \in \text{Irr}(W)$ be the unique representation corresponding to a smooth point of $T^{-1}(0)$ in $X_c$ i.e. $\Theta(\lambda)$ is smooth in $X_c$. Then the Poincaré polynomial of $L(\lambda)$ as a graded vector space is given by

\[
(2) \quad P(L(\lambda), t) = \frac{\dim(\lambda) t^{b_{\lambda^*}} P(\mathbb{C}[h]^\text{colW}; t)}{f_{\lambda^*}(t)}
\]

where $\lambda^*$ is the irreducible $W$-module dual to $\lambda$, and $b_{\lambda}$ the trailing degree of the fake polynomial $f_{\lambda}(t)$.

**Proof.** By [Go, Lemma 4.4, paragraphs (5.2) and (5.4)], the graded composition factors of $M(\lambda)$ are all of the form $L(\lambda)[i]$, for some $i \geq 0$. Therefore we can find a multiset $\{i_1, \ldots, i_k\}$ such that as a graded $W$-module

\[
M(\lambda) \cong L(\lambda)[i_1] \oplus L(\lambda)[i_2] \oplus \cdots \oplus L(\lambda)[i_k].
\]

Since $\Theta(\lambda)$ is a smooth point in $X_c$, [EG, Theorem 1.7] says that $L(\lambda) \cong CW$ as a $W$-module. Hence it contains a unique copy of the trivial representation $T$. Assume this copy occurs in degree $a$ in $L(\lambda)$. Then it will occur in degree $a - i_j$ in $L(\lambda)[i_j]$. As a graded $W$-module, $M(\lambda) \cong \mathbb{C}[h^*]^{\text{colW}} \otimes \lambda$. The fact that $[\tau \otimes \lambda : T] = \delta_{s,\lambda^*}$ implies that the graded multiplicity of $T$ in $M(\lambda)$ equals the graded multiplicity of $\lambda^*$ in $\mathbb{C}[h^*]^{\text{colW}}$. The graded multiplicity of $\lambda^*$ in $\mathbb{C}[h^*]^{\text{colW}}$ is $f_{\lambda^*}(t)$. Hence $P(M(\lambda), t) = t^{-a} f_{\lambda^*}(t) P(L(\lambda), t)$. The lowest nonzero term of $P(L(\lambda), t)$ occurs in degree zero implying that $a = b_{\lambda^*}$. The formula follows by noting that $P(M(\lambda), t) = \dim(\lambda) P(\mathbb{C}[h^*]^{\text{colW}})$. \qed

Since $L(\lambda)$ is a finite dimensional module, the above lemma shows that the right hand side of equation (2) is a polynomial in $\mathbb{Z}[t, t^{-1}]$ with integer coefficients. Moreover, [Go, Lemma 4.4] shows that it is actually in $\mathbb{Z}[t]$ and that the degree 0 coefficient is 1.
We give a description of the parameterization of irreducible $G(m, p, n) = G_2(2, 3)$ by Lemma 3.3 does not hold. Thus $L(\lambda)$ will have dimension $\langle G(m, p, n) \rangle$, proving Proposition 3.2 in this case. The group $G(2, 2, 3)$ is the Weyl group corresponding to the Dynkin diagram $D_3 = A_3$ and hence $G(2, 2, 3) \cong S_4$. By [EG, Corollary 16.2], $X_c$ is smooth for generic and hence all non-zero $c$ in this case.

Let $(t)_{(n)} = (1 - t) \cdots (1 - t^{n-1})(1 - t^n)$ and for $\lambda$ a partition of $n$, denote by $n(\lambda) = \sum i(i - 1)\lambda_i$ the partition statistic. The young diagram $D_\lambda$ of a partition $\lambda$ is the finite subset of $\mathbb{N} \times \mathbb{N}$, justified to the south west (in the French style), representing $\lambda$. For $(i, j) \in D_\lambda$, we denote by $h(i, j)$ the hook length at $(i, j)$. The hook polynomial is defined to be

$$H_\lambda(t) = \prod_{(i, j) \in D_\lambda} (1 - t^{h(i, j)})$$

[Ste, Corollary 6.4] states that the fake polynomial of the irreducible representation labelled by $(\{\Delta\}, \epsilon)$ is

$$f_\Delta(t) = \frac{1 - t^{dn}}{1 - t^{mn}} R_\Delta(t) I_\Delta(t^m)$$

where

$$R_\Delta(t) = \sum_{\mu \in \Delta} t^{r(\mu)} \text{ with } r(\mu) = \sum_{i=0}^{m-1} i|\mu^i| \text{ and } I_\Delta(t) = (t)_{(n)} \prod_{i=1}^{m} \frac{t^{\mu_i(\lambda)}}{H_{\lambda_i}(t)}$$

Note that the formula only depends on the orbit and not on the choice of stabiliser.

We wish to calculate the rational function (2) for a well chosen representation $(\{\mu\}, \epsilon)$ of the irreducible representations of $G(m, p, n)$. By [Hu, Theorem 3.15], the Poincaré polynomial of the coinvariant ring of $W$ is given by

$$P(\mathbb{C}[h^1]^{G W}, t) = \prod_{i=1}^{n} \frac{1 - t^{d_i}}{1 - t}$$
where \(d_1, \ldots, d_n\) are the degrees of a set of fundamental homogeneous invariant polynomials of \(W\) \((d_1, \ldots, d_n\) are independent, up to reordering, of the polynomials chosen). By [ST, page 291], \(d_1, \ldots, d_n = m, 2m, \ldots, (n-1)m, dn\) when \(W = G(m, p, n)\).

Hence, if the dual representation of \(\{\mu\}, \epsilon\) is \(\{\lambda\}, \eta\), equation (2) becomes

\[
P(L(\{\mu\}, \epsilon), t) = \frac{\dim(\{\mu\}, \epsilon) t^m (1-t)^{(n-1)m} \prod_{i=0}^{m-1} H_{\lambda_i}(t^m) (1-t^{mn})}{(1-t)^n R_{\{\lambda\}}(t) \prod_{i=0}^{m-1} t^{n(\lambda_i)m}}
\]

(4)

Let \(k \in \mathbb{N}\) such that \(t^k | R_{\{\lambda\}}(t)\) but \(t^{k+1} \nmid R_{\{\lambda\}}(t)\) in \(\mathbb{Z}[t]\) and write \(R_{\{\lambda\}}(t) = t^k R_{\{\lambda\}}(t)\). Then rearrange equation (3) as

\[
f_{\{\lambda\}}(t) = \left(t^k \prod_{i=0}^{m-1} t^{n(\lambda_i)m}\right) \tilde{R}_{\{\lambda\}}(t) \left(\frac{1-t^{dn}}{1-t^{mn}} \prod_{i=1}^{m} \frac{1}{H_{\lambda_i}(t^m)}\right)
\]

(5)

Since each \(H_{\lambda_i}(t^m)\) is a product of factors of the form \((1-t^j)\), the product in the right most bracket consists entirely of factors of the form \((1-t^i)\). Therefore

\[
t^k = t^k \prod_{i=0}^{m-1} t^{n(\lambda_i)m}
\]

and equation (4) becomes

\[
P(L(\{\mu\}, \epsilon), t) = \frac{\dim(\{\mu\}, \epsilon) \prod_{i=1}^{m} H_{\lambda_i}(t^m)}{(1-t)^n R_{\{\lambda\}}(t)}.
\]

(6)

To contradict Lemma 3.3 and hence prove Proposition 3.2 we have

**Lemma 3.4.** Let \(p \neq 1\) and \(W = G(m, p, n)\) with \(W \neq G(2, 2, 3)\). Then there exists \((\{\mu\}, \epsilon) \in \text{Irr}(W)\) such that the right hand side of equation (6) is not an element of \(\mathbb{C}[t]\).

**Proof.** We consider the cases \(n = 2, 3\) and \(n > 3\) separately. For \(n > 3\) choose \((\{\mu\}, \epsilon)\) such that its dual representation is \(\lambda = (\lambda^0, 0, \ldots, 0)\), where \(\lambda^0 = (2, 2, 1, 1, \ldots, 1)\). Then

\[
\tilde{R}(t) = R(t) = 1 + t^{dn} + t^{2dn} + \cdots + t^{(p-1)dn} = \frac{1-t^{mn}}{1-t^{dn}}
\]

and for this particular \(m\)-multipartition we have
\[
\prod_i H_{\lambda_i}(t^m) = H_{\lambda}(t^m) = (1 - t^{2m})(1 - t^m)(1 - t^{(n-1)m})(1 - t^{(n-2)m}) \prod_{i=1}^{n-4} (1 - t^{im})
\]

Equation (6) becomes

\[
P(L(\{\mu\}, \epsilon), t) = \frac{\dim(\{\mu\}, \epsilon)(1 - t^{2m})(1 - t^m)(1 - t^{(n-1)m})(1 - t^{(n-2)m}) \prod_{i=1}^{n-4} (1 - t^{im})}{(1 - t^{im})(1 - t)^n}.
\]

The numerator of (7) factorises over \( \mathbb{C} \) as a product of factors \((1 - \omega t)\), where \( \omega \) is a primitive \( k \)th root of unity with \( 1 \leq k < mn \), whereas the denominator contains at least one factor of the form \((1 - \sigma t)\), where \( \sigma \) is a primitive \( mn \)th root of unity. Therefore, since \( \mathbb{C}[t] \) is an Euclidean domain, the right hand side of (7) cannot lie in \( \mathbb{C}[t] \).

For \( n = 2 \) and \( m \geq n \), take \( \lambda = ((1), (1), \emptyset \ldots \emptyset) \). Then

\[
\prod_i H_{\lambda_i}(t^m) = (1 - t^m)^2 \quad \text{and} \quad R(t) = \frac{t(1 - t^2m)}{1 - t^2d} \quad \text{and} \quad \tilde{R}(t) = \frac{1 - t^{2m}}{1 - t^{2d}}.
\]

Substituting into (6)

\[
P(L(\{\mu\}, \epsilon), t) = \frac{\dim(\{\mu\}, \epsilon)(1 - t^{2m})^2(1 - t^{2d})}{(1 - t^{2m})(1 - t)^2}.
\]

By the same reasoning as above, since \( 2m > 2d, m \), this rational function is not a polynomial.

Similarly, for \( n = 3 \) and \( m \geq n \), take \( \lambda = ((1), (1), (1), \emptyset \ldots \emptyset) \). Then

\[
\prod_i H_{\lambda_i}(t^m) = (1 - t^m)^3 \quad \text{and} \quad R(t) = \frac{t^3(1 - t^3m)}{1 - t^3d} \quad \text{and} \quad \tilde{R}(t) = \frac{1 - t^{3m}}{1 - t^{3d}}.
\]

Substituting into (6)

\[
P(L(\{\mu\}, \epsilon), t) = \frac{\dim(\{\mu\}, \epsilon)(1 - t^m)^3(1 - t^{3d})}{(1 - t^3m)(1 - t)^3}.
\]

Once again, this rational function is not a polynomial because \( 3m > 3d, m \).

Therefore, for all \( W = G(m, p, n), p > 1 \), and with \( W \not\cong G(2, 2, 3) \), we have found an irreducible representation \((\{\mu\}, \epsilon)\) of \( W \) such that the Poincaré polynomial of the corresponding irreducible \( \tilde{H}_0, c(W) \)-module \( L(\{\mu\}, \epsilon) \) cannot be of the form given in Lemma 3.3. Hence the dimension of \( L(\{\mu\}, \epsilon) \) must be less than \( |W| \). Our argument is independent of the parameter \( c \), therefore we have proved Proposition 3.2 in this case.

3.3. The Exceptional Groups. Using the computer algebra program [GAP, GAP] together with the package [CHE, CHEVIE] we calculate for each exceptional complex reflection group \( W \) (excluding \( G_4 \)), the number of irreducible representations \( \lambda \) for which the polynomial \( t^{-b} f_{\lambda^*}(t) \) does not divide \( P(\mathbb{C}[t]/\mathfrak{co}W, t) \)
in \( \mathbb{C}[t] \). Table (3.3) gives the results of these calculations. For each of these \( \lambda \), Lemma 3.3 does not hold and hence \( \dim L(\lambda) < |W| \) for all values of \( c \). Since this number is always positive, Proposition 3.2 is proved for the exceptional groups.

**Table 1.** Number of irreducibles that fail Lemma 3.3

<table>
<thead>
<tr>
<th>Group</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td># failures</td>
<td>3</td>
<td>6</td>
<td>13</td>
<td>2</td>
<td>16</td>
<td>15</td>
<td>43</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>18</td>
<td>15</td>
<td>43</td>
<td>1</td>
<td>18</td>
<td>15</td>
<td>43</td>
<td>1</td>
<td>18</td>
</tr>
</tbody>
</table>

The code used to produce the data in Table (3.3) is available on the author’s website [Be]. For every exceptional group, the fake polynomials of the irreducible characters are listed there. The remainder of \( P(\mathbb{C}[\mathfrak{h}]^{coW}, t) \) on division by \( t - b^* f_\lambda(t) \) is also listed. In addition, this information is available for many of the groups \( G(m, p, n) \) of rank \( \leq 5 \).

4. **The exceptional group \( G_4 \)**

The group \( G_4 \), as labelled in [ST], is the binary tetrahedral group. It can be realised as a finite subgroup of the group of units in the quaternions

\[
G_4 = \{ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k) \}
\]

and has order 24. It is generated by the elements \( s_1 = \frac{1}{2}(-1 + i + j - k) \) and \( s_2 = \frac{1}{2}(-1 + i - j + k) \) and has presentation \( G_4 = \langle s_1, s_2 | s_1^3 = s_2^3 = (s_1s_2)^6 = 1 \rangle \). It has seven conjugacy classes which we label \( Cl_1 = \{ 1 \}, Cl_2, Cl_3, Cl_4, Cl_5, Cl_6, \) and \( Cl_7 \). The character table is

<table>
<thead>
<tr>
<th>Class</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Order</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>( T )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( V_1 )</td>
<td>1</td>
<td>1</td>
<td>( \omega^2 )</td>
<td>( \omega )</td>
<td>1</td>
<td>( \omega^2 )</td>
<td>( \omega )</td>
</tr>
<tr>
<td>( V_2 )</td>
<td>1</td>
<td>1</td>
<td>( \omega )</td>
<td>( \omega^2 )</td>
<td>1</td>
<td>( \omega )</td>
<td>( \omega^2 )</td>
</tr>
<tr>
<td>( W )</td>
<td>2</td>
<td>-2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \mathfrak{h} )</td>
<td>2</td>
<td>-2</td>
<td>-( \omega^2 )</td>
<td>-( \omega )</td>
<td>0</td>
<td>( \omega^2 )</td>
<td>( \omega )</td>
</tr>
<tr>
<td>( \mathfrak{h}^* )</td>
<td>2</td>
<td>-2</td>
<td>-( \omega )</td>
<td>-( \omega^2 )</td>
<td>0</td>
<td>( \omega )</td>
<td>( \omega^2 )</td>
</tr>
<tr>
<td>( U )</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where \( \omega \) is a primitive cube root of unity. Note that the reflection representation \( \mathfrak{h} \) has dimension 2, therefore \( G_4 \) is a rank 2 complex reflection group.

The group \( G_4 \) has two classes which consist of complex reflections and we label these reflections as

\[
Cl_3 = \{ s_1, s_2, s_3, s_4 \}
\]
Theorem 4.1. For generic values of $c$, the generalised Calogero-Moser space $X_c$ associated to $G_4$ is a smooth variety.

Proof. The theorem is proved by showing that each irreducible $H_{0,c}$-module is isomorphic to the regular representation of $G_4$. By [EG, Proposition 3.8], this is equivalent to the statement of the theorem. Let $E = T \oplus V_1 \oplus V_2 \oplus 3U$ and $F = \mathfrak{g} \oplus \mathfrak{h}^* \oplus W$, two $G_4$-modules.

Claim 1
Let $L$ be a finite dimensional $H_{0,c}$-module, then $L \cong aE \oplus bF$, for some $a, b \in \mathbb{Z}_{\geq 0}$.

To prove Claim 1 we use an argument similar to that of [EG, Proposition 16.5]. Let $\rho : H_{0,c} \to \text{End}_C(L)$ realise the action of $H_{0,c}$ on $L$. Then, for all $x, y \in \mathfrak{g} \oplus \mathfrak{h}^*$, we have the commutation relation

$$[\rho(x), \rho(y)] = c_1 \sum_{i=1}^{4} \omega_{s_i}(x, y) \rho(s_i) + c_2 \sum_{j=1}^{4} \omega_{t_j}(x, y) \rho(t_j)$$

By Lemma 2.1, $\sum_{i=1}^{4} \omega_{s_i} = \sum_{j=1}^{4} \omega_{t_j} = 2\omega$. Taking traces on both sides of equation (8)

$$0 = c_1 2\omega(x, y)Tr_{L}(s_1) + c_2 2\omega(x, y)Tr_{L}(t_1) \quad \forall x, y \in \mathfrak{g} \oplus \mathfrak{h}^*$$

Since $c_1$ and $c_2$ are generic i.e. take values in a dense open subset of $\mathbb{C}^2$, and equation (9) is linear, we have $0 = 2\omega(x, y)Tr_{L}(s_1) = 2\omega(x, y)Tr_{L}(t_1)$. The fact that $\omega$ is nondegenerate implies that $Tr_{L}$ is zero on $Cl_3$ and $Cl_4$.

Using the fact that $s_1$ is a complex reflection and $\dim \mathfrak{h}^* = 2$, we can choose a nonzero $x_1 \in \mathfrak{h}^*$ such that $s_1(x_1) = x_1$. Then $s_1[x_1, y] = [x_1, s_1(y)]$ for all $y \in \mathfrak{g}$. Since $s_1(x_1) = x_1$, $x_1 \in \text{Ker}(1 - s_1)$ and hence $\omega_{s_i}(x_1, y) = 0$ for all $y \in \mathfrak{g}$. Similarly, $s_1 t_1 = 1$ implies that $x_1 \in \text{Fix}(t_1)$ and hence $\omega_{t_1}(x_1, y) = 0$. Therefore, multiplying both sides of equation (8) on the left by $\rho(s_1)$ and taking traces

$$0 = c_1 \sum_{i=2}^{4} \omega_{s_i}(x_1, y)Tr_{L}(s_1 s_i) + c_2 \sum_{j=2}^{4} \omega_{t_j}(x_1, y)Tr_{L}(s_1 t_j)$$

Unlike all other exceptional irreducible complex reflection groups we have

$$= \{ \frac{1}{2}(-1 + i + j - k), \frac{1}{2}(-1 + i - j + k), \frac{1}{2}(-1 - i + j + k), \frac{1}{2}(-1 - i - j - k) \}$$

and

$$Cl_4 = \{ t_1, t_2, t_3, t_4 \}$$

$$= \{ \frac{1}{2}(-1 - i - j + k), \frac{1}{2}(-1 + i - j - k), \frac{1}{2}(-1 - i + j - k), \frac{1}{2}(-1 + i + j + k) \}$$
Again, using the fact that $c_1, c_2$ are generic, we get
\[ 0 = \sum_{i=2}^{4} \omega_{s_i}(x_1, y) Tr_L(s_1 s_i) = \sum_{j=2}^{4} \omega_{t_j}(x_1, y) Tr_L(s_1 t_j) \]

Since $s_1 s_2, s_1 s_3$ and $s_1 s_4$ all belong to $Cl_7$ and $s_1 t_2, s_1 t_3, s_1 t_4$ all belong to $Cl_5$ we have
\[ 0 = \sum_{i=2}^{4} \omega_{s_i}(x_1, y) Tr_L(s_1 s_i) = 2\omega(x_1, y) Tr_L(s_1 s_2) \]
\[ 0 = \sum_{j=2}^{4} \omega_{t_j}(x_1, y) Tr_L(s_1 t_j) = 2\omega(x_1, y) Tr_L(s_1 t_2) \]

Therefore $Tr_L$ is zero on $Cl_7$ and $Cl_5$.

We can also multiplying both sides of equation (8) on the left by $\rho(t_1)$ instead of $\rho(s_1)$. Noting that $t_1^2 \in Cl_3, t_1 t_2, t_1 t_3, t_1 t_4 \in Cl_6$ and repeating the above argument shows that $Tr_L$ is also zero on $Cl_6$.

Therefore any element of $G_4$ that has nonzero trace on $L$ must belong to $Cl_1$ or $Cl_2$. Hence the character associated to $L$ must take values $(n, m, 0, 0, 0, 0, 0)$, for some $n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}$, on the classes $Cl_1, Cl_2, \ldots, Cl_7$.

Taking inner products shows that
\[ L \cong \frac{1}{|G_4|} (n + m) E \oplus \frac{2}{|G_4|} (n - m) F \]

Setting $a = \frac{1}{|G_4|} (n + m)$ and $b = \frac{2}{|G_4|} (n - m)$ proves Claim 1.

**Claim 2**

Let $L$ be an irreducible representation of $H_{0,e}$, with $e$ generic. Then $L$ must be isomorphic to $E \oplus F$ or $CG_4$ as a $G_4$-module.

If $L$ is irreducible then $\dim L \leq 24$. Therefore Claim 1 implies that $L \cong E, 2E, nF, 1 \leq n \leq 4, E \oplus F$ or $CG_4$. Assume that $L$ is isomorphic to $E$ as a $G_4$-module. The action of $h^*$ on $L$ defines a linear map $\phi : h^* \rightarrow \text{End}_C(E)$. For $w \in G_4$ and $x \in h^*$, $wxw^{-1} = wx$ in $H_{0,e}$. Therefore $\phi(wx)(e) = wx^e = wxw^{-1}e = w(x)(w^{-1}e) = w(\phi(x)(w^{-1}e))$. The action of $w \in G_4$ on $f \in \text{End}_C(E)$ is defined by $(wf)(e) = w(f(w^{-1}e))$. Therefore the map $\phi : h^* \rightarrow \text{End}_C(E)$ is $G_4$-equivariant. The $G_4$-module $\text{End}_C(E)$ decomposes as

\[ \text{End}_C(E) \cong (T \otimes T) \oplus 2(T \otimes V_1) \oplus 2(T \otimes V_2) \oplus 6(T \otimes U) \oplus (V_1 \otimes V_1) \oplus 2(V_1 \otimes V_2) \oplus 6(V_1 \otimes U) \oplus (V_2 \otimes V_2) \oplus 6(V_2 \otimes U) \oplus 9(U \otimes U) \cong 12T \oplus 12V_1 \oplus 12V_2 \oplus 36U \]

This shows that $h^*$ is not a summand of $\text{End}_C(E)$. Therefore $\phi$ must be the zero map. Similarly, the action of $h^*$ must also be zero on $E$. Therefore the right hand side of equation (8) must also act as zero on $E$. In particular, it must act as zero on $T \subset E$. This means that
\[
0 = c_1 \sum_{i=1}^{4} \omega_{s_i}(x,y) + c_2 \sum_{j=1}^{4} \omega_{t_j}(x,y) = 2(c_1 + c_2)\omega(x,y)
\]

This is a contradiction because \(c_1, c_2\) are generic and \(\omega\) is nondegenerate. Hence \(L\) cannot be isomorphic to \(E\). Repeating the above argument for \(F\) we have

\[
\text{End}_{\mathbb{C}}(F) \cong (\mathfrak{h} \otimes \mathfrak{h}) \oplus 2(\mathfrak{h} \otimes W) \oplus \\
(\mathfrak{h}^* \otimes \mathfrak{h}^*) \oplus 2(\mathfrak{h}^* \otimes W) \oplus (W \otimes W) \cong 3T \oplus 3V_1 \oplus 3V_2 \oplus 9U
\]

Therefore \(\mathfrak{h}^*\) and \(\mathfrak{h}\) must act as zero on \(F\). If we consider the right hand side of equation (8), this time restricted to \(W \subset F\) then we have

\[
0 = c_1 \sum_{i=1}^{4} \omega_{s_i}(x,y)\rho|_{W}(s_i) + c_2 \sum_{j=1}^{4} \omega_{t_j}(x,y)\rho|_{W}(t_j)
\]

Taking the trace of this equation gives \(0 = -2(c_1 + c_2)\omega(x,y)\), which is a contradiction because \(c_1, c_2\) are generic and \(\omega\) is nondegenerate. Therefore \(L \not\cong F\). The same reasoning shows that \(L\) cannot be isomorphic to \(2E\) or \(nF, 2 \leq n \leq 4\) either. This proves Claim 2.

**Claim 3**

Let \(L\) be an irreducible \(H_{0,c}\)-module. Then \(L\) cannot be isomorphic to \(E \oplus F\) as a \(G_4\)-module.

By Lemma 2.3, there exists a generalised Verma module \(\Delta_c(M; p, q)\) and a surjective homomorphism \(\phi : \Delta_c(M; p, q) \to L\). As a \(G_4\)-module we have

\[
\Delta_c(M; p, q) = H_{0,c}(W) \otimes \mathbb{C}[\mathfrak{h}]^w \otimes \mathbb{C}[\mathfrak{h}^*] \rtimes W_q M \cong \mathbb{C}G_4 \otimes \operatorname{Ind}_{(G_4)_q}^{G_4} M \cong kCG_4
\]

where \((G_4)_q\) is the stabiliser of \(q \in \mathfrak{h}^*\) and \(k = [G_4 : (G_4)_q]\dim M\). The generalised Verma module \(\Delta_c(M; p, q)\) has a finite composition series. Each factor of this series must have dimension \(\leq 24\). Therefore, by Claim 2, each factor is isomorphic to either \(CG_4\) or \(E \oplus F\) as a \(G_4\)-module. Hence there exist \(m, n \in \mathbb{N}\) such that \(kCG_4 \cong mCG_4 \oplus n(E \oplus F)\) with \(n \geq 1\). But then \(n(E \oplus F) \cong (k - m)CG_4\), which is a contradiction. This completes the proof of Claim 3 and the theorem.

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References


School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, James Clerk Maxwell Building, Kings Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland
E-mail address: G.E.Bellamy@sms.ed.ac.uk