Solutions: The Platonic solids

1. If the Schl"afli symbol of the Platonic solid $P$ is $\{p,q\}$, use Euler’s formula $V - E + F = 2$ to show that

$$V = \frac{4p}{4 - (p-2)(q-2)}, \quad E = \frac{2pq}{4 - (p-2)(q-2)}, \quad F = \frac{4q}{4 - (p-2)(q-2)},$$

where $V$, $E$ and $F$ are the number of vertices, edges and faces respectively of $P$.

Solution. By definition, there are $p$ faces meeting at each vertex, each of which is a $q$-gon. Therefore, each vertex has $p$ edges connected to it and every edge connects 2 vertices implying that $2E = pV$. Similarly, each face has $q$ edges and every edge belongs to 2 faces. Therefore $2E = qF$. This gives us three equations

$$V - E + F = 2, \quad 2E = pV, \quad 2E = qF.$$

Solving for $V, E$ and $F$ give the required equations.

2. Recall from the first lecture that a reflection on $\mathbb{R}^n$ is an orthogonal transformation $s \in O(\mathbb{R}, n)$ such that $\dim \text{Fix}_{\mathbb{R}^n}(s) = n - 1$ and $s^2 = \text{id}$.

(a) Show that $\text{Fix}_{\mathbb{R}^n}(s) = \text{Ker}(\text{id} - s)$.

(b) Prove that $s$ is diagonalizable. What are the eigenvalues of $s$?

(c) Deduce that $\det(s) = -1$.

(d) Choose $\alpha$ such that $\text{Fix}_{\mathbb{R}^n}(s) = H_\alpha$. Derive the formula

$$s_\alpha(x) = x - \frac{2(x, \alpha)}{(\alpha, \alpha)} \alpha$$

for a reflection.

Hint: For part (b), if $H := \text{Ker}(\text{id} - s)$, consider the space $H^\perp$. Show that $s$ acts on $H^\perp$.

Solution. (a) Recall that $\text{Fix}_{\mathbb{R}^n}(s) = \{v \in \mathbb{R}^n \mid s(v) = v\}$. But $s(v) = v$ if and only if $(\text{id} - s)(v) = 0$ i.e. if and only if $v \in \text{Ker}(\text{id} - s)$.

(b) If $\alpha \in H^\perp$ then $(y, s(\alpha)) = (s(y), s(\alpha)) = (y, \alpha) = 0$ for all $y \in H$. This implies that $s(\alpha) \in H^\perp$. Hence $s$ acts on $H^\perp$. Now $H \cap H^\perp = 0$ and $\dim H + \dim H^\perp = n$. Therefore, $H^\perp$ is a line perpendicular to $H$. This means that $s(\alpha) = \lambda \alpha$ for some $\lambda$. 

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Since $s^2 = 1$, $\lambda = \pm 1$. But $\lambda = 1$ would imply that $\alpha \in H$; a contradiction. Thus, $\lambda = -1$. Therefore if $x_1, \ldots, x_{n-1}$ is a basis of $H$, then $x_1, \ldots, x_{n-1}, \alpha$ is a basis of $\mathbb{R}^n$ with respect to which $s$ is diagonal, and the eigenvalues of $s$ are $1, -1$, where $-1$ occurs with multiplicity one.

(c) The determinant of $s$ is just the product of its eigenvalues, which is clearly $-1$.

(d) The reflection $s$ is the unique orthogonal map $\mathbb{R}^n \to \mathbb{R}^n$ such that $s(x) = x$ for all $x \in H = \{x \in \mathbb{R}^n \mid (x, \alpha) = 0\}$ and $s$ maps $\alpha$ to its negative. Therefore it suffices to check that the map $L(x) = x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha$ has both of these properties. This is clear.

3. There is a purely topological proof of the fact that there are are only five Platonic solids. The key topological fact is that Euler’s formula holds: $V - E + F = 2$. Using this, together with the relations $pF = 2E = qV$, show that

\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}.
\]

Deduce that there are only five Platonic solids.

Solution. First divide $V - E + F = 2$ through by $2E$ and then substitute in $pF = 2E = qV$ to get the equation. Since $\frac{1}{E} > 0$, we deduce that

\[
\frac{1}{p} + \frac{1}{q} > \frac{1}{2}.
\] (1)

Therefore we are looking for pairs $\{p, q\}$, with $p, q \geq 3$ (clear $p$ or $q$ equal to 2 cannot produce a Platonic solid) such that the inequality (1) holds. Listing small examples quickly shows that this only happens for $\{p, q\} = \{3, 3\}, \{3, 4\}, \{3, 5\}, \{4, 3\}$ or $\{5, 3\}$.

4. Using the fact that $g \in W(P)$ is a reflection if and only if it has one eigenvalue equal to $-1$ and two eigenvalues equal to 1, count the number of reflections in $W(H)$ and $W(D)$.

Solution. For both the cube $H$ and the dodecahedron $D$, we have seen that $W(P) = W_0(P) \times \{\pm \text{Id}\}$. Therefore, since $W_0(P)$ consists of rotations, each reflection must be of the form $(r, -1)$ for some rotation. Moreover, we have seen in exercise 2. that a reflection has one eigenvalue which is $-1$ and two that equal 1. Thus, $r$ must be a rotation with eigenvalues $1, -1, -1$ i.e. $r$ must be a rotation of order 2 (or a rotation of $\pi$ about its axis of rotation).

Looking at the list (A), (B) and (C) for the cube, we see that there are $3 + 6 + 0 = 9$ such rotations. Hence $W(H)$ contains 9 reflections.

Similarly, for dodecahedron $D$, we have $0 + 15 + 0 = 15$ reflections in total.
Solutions: Reflection groups and root systems

1. Let

$$E = \left\{ x = \sum_{i=1}^{n+1} x_i \epsilon_i \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \right\},$$

where \(\{\epsilon_1, \ldots, \epsilon_{n+1}\}\) is the standard basis of \(\mathbb{R}^{n+1}\) with \((\epsilon_i, \epsilon_j) = \delta_{i,j}\). Let \(R = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n + 1\}\).

(i) Show that \(R\) is a crystallographic root system.

(ii) By considering the action of the reflections \(s_{\epsilon_i - \epsilon_j}\) on the basis \(\{\epsilon_1, \ldots, \epsilon_{n+1}\}\) of \(\mathbb{R}^{n+1}\), show that the Weyl group of \(R\) is isomorphic to \(S_{n+1}\).

(iii) Construct two different sets of simple roots for \(R\).

Solution. (i) It is clear that the only multiples of a root \(\alpha\) in \(R\) are \(\pm \alpha\). A case by case check shows that \(s_{\epsilon_i - \epsilon_j}(\epsilon_k - \epsilon_l) \in R\) for all \(i, j, k, l\). Finally,

$$\langle \epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l \rangle = \frac{2(\epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l)}{\langle \epsilon_i - \epsilon_j, \epsilon_k - \epsilon_l \rangle} = \begin{cases} \pm 2 & \{i, j\} = \{k, l\}, \\ \pm 1 & |\{i, j\} \cap \{k, l\}| = 1, \\ 0 & \text{otherwise} \end{cases}$$

(ii) A direct calculation shows that

$$s_{\epsilon_i - \epsilon_j}(\epsilon_k) = \begin{cases} \epsilon_j & k = i, \\ \epsilon_i & k = j, \\ \epsilon_k & \text{otherwise} \end{cases}$$

Therefore the reflection group \(W\) is isomorphic to the subgroup of \(S_{n+1}\) generated by all transpositions \((i, j)\), but this is clearly the whole of \(S_{n+1}\).

(iii) An obvious set of simple roots is \(\Delta = \{\epsilon_i - \epsilon_{i+1} \mid i = 1, \ldots, n\}\). To get another set of simple roots we can take any element of \(S_{n+1}\) and apply it to this set. For instance, if we take the transposition swapping \(\epsilon_2\) and \(\epsilon_3\) then

$$\{\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_2, \epsilon_2 - \epsilon_4\} \cup \{\epsilon_i - \epsilon_{i+1} \mid i = 4, \ldots, n\}$$

is another set of simple roots.
2. Show that the symmetric matrix

\[
A = \begin{pmatrix}
1 & -\cos \frac{\pi}{5} & 0 \\
-\cos \frac{\pi}{5} & 1 & -\cos \frac{\pi}{3} \\
0 & -\cos \frac{\pi}{3} & 1
\end{pmatrix}
\]

corresponding to the Coxeter graph \[\begin{array}{c}
\bullet \\
5
\end{array}\] of type \(H_3\) is positive definite. What is the determinant of \(A\)? Hint: recall that \(\cos \frac{\pi}{3} = \frac{1}{2}\) and \(\cos \frac{\pi}{5} = \frac{1+\sqrt{5}}{4}\).

**Solution.** We need to show that all the eigenvalues of \(A\) are strictly positive real numbers. The characteristic polynomial of \(A\) is

\[
(1-t) \left( (1-t)^2 - \frac{1}{4} - \left( \frac{1+\sqrt{5}}{4} \right)^2 \right)
\]

and hence the eigenvalues of \(A\) are \(1, 1 \pm \frac{1}{4} \sqrt{10+2\sqrt{5}}\). Notice that \(1 - \frac{1}{4} \sqrt{10+2\sqrt{5}} > 0\). The determinant of \(A\) is the product of eigenvalues, which is \(\frac{3-\sqrt{5}}{8}\).

3. The angle between roots in a crystallographic reflection groups. Recall the following table in section 2.5 of the lecture notes. The only possible values of \(\langle \alpha, \beta \rangle\) are:

<table>
<thead>
<tr>
<th>(\langle \beta, \alpha \rangle)</th>
<th>(\langle \alpha, \beta \rangle)</th>
<th>(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(\frac{\pi}{2})</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(\ast)</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>(\frac{2\pi}{3})</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(\ast\ast)</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>(\frac{3\pi}{4})</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>(\frac{\pi}{6})</td>
</tr>
<tr>
<td>-1</td>
<td>-3</td>
<td>(\ast\ast\ast)</td>
</tr>
</tbody>
</table>

(a) What are the angles \(\theta\) in \((\ast)\), \((\ast\ast)\) and \((\ast\ast\ast)\)?

(b) What about \(\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle = \pm 2\)?

(c) Let \(\alpha, \beta \in \mathbb{R}^n\). Show that \(s_\beta s_\alpha\) is a rotation of \(\mathbb{R}^n\). Hint: decompose \(\mathbb{R}^n = \mathbb{R}\{\alpha, \beta\} \oplus H_\alpha \cap H_\beta\) and consider \(s_\beta s_\alpha\) acting on \(\mathbb{R}\{\alpha, \beta\}\). If \(e_1, e_2\) is an orthonormal basis of \(\mathbb{R}^2\), write out \(s_\alpha\) and \(s_\beta\) explicit.

**Solution.** (a) Recall that \(\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4 \cos^2 \theta\). Therefore, in \((\ast)\) we have \(\cos^2 \theta = \frac{1}{4}\) and...
\[ \theta = \frac{\pi}{3} \text{. In (⋆⋆), we have } \cos^2 \theta = \frac{1}{2} \text{ and hence } \theta = \frac{\pi}{4} \text{. Finally, in (⋆⋆⋆), we have } \cos^2 \theta = \frac{3}{4} \text{ and hence } \theta = \frac{\pi}{6} \text{.} \]

(b) In this case we would have \( \theta = 0 \) and hence \( \beta = \pm \alpha \).

(c) The hint implies that it is enough to assume that \( n = 2 \) and \( \mathbb{R}^2 \) is spanned by \( \alpha, \beta \). Therefore we are given reflections in \( \mathbb{R}^2 \) perpendicular to \( \alpha \) and \( \beta \). We may assume without loss of generality that \( \alpha \) and \( \beta \) have length one. Moreover, after an orthonormal change of basis we may assume \( \alpha = (1, 0) \). If \( R_\theta \) denotes rotation anticlockwise by \( \theta \) then

\[
R_\theta = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]

To see that this is correct, simply draw the picture with \((1, 0), (0, 1), R_\theta(1, 0)\) and \( R_\theta(0, 1) \). If you don’t want to do a very long explicit computation (or like me, you have forgotten all your trig identities) then here is a quick and easy argument. The key is to notice that if \( \theta \) is the angle between \( \alpha \) and \( \beta \) then to reflect a vector \( v \) in the line perpendicular to \( \beta \), one can first rotate \( v \) (and \( \beta \)) by \( -\theta \), then reflect perpendicular to \( \alpha \) and then rotate back by \( \theta \). Thus, \( s_\beta = R_\theta \circ s_\alpha \circ R_{-\theta} \) and hence \( s_\beta \circ s_\alpha = R_\theta \circ s_\alpha \circ R_{-\theta} \circ s_\alpha \).

We calculate

\[
s_\alpha \circ R_{-\theta} \circ s_\alpha = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} = R_\theta.
\]

Hence

\[
s_\beta \circ s_\alpha = R_\theta \circ s_\alpha \circ R_{-\theta} \circ s_\alpha = R_\theta \circ R_\theta = R_{2\theta}.
\]

4. The **hypercube** \( H_n \) is the \( n \)-dimensional analogue of the square \((n = 2)\), or cube \((n = 3)\). Concretely, we can realize \( H_n \) in \( \mathbb{R}^n \) as the set of points

\[
H_n = \{ v \in \mathbb{R}^n \mid -1 \leq v_i \leq 1 \text{ for } i = 1, \ldots, n \}.
\]

The group of symmetries of \( H_n \) is denoted \( BC_n \). It is called the **hyperoctahedral group**.

(i) How many vertices does the \( H_n \) have? How about edges, or faces?

(ii) The \((n - 1)\)-dimensional faces of \( H_n \) are the copies \( F_i^{\pm} \) of \( H_{n-1} \) given by \( \{v \in H_n \mid v_i = \pm 1\} \). Since \( BC_n \) permutes these \((n - 1)\)-dimensional faces, it will permute their mid-
points \( \{e_i^\pm | i = 1, \ldots, n\} \), where
\[
e_i^\pm = (0, \ldots, 0, \pm 1, 0, \ldots, 0).
\]

Deduce that \( w \) is a sign permutation matrix i.e. a matrix where each row has only one non-zero entry which is either a 1 or \(-1\), and similarly for the columns.

(iii) What is the order of the group \( BC_n \)?

(iv) The hyperoctahedron is dual to the hypercube. It is defined to be
\[
O_n = \{ x \in \mathbb{R}^n \mid (x, v) \leq 1 \text{ for all vertices } v \text{ of } H_n \}.
\]

Check for \( n = 2 \) and \( n = 3 \) that one gets the (rotated by \( \frac{\pi}{4} \)) square and octahedron respectively.

(v) Show directly from the definition that the symmetries of \( H_n \) are also symmetries of \( O_n \).

This shows that \( W(H_n) \subset W(O_n) \).

(vi) Notice that the \( e_i^\pm \) are the vertices of \( O_n \). Deduce that \( W(O_n) = BC_n \).

Solution. (i) The vertices of \( H_n \) are the points on \( \mathbb{R}^n \) with all entries either 1 or \(-1\). There are \( 2^n \) of them. There is an edge between two vertices if the two vertices only differ in one entry (where one vertex has a 1 and the other a \(-1\)). Hence there are \( n \) edges connected to each vertex. Since each edge connects two vertices, this implies that there are \( n2^{n-1} \) edges. Similarly, each face has four edges and each edge is part of two faces. Therefore there are \( n2^{n-2} \) faces to \( H_n \).

(ii) If \( w(e_i^+) = e_j^\pm \), then the \( i \)th column of \( w \), thought of as a matrix, has all zeros except in the \( i \)th entry were it has a \( \pm 1 \). Since \( w \) is invertible, its rows must have the same property.

(iii) If \( P_n \) is the group of all signed permutation matrices then we have shown that \( BC_n \subset P_n \).

On the other hand, it is clear that any element of \( P_n \) preserves \( H_n \). Therefore \( BC_n = P_n \).

This implies that \( |BC_n| = 2^n n! \).

(iv) The case \( n = 2 \) is clear from drawing a picture. For \( n = 3 \) it suffices to note that the vertices \( (1, 1, 1), (1, -1, 1), (-1, 1, 1), (-1, -1, 1), (1, 1, -1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1) \) and \( (-1, -1, -1) \) of \( H_3 \) span the line through the midpoints of the faces of the octahedron. Therefore, the face with midpoint on the line \( \mathbb{R}_{\geq 0} \cdot v \) is defined by the equation \( (x, v) = 1 \).

It follows that the octahedron is the intersection of the half-spaces \( (x, v) \leq 1 \).
(v) If \( w \in W(H_n) \) then \( w(v) \) is a vertex of \( H_n \) for any vertex \( v \). Thus, \( (x, v) \leq 1 \) if and only if \( (w(x), w(v)) \leq 1 \), which implies that \( w(O_n) \subseteq O_n \).

(vi) Since \( \{ e_i^\pm \mid i = 1, \ldots, n \} \) are the vertices of \( O_n \), the group \( W(O_n) \) acts faithfully on this set. But we have already shown in (ii) that any element \( w \in GL(\mathbb{R}^n) \) that maps this set to itself must be a signed permutation matrix i.e. \( w \in BC_n \). Thus, \( W(O_n) = BC_n \).
Solutions: Quivers

1. A homomorphism between representations. Let \( M = \{ (\mathbb{C}^v, \varphi_\alpha) \} \) and \( N = \{ (\mathbb{C}^w, \psi_\alpha) \} \) be representations of a quiver \( Q \). Then a homomorphism \( f : M \to N \) is a collection of linear maps \( f_i \in \text{Hom}_\mathbb{C}(\mathbb{C}^v, \mathbb{C}^w) \) for each \( i \in Q_0 \) such that the diagrams

\[
\begin{array}{ccc}
\mathbb{C}^{v_{t(\alpha)}} & \xrightarrow{\varphi_\alpha} & \mathbb{C}^{v_{h(\alpha)}} \\
\downarrow f_{t(\alpha)} & & \downarrow f_{h(\alpha)} \\
\mathbb{C}^{w_{t(\alpha)}} & \xrightarrow{\psi_\alpha} & \mathbb{C}^{w_{h(\alpha)}}
\end{array}
\]

commute for all \( \alpha \in Q_1 \). The space of all homomorphisms from \( M \) to \( N \) is denoted \( \text{Hom}_Q(M, N) \).

(a) Consider the representations

\[
M : \quad \begin{array}{c}
\mathbb{C}^2 \\
\downarrow (a,b) \\
\mathbb{C}
\end{array} \quad \xrightarrow{x} \quad \begin{array}{c}
\mathbb{C}
\downarrow (c,d) \\
\mathbb{C}
\end{array} \quad \text{and} \quad \begin{array}{c}
\mathbb{C} \\
y \end{array}
\]

where \( a, b, c, d, x, y \in \mathbb{C} \). If \( (a,b) = (2,1) \), \( (c,d) = (6,3) \), \( x = 1 \) and \( y = 3 \), construct a non-zero homomorphism \( f : M \to N \). Are there any homomorphisms \( f : M \to N \) when \( (a,b) = (2,2) \), \( (c,d) = (6,4) \), \( x = 2 \) and \( y = 2 \) ? In general, what conditions do \( a, b, c, d, x, y \) need to satisfy for \( \text{Hom}_Q(M, N) \) to be non-zero? What is the dimension of \( \text{Hom}_Q(M, N) \) in this case?

(b) Recall that the representations of the quiver \( e_1 \xrightarrow{\alpha} \) are simply pairs \( (\mathbb{C}^n, A) \),

where \( A : \mathbb{C}^n \to \mathbb{C}^n \) is an \( n \times n \) matrix. If \( M = (\mathbb{C}^n, A) \), show that \( \text{Hom}_Q(M, M) = \{ B : \mathbb{C}^n \to \mathbb{C}^n \mid [A, B] = 0 \} \), where \( [A, B] := AB - BA \) is the commutator of \( A \) and \( B \).

Solution. (a) When \( (a,b) = (2,1) \), \( (c,d) = (6,3) \), \( x = 1 \) and \( y = 3 \), we define \( f : M \to N \) by \( f_1 = (2,1) : \mathbb{C}^2 \to \mathbb{C} \) and \( f_2 = 1 : \mathbb{C} \to \mathbb{C} \). Then

\[
x \circ f_1 = f_2 \circ (a,b), \quad y \circ f_1 = f_2 \circ (c,d).
\]

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Write \( f_1 = (u, v) \) and \( f_2 = w \). In general, the conditions one must satisfy are
\[
(xu, xv) = (wa, wb), \quad (yu, yv) = (wc, wd).
\]

Unless \((u, v) = (0, 0)\) and \( w = 0 \), it is not possible to satisfy these conditions when \((a, b) = (2, 2), (c, d) = (6, 4), x = 2 \) and \( y = 2 \). There are several cases to consider in general, depending on whether \( x \) or \( y \) is non-zero etc. The generic situation is where \( x, y, a, b, c, d \) are all non-zero. In this case \( \text{Hom}_Q(M, N) \neq 0 \) if and only if \( \frac{a}{b} = \frac{c}{d} \) and \( \frac{x}{y} = \frac{a}{c} \). When these conditions are satisfied, one can choose \( w \) freely, which forces \( u = wax^{-1} \) and \( v = wbx^{-1} \). Hence \( \text{dim} \text{Hom}_Q(M, N) = 1 \).

(b) Let \( B : \mathbb{C}^n \to \mathbb{C}^n \) be an element of \( \text{Hom}_Q(M, M) \). Then the following diagram must commute
\[
\begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n \\
\downarrow{B} & & \downarrow{B} \\
\mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n
\end{array}
\]

This means that \( B \circ A = A \circ B \) i.e. \([A, B] = 0\).

2. Let \( Q \) be a quiver. Recall that, for each \( i \in Q_0 \), we have defined the representation \( E(i) \) of \( Q \).

(a) Show that the representation \( E(i) \) is simple.

(b) If \( Q \) has no oriented cycles, show that every simple representation equals \( E(i) \) for some \( i \in Q_0 \).

(c) Consider the quiver \( e_1 \xleftarrow{\alpha} e_2 \). Show that the representation \( M : \mathbb{C} \xrightarrow{3} \mathbb{C} \) is simple.

Solution. (a) Notice that \( \text{dim} E(i) = 0 \). Any subrepresentation of \( E(i) \) must in particular be a subspace too. But the only proper subspace of a one-dimensional space is \( \{0\} \). Thus, \( E(i) \) is simple.

(b) If \( Q \) has no oriented cycles then we can relabel the vertices \( Q_0 = \{1, \ldots, k\} \) of \( Q \) such that \( h(\alpha) > t(\alpha) \) for all \( \alpha \in Q_1 \). Now let \( S = \{(S_i, \varphi_\alpha)\} \) be a simple representation of \( Q \). Then there exists a smallest integer \( i \in Q_0 \) such that \( S_i \neq 0 \). Let \( S' = \{(S'_j, \psi_\alpha)\} \) be the subrepresentation of \( S \) such that \( S'_j = 0 \) if \( j \neq i \) and \( S'_i = S_i \), and \( \psi_\alpha = 0 \) for all \( \alpha \). This is well-defined because all \( \varphi_\alpha \) such that \( h(\alpha) = i \) must be zero, since \( S_{t(\alpha)} = 0 \) for all such \( \alpha \). Since \( S \) is simple \( S' = S \), and \( S_j = 0 \) for all \( j \neq 0 \). Now,
since all maps $\varphi_\alpha$ are zero, if $\dim S_i > 1$, then any one-dimensional subspace of $S_i$ is a proper subrepresentation. This cannot happen since $S$ is simple. Thus, $\dim S_i = 1$ and $S = E(i)$.

(c) Since $\dim M = (1, 1)$, a proper subrepresentation $S$ of $M$ must have dimension vector $(1, 0)$ or $(0, 1)$. However, if we take $S$ with dimension vector $(1, 0)$ say then this is not a subrepresentation since the map of multiplication by 3, $3 : \mathbb{C} \to 0$ is not well-defined. Similarly, if we consider $(0, 1)$.

3. Let $Q$ be the quiver

$$
e_1 \xrightarrow{\beta} \ne_5 \xrightarrow{\gamma} \ne_3$$

Write down the basis of paths for the path algebra $\mathbb{C}Q$. What is $\dim \mathbb{C}Q$?

Solution. The path algebra $\mathbb{C}Q$ has basis $\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha \gamma, \alpha \delta, \beta \gamma, \beta \delta\}$. Therefore $\dim \mathbb{C}Q = 13$.

4. Let $Q$ be the quiver $e_1 \xrightarrow{\alpha} e_2 \xrightarrow{\beta} e_3$ and let

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{C} \right\}$$

be the algebra of upper triangular $3 \times 3$ matrices, where multiplication is just the usual matrix multiplication. Construct an explicit isomorphism of algebras $\mathbb{C}Q \sim A$.

Solution. The path algebra $\mathbb{C}Q$ has basis $\{e_1, e_2, e_3, \alpha, \beta, \alpha \beta\}$. If we write $E_{i,j}$ for the matrix with all entries zero except for a 1 in the $(i, j)$th entry, then $A$ has a basis

$$\{E_{1,1}, E_{1,2}, E_{1,3}, E_{2,2}, E_{2,3}, E_{3,3}\}.$$ 

We leave it to the reader to check that $e_i \mapsto E_{i,i}$, $\alpha \mapsto E_{1,2}$, $\beta \mapsto E_{2,3}$ and hence $\alpha \beta \mapsto E_{1,2}E_{2,3} = E_{1,3}$ is the required isomorphism.
**Solutions: Gabriel’s Theorem**

1. You’ll notice that the positive definite Euler graphs are precisely the positive definite Coxeter graphs that are *simply laced* i.e. have at most one edge between any two vertices. Let $(-,-)_C$, resp. $(-,-)_E$, be the Coxeter form, resp. the Euler form, associated to a graph $\Gamma$.

(a) Show that if $\Gamma$ is simply laced then $(-,-)_E = 2(-,-)_C$.

(b) If $\Gamma$ is not simply laced, show that there is no $\lambda \in \mathbb{R}$ such that $(-,-)_E = \lambda (-,-)_C$.

(c) Show that the symmetric matrix

$$
\begin{pmatrix}
2 & -m \\
-m & 2
\end{pmatrix}
$$

corresponding to the Euler graph $\bullet \overset{m}{\rightarrow} \bullet$ is positive definite if and only if $m = 1$. When is it positive semi-definite?

(d) By considering the subgraphs $\bullet \overset{m}{\rightarrow} \bullet$ with $m > 1$ of $\Gamma$, show that a non-simply laced Euler graph is not positive definite.

(e) Deduce Theorem 4.8 from Theorem 2.18.

**Solution.** (a) Let $A$ be the symmetric matrix representing $(-,-)_C$ and $B$ the symmetric matrix representing $(-,-)_E$. Notice that the diagonal entries of $A$ are all 1 and the diagonal entries of $B$ are all 2. If $\Gamma$ is simply laced then every non-zero off-diagonal entry of $A$ is $-\cos \frac{\pi}{3} = -\frac{1}{2}$. Similarly, every non-zero off-diagonal entry of $B$ is $-1$. The off-diagonal entries of $A$ are non-zero if and only if there is an edge between the corresponding vertices. The same is true for $B$. Thus, $B = 2A$.

(b) By comparing diagonal entries again, we must have $\lambda = 2$. But then if vertices $i$ and $j$ are connected by an edge labeled by $m > 3$, we have $a_{i,j} = -\cos \frac{\pi}{m}$ and $b_{i,j} = -m$. Clearly $-m \neq -2 \cos \frac{\pi}{m}$.

(c) We calculate the eigenvalues of the matrix. The characteristic polynomial is $(2-t)^2 - m^2$, which implies that $t = 2 \pm m$. Since $m \geq 1$, $2 - m > 0$ if and only if $m = 1$. When $m = 2$, $2 - m = 0$ and hence the symmetric matrix is positive semi-definite. For $m > 2$ the symmetric form is not positive semi-definite.

(d) Let $\Gamma$ be non-simply laced. Then there exists vertices $i$ and $j$ such that there are $m > 1$ edges between $i$ and $j$. We let $v = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0 \ldots)$ be the vector with 1
in the $i$ and $j$th positions and all other entries zero. If $A$ is the symmetric matrix corresponding to $\Gamma$, then

$$v^T Av = (1, 1) \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4 - 2m < 0.$$ 

Thus, $A$ is not positive (semi)-definite.

(c) We have seen that if $\Gamma$ is not simply laced then $(-,-)_E$ is not positive definite. On the other hand, when $\Gamma$ is simply laced, $(-,-)_E$ will be positive definite if and only if $(-,-)_C$ is positive definite. Therefore Theorem 4.8 follows from Theorem 2.18.

2. Let $i \in Q_0$ be a sink. Show that $S_i^+(E(i)) = 0$.

Solution. Let $T = \{(U_j, \psi_\alpha)\}$ equal $S_i^+(E(i))$. In the definition of $S_i^+(E(i))$, $U_j = E(i)_j = 0$ for all $j \neq i$. Therefore it suffices to check that $U_i = 0$. By definition $U_i$ is the kernel of the maps

$$\bigoplus_{h(\alpha) = i} E(i)_{t(\alpha)} \to E(i)_i.$$ 

In particular, $U_i$ is a subspace of $\bigoplus_{h(\alpha) = i} E(i)_{t(\alpha)} = 0$ (here we have used the fact that $E(i)_{t(\alpha)} \neq E(i)_i$ for all $\alpha$ since $i$ is a sink).

3. Consider the representation $M$ given by

$$
\begin{array}{c}
\mathbb{C} \\
\mathbb{C}^2 \\
\mathbb{C}
\end{array}
\xleftarrow{(1,0)} \xrightarrow{(0,1)} \xrightarrow{(1,1)} \mathbb{C}
$$

If we label the central vertex by $i$, what is $S_i^- (M)$?

Solution. Let $T = \{(U_j, \psi_\alpha)\}$ equal $S_i^- (M)$. This means that $U_j = \mathbb{C}$ for all $j \neq i$. At the vertex $i$, the space $U_i$ is the cokernel of the map

$$\mathbb{C}^2 \xrightarrow{A} \mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$
It is clear from the top four entries of $A$ that the map is injective. Therefore the cokernel is two-dimensional. It is spanned by the images of

$$v_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$  

Using this we can calculate the maps $\mathbb{C} \to \mathbb{C}^2$. For instance, the dual of the map $\mathbb{C}^2 \overset{(0,1)}{\to} \mathbb{C}$ sends $1 \in \mathbb{C}$ to the image of $(0, 1, 0, 0)^T$ in $\mathbb{C}^4/\text{Im } A$. Since

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -2v_1 - v_2 \mod \text{Im } A,$$

this map is given by $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$. Similarly, the dual of the map $\mathbb{C}^2 \overset{(1,2)}{\to} \mathbb{C}$ sends $1 \in \mathbb{C}$ to the image of $(0, 0, 1, 0)^T$ in $\mathbb{C}^4/\text{Im } A$. Since

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = v_1 \mod \text{Im } A,$$

this map is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Computing all maps gives

$$\begin{pmatrix} \mathbb{C} \\ (0,1) \end{pmatrix} \xrightarrow{\begin{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ (0,1) \end{pmatrix}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\begin{pmatrix} \mathbb{C}^2 \\ \begin{pmatrix} -2 \\ -1 \end{pmatrix} \end{pmatrix}} \begin{pmatrix} \mathbb{C} \end{pmatrix}.$$  

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4. Let $Q$ be the quiver $e_1 \xrightarrow{\alpha} e_2 \xleftarrow{\beta} e_3$ of type $A_3$. The corresponding root system, with reflection group $\mathfrak{S}_4$ was considered in the first exercise on reflection groups and root systems. Thus, the positive roots are

$$R^+ = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_1 - \epsilon_4\},$$

which, under the identification $e_i \mapsto \epsilon_i - \epsilon_{i+1}$, corresponds to

$$R^+ = \{e_1, e_2, e_3, e_1 + e_2, e_2 + e_3, e_1 + e_2 + e_3\}.$$

For each of the above dimension vectors construct an explicit indecomposable representation of $Q$.

Solution. The simple representations $E(1), E(2)$ and $E(3)$ are of course indecomposable and have dimension vector $e_1, e_2$ and $e_3$ respectively. The following are easily checked to be examples of indecomposables

(a) $\begin{array}{c} \mathbb{C} \xrightarrow{2} \mathbb{C} \\ \xleftarrow{0} \end{array}$ of dimension vector $e_1 + e_2$.

(b) $\begin{array}{c} 0 \xrightarrow{0} \mathbb{C} \\ \xleftarrow{3} \end{array}$ of dimension vector $e_2 + e_3$.

(c) $\begin{array}{c} \mathbb{C} \xrightarrow{5} \mathbb{C} \\ \xleftarrow{-4} \end{array}$ of dimension vector $e_1 + e_2 + e_3$.

In fact, in (a) one can take $\alpha$ to be any non-zero number and $\beta = 0$ and similarly for (b) and (c).