HOW TO COUNT CURVES: FROM 19^{th} CENTURY PROBLEMS TO 21^{st} CENTURY SOLUTIONS

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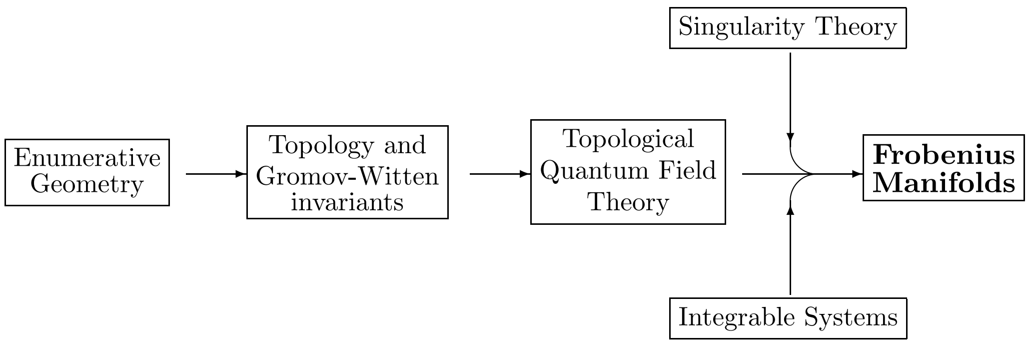
1. Introduction

Guess the next term in the sequence:

$$1\,,\quad 1\,,\quad 12\,,\quad 620\,,\quad 87304\,,\quad 26312976\,.$$

This problem belongs to an area of mathematics known as enumerative geometry, the origins of which date from the 19^{th} century, when much progress was made, and even earlier to classical Euclidean geometry. At the International Congress of Mathematicians held in Paris in 1900, Hilbert listed a set of 23 problems in mathematics, one of which, the 15^{th}, asked questions on the general theory of enumerative geometry.

Despite this promising start, progress in the 20^{th} century was slow and it was not until the closing decade of the century that spectacular advances were achieved using ideas not from geometry but from quantum field theory and string theory. This article will describe some of the ancient and modern mathematics behind this breakthrough and how this is being used in the opening years of the 21th century. The story will take us from the original problem, via topology, string theory and the banks of a canal in Scotland, to a fusion of apparently disparate areas of mathematics and physics.



The central idea is that of a Frobenius manifold - introduced by Boris Dubrovin as a geometric way to understand the equations of Topological Quantum Field Theory derived by Ed Witten, Robert Dijkgraaf and the Verlinde brothers - which is crucial to the solution of the opening puzzle, which was solved in a ground breaking work by Manin and Kontsevich.

To experts in this field I apologize. This description is not historically or indeed mathematically correct. I have tried, constrained by page limits and the editorial requirement that 'detailed mathematics should be kept to a minimum' , to produce an account that is, at worst, not wholly wrong.

2. Enumerative Geometry - how to count curves

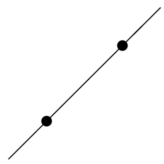
Enumerative geometry seeks to answer questions of the form:

> How many geometric structures of given type are there which satisfy some given geometric conditions?

The simplest such question is just

> How many straight lines can be drawn on the plane going through two distinct points?

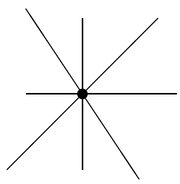
Here the 'geometric structure' is a straight line and the 'geometric conditions' are 'going through two points'. The answer - a result from Euclidean geometry - is clearly one:



Note that if the conditions on the lines change the answer can change dramatically. So consider the related questions:

> How many straight lines can be drawn on the plane going through one point?

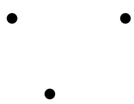
The answer to this is that there are an infinite number of straight lines going through a single point:

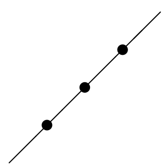


If more points are added the answer will change again:

> How many straight lines can be drawn on the plane going through three distinct, generic points?

In general, no straight line may be drawn through three or more distinct, generic points - as the figure on the left shows. Clearly there are special configurations of three points - as the figure on the right shows





but these are excluded by the use of the word generic - this is a special configuration and hence is not generic.

After lines, which are defined by the equation

$$ax + by + c = 0$$

one may move on in complexity and consider 'geometric structures' given by conic sections - circles, ellipses, parabolas and hyperbolas - these being defined by the equation

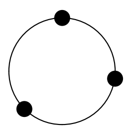
$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

(in general, the degree of such a curve is the highest power that appears in such equations, so lines are defined by degree 1 equations, conics by degree 2 equations).

Thus one may ask

How many circles can be drawn on the plane going through three district points?

The answer again is a classical result from Euclidean geometry - one - the circle is the circumcircle of the triangle defined by the three points.



If three is replaced by four then the answer is zero, or if the three is replaced by two then the answer is infinite.

In each of these cases there is a critical number of points where one gets an interesting answer - below that the answer is infinite, and above it the answer is zero, so for lines this critical number is 2 and for circles this number is 3.

One may clearly continue this game, looking at cubics (given by degree 3 equations), quartics (given by degree 4 equations), quintics (given by degree 5 equations) and on to curves given by equations of arbitrary degree. The critical number of points that gives an interesting answer will also increase as the degree, or complexity, of the curve increases.

Let us now return to the opening sequence, and introduce the notation $N(d)$ to denote the d^{th} number in the sequence, so $N(1) = 1\,, N(2) = 1\,, N(3) = 12\,,$ etc.. We may now define how the opening sequence is produced: it describes the number of curves of a given degree - denoted by d - going through a critical number of generic points. Let

$$\boxed{N(d) = \left\{ \begin{array}{c} \text{number of curves of degree } d \text{ passing through} \\ 3d-1 \text{ generic points in the complex projective plane} \end{array} \right\}.}$$

The number of points $3d-1$ that appears in this formula is the critical number of points for this problem; if this was less the answer would be infinite, if it was more the answer would be zero.

We now have now a well defined problem, but simple questions may have deceptively complicated solutions. For example, 'is this number the product of two prime numbers?' is a simple question, but answering the question for some specific number is extremely time consuming, even with the aid of modern computers. Our problem is like this; understanding what the problem is does not give us any clues as to how to solve it. We may now rephrase the opening question, which asks what is the 7^{th} number in the sequence, as:

How many curves of degree 7 passing through $3.7-1=20$ generic points are there in the complex projective plane?

The huge, in fact exponential, growth in these numbers shows that counting such curves 'by hand' would be an impossible task - other tools are required. Before describing how these tools were developed, there are two words in the definition of $N(d)$ that have not been defined: *complex* and *projective*.

The first calculations with complex numbers - numbers involving the square root of -1, or $\sqrt{-1}$ - appeared in the book Ars Magna written by the Italian mathematician Cardano in the mid 16^{th} century. He wrote

> *Dismissing mental tortures, and multiplying* $5+\sqrt{-15}$ *by* $5-\sqrt{-15}$*, we obtain* $25-(-15)$*. Therefore the product is* $40,\ldots$ *and thus far does arithmetical subtlety go, of which this, the extreme, is, as I have said, so subtle that it is useless.*

Far from being useless, complex numbers are extremely useful; they make many calculations easier, not more complicated.

For example, consider the solution to the quadratic equation

$$ax^2+bx+c=0$$

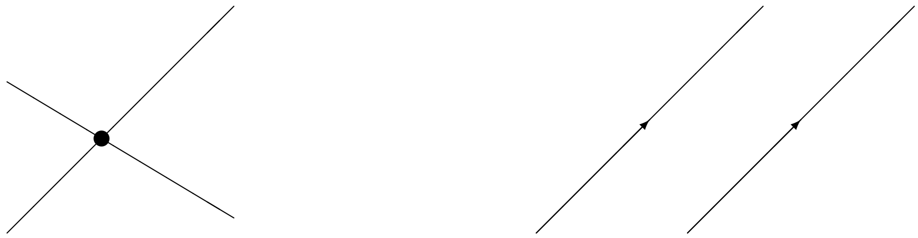
given by the formula, attributed to Omar Khayyam (of *Rubaiyat* fame) in the $11^{\text{th}}/12^{\text{th}}$ century,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\,.$$

If one restricts to only real numbers, there are two solutions to the quadratic equation (if $b^2-4ac>0$), one solution (if $b^2-4ac=0$) or no solutions (if $b^2-4ac<0$). However with complex numbers the answer is simpler - there are always two solutions, and this simplicity persists if one looks at higher degree equations - a degree n equation has n solutions if one uses complex numbers - a result known as the fundamental theorem of algebra. Thus counting complex objects is easier than counting real objects.

Moving from real to complex numbers means that one must now consider complex curves rather than just real curves. A complex curve just consists of points (x,y) satisfying some equation $P(x,y)=0$ (such as the equation for conics given above): the important new point is that x and y may now be complex. Such complex curves behave more like surfaces than curves: one has 4 real numbers (the real and imaginary parts of both the complex numbers x and y) constrained by 2 real equations (the real and imaginary parts of the equation $P=0$). A parameter count show that a curve depends on $(4-2)=2$ real numbers, i.e. a complex curve is 2-dimensional and hence 'looks' more like a surface.

Projective geometry dates from the development of perspective during the Renaissance, and a key idea is that of a point at infinity. On the plane two lines will, in general, intersect at a single point, but if the lines are parallel then they do not meet:



Thus the answer to the simple question - at how many points can two straight lines on the plane meet - can be one or zero. In projective geometry one adds to the usual plane an additional point, called the point at infinity, and parallel lines meet at such a point. Thus the answer to the question - at how many points do two straight lines on the projective plane met - is always one. Thus counting objects in projective geometry, where two lines always meet, is easier than problems on the ordinary plane, where pairs of lines may or may not meet.

Thus looking at complex curves in the complex projective plane is much easier than looking at real curves in the ordinary plane - one does not have difficulties with counting real solutions or difficulties with solutions that are infinite. The next step in the solution of the opening puzzle is to regard it as a *topological* rather than a geometric problem.

3. From counting curves to Topology: Gromov-Witten invariants

The origins of topology date from the fundamental work of Euler in the 18^{th} Century - and the most basic topological invariant is still named after him. Topology is often referred to as rubber sheet geometry. Consider a ring made from rubber. One can bend and stretch it (cutting is not allowed) and its shape will change, but the number of holes in the ring will not change. Thus the number of holes - grandly referred to as the genus - is a topological invariant; it is invariant under certain bending and stretching transformations. Topological invariants can be used to distinguish objects - if you want to see if your mint is a Polo or a Trebor mint, just count the number of holes.

Topological invariants are hard to find, and even harder to compute. In the 1980s a new class of topological invariants were defined by Mikhael Gromov and Ed Witten, which are now named Gromov-Witten invariants. A full description of what these are would take up far too much space - it suffices to say that they may be defined as the number of curves in a geometric object that satisfy certain geometric relations. In other words the opening sequence is nothing more than the Gromov-Witten invariants of the complex projective plane. This reformulation gave a new way to think about the problem, but the problem of actually calculating the invariants remained.

A convenient way to study such Gromov-Witten invariants is to combine them into a function and then study the properties of a single function rather than the properties of an infinite number of separate numbers. One can define a function $\psi(x)$ as

$$\begin{aligned}\psi(x) &= N(1)\frac{x}{2.1} + N(2)\frac{x^2}{5.4.3.2.1} + N(3)\frac{x^3}{8.7.6.5.4.3.2.1} + \dots\,, \\ &= \sum_{d=1}^{\infty} N(d)\frac{x^d}{(3d-1)!}\,.\end{aligned}$$

Since $N(1) = 1\,, N(2) = 1\,, N(3) = 12$ etc., this begins

$$\psi(x) = \frac{x}{2} + \frac{x^2}{120} + \frac{x^3}{3360} + \dots\,.$$

This single function turned out to be a key ingredient in defining a topological quantum field theory. Such theories must obey certain conditions and these give you information about ψ you did not know before. This information, since ψ is defined in terms of the Gromov-Witten invariants $N(d)$, then gives a way to calculate the invariants themselves. To describe this one must first understand how objects may be combined to give new objects.

4. How to combine objects

One of the most important notions in mathematics is that of combining objects to form new objects. The simplest case is that of a so-called binary operation where two objects are combined - for example in addition two numbers are combined under a process called ‘adding’ to form a new number. Addition, multiplication, subtraction and division are all examples of binary operations. But what properties does the operation of combining have? In what follows the objects will be denoted by letters $a\,, b\,, c$ and the object formed by combining a and b will be denoted $a.b$. At first reading, one should think of these objects as just numbers - later we will see that other objects can by combined, and this will give the link to string theory.

Commutativity In general, combining a with b may be different from combining b with a. For example, if the combining operation is subtraction, then

$$a - b \neq b - a\,.$$

If, however,

$$a.b = b.a$$

then the operation is said to be commutative. Examples are addition and multiplication:

$$\begin{aligned}a + b &= b + a\,, \\ a \times b &= b \times a\,.\end{aligned}$$

Associativity Three objects cannot be combined directly with a binary operation, but they may be combined via a two-step process: combine two, then combine this with the third object. Thus we can form $(a.b).c$ by first combining a with b, then combining this with c. Alternatively we could form $a.(b.c)$ where a is combined with the result of combining b with c. In general there is no reason why these two ways of combining three objects should give the same answer. If they are the same, that is, if

$$(a.b).c = a.(b.c)$$

then the operation is said to be associative. Examples of associative operations are again addition and multiplication:

$$\begin{aligned}(a+b)+c &= a+(b+c)\,,\\ (a\times b)\times c &= a\times(b\times c)\end{aligned}$$

but not subtraction and division (for example, $(5-1)-2 \neq 5-(1-2)$.)

Identity Is there a special object that does not change another object when combined with it? If such an object exists it is known as the identity and will be denoted by the letter e. Thus e is the identity if

$$\begin{aligned}e.a &= a\,,\\ a.e &= a\end{aligned}$$

irrespective of which object a is. What the identity is depends on the nature of the combining operation, not the objects themselves, so for example 0 is the identity for addition:

$$0+a=a$$

and 1 is the identity for multiplication:

$$1.a=a\,.$$

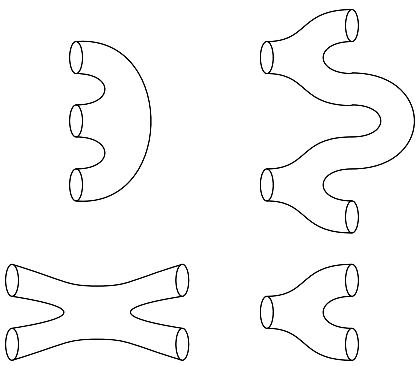
These properties are distinct - one can have commutative operations that are not associative and visa versa. Some of the most interesting combining operations are those which satisfy all of the above three conditions. These are known as Frobenius algebras.

Definition *A Frobenius algebra consists of a set of objects and a binary operation with the properties that it:*

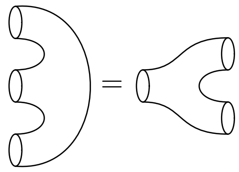
- *is commutative;*
- *is associative;*
- *has a unit.*

These definitions have all used the word ‘object’ - one may combine many different types of objects, not just numbers.

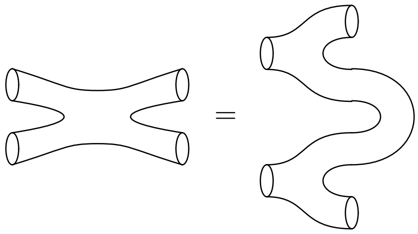
The objects required here are two dimensional surfaces with an arbitrary number of holes and some (non-zero) numbers of boundaries or edges, these being circles. Thus the following are all examples of these objects:



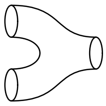
Two objects are said to be the same if one can be continuously deformed (no cutting allowed) into another. Thus the following two surfaces should be regarded as the same object:



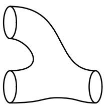
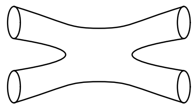
and similarly



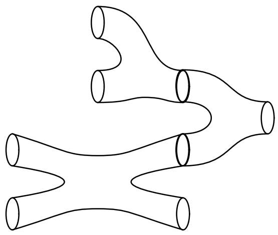
Having defined the class of objects, one can define a way of combining them. This can be achieved with the following surface, denoted c:

$c =$ 

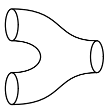
Thus to combine two surfaces a and b one joins up an edge of a with an edge of c and an edge of b with an edge of c. The resulting surface is then $a.b$. Thus if

$a =$  $b =$ 

then

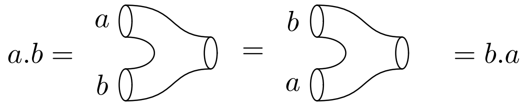
$a.b =$ 

This can an abbreviated as

$a.b =$ 

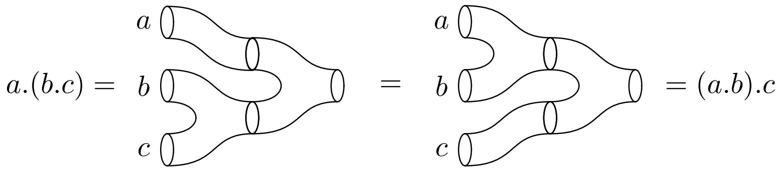
Note that since everything may be continuously deformed, it does not matter which edge of, say, the surface a is used. We thus have a class of objects, together with a way of combining such objects together. It is not hard to see that this is, in fact, a Frobenius algebra; that is, the operation of combining surfaces this way is commutative, associative and there is a special surface which plays the role of an identity object.

Commutativity Let a and b be arbitrary surfaces. Then



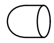
since one can just rotate the surfaces about in three dimensions.

Associativity Let a , b and c be arbitrary surfaces. Then

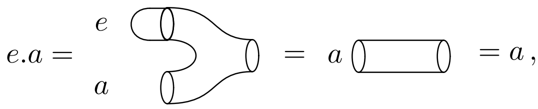


since one can just bend and stretch one surface into the other.

Identity The object corresponding to the identity is a 'cap' :



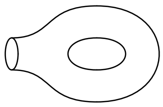
This 'plugs' a hole in a surface, so:



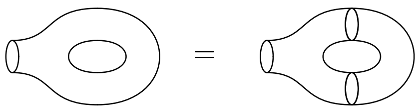
since adding a tube to a hole does not change the surface, as this just corresponds to stretching it.

Thus the three conditions in the definition of a Frobenius algebra are satisfied and hence we have a Frobenius algebra.

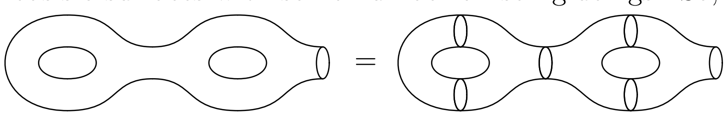
All the surfaces drawn above have no holes. Such surfaces, for example



may be decomposed into simpler components, each based on the c-surface. In the above example the surface decomposes into one basic c-surface, and one 'self-glueing' surface which joins two edges of c together:



In fact, any surface with some arbitrary number of holes may be thought of as a combination of some number of basis c surfaces with some number of 'self-glueings'. So, for example



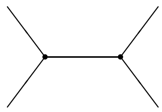
where one uses 3 basic c-surface and one self-gluing.

The last step is to go from a Frobenius algebra to a Frobenius manifold. The theory of manifolds grew out of the study of the geometry of surfaces, where one considers surfaces as divorced from the bigger space in which they sit. One then studies properties of the surface that depend only on the surface and not on how that surface is embedded. Manifolds come in all shapes and sizes and in all dimensions - surfaces being examples of two dimensional manifolds.

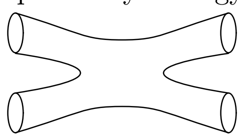
In a Frobenius manifold - which can be of any dimension you like - one has, at each point, a Frobenius algebra, and as the point moves around the manifold the detailed properties of the Frobenius algebra change. [1] There are other ingredients, but this is the fundamental one. Such Frobenius manifolds were introduced by Boris Dubrovin about a decade ago and they have had an enormous impact on mathematics and physics. Historically, the equations that the c-object must satisfy in order to obtain a Frobenius algebra were written down - without reference to any manifold structure - by Witten, Dijkgraaf and the Verlinde brothers, and the reformulation of these equations as geometric objects was the spur for much progress in this area.

So what is the connection between such algebras and enumerative geometry? The connection is via quantum field theory. Such theories came from applying the ideas of quantum mechanics not to particles but to fields, such as to electromagnetic fields, resulting in a theories such as quantum electrodynamics (or QED for short).

Fundamental in quantum field theories are objects called correlators. They describe how states transform and interact, and are, in a sense, analogous to the role of observables in quantum mechanics. In field theories, like QED, interaction diagrams - Feynman diagrams - are point-like:



In string theory these diagrams are replaced by ‘stringy’ ones:



While there are still many mathematical problems remaining in string theory it is clear that Frobenius algebra type structures will start to appear. The idea which emerged in the 1980/90s from the work of Michael Atiyah, Ed Witten, Simon Donaldson and Vaughan Jones was that in certain special quantum field theories the correlators encode topological information, and in particular, that Gromov-Witten invariants could be encoded as the correlators in a topological quantum field theories.

What Kontsevich and Manin showed is that the function ψ plays the role of the object c in the Frobenius algebra (recall, it defines how objects are combined). The commutativity and associativity conditions then implies that ψ satisfies certain conditions and since ψ is defined in terms of the Gromov-Witten invariants, these conditions then supply previously unknown conditions on the invariants themselves, namely the new recursion relations:

$$\boxed{N(n) = \sum_{i+j=n} \left[\begin{pmatrix} 3n-4 \\ 3i-2 \end{pmatrix} i^2 j^2 - i^3 j \begin{pmatrix} 3n-4 \\ 3i-1 \end{pmatrix} \right] N(i)N(j)}\,.$$

[1]The surfaces shown above to illustrate the idea of a Frobenuius algebra are themselves two-dimensional manifolds; these are not Frobenius manifolds - they show pictorially the algebraic structure at each point on a multidimensional Frobenius manifold.

Unpacking this, for small values of d gives:

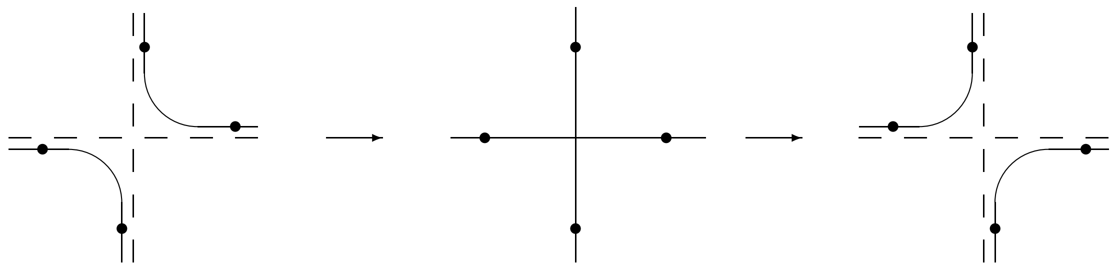
$$\begin{aligned}
N(2) &= N(1)\,,\\
N(3) &= 12N(2)N(1)\,,\\
N(4) &= 33N(1)N(3)+224N(2)^2\,,\\
N(5) &= 68N(1)N(4)+3762N(2)N(3)\,,\\
N(6) &= 120N(1)N(5)+17472N(2)N(4)+34749N(3)^2\\
N(7) &= 192N(1)N(6)+58480N(2)N(5)+599352N(3)N(4)\,.
\end{aligned}$$

Thus by knowing only $N(1)$ - the number of lines between two points on the plane - one may, term-by-term - calculate all other numbers. Thus the number $N(d)$ may be calculated for any number d. Thus the answer to the original question (the value of $N(7)$) is:

$$\boxed{\text{The next term in the sequence is 14616808192.}}$$

The sequence continues: $13525751027392, 19385778269260800\,, \ldots$.

This recursion formula also encodes how curves degenerate into collections of simpler curves as the points move to a nongeneric configuration. For example, conics can degenerate into pairs of straight lines, as the following sequence shows (the dashed lines are just to aid the eye): as the points move, the curves slip in and out of a degenerate configuration consisting of two intersecting lines.



This type of argument shows how $N(2)$ can be determined from $N(1)$, and similar arguments holds for higher $N(d)$, for example a cubic curve can degenerate into a line and a conic, or even into three lines. Enumerating by hand all such degenerate configurations for arbitrary degree curves is as difficult as the original problem.

There are many more influences in the theory of Frobenius manifolds than can be fully described here. Early simple examples turned out to be related to much earlier work by V.I. Arnold and K. and M. Saito in the area of singularity theory - the mathematics underpinning catastrophe theory. Much of the current theory of Frobenius manifolds owes as much to singularity theory as it does to quantum field theories, and tools developed in one area are now being routinely used in the other. As the area develops the dividing lines between all these areas are becoming less distinct and as a result, collaboration and cross fertilization is becoming more common. My own vision of the future is one in which the distinction between pure and applied mathematics, between geometry and topology become to be seen as artificial and, in time, redundant.

It is important to realize that a Frobenius manifold only encodes data on the numbers of curves of the simplest type - so-called genus zero curves. Recall that complex curves are

2-dimensional and so are more like surfaces than curves. The simplest complex curves are topologically - that is, up to bending and stretching - spheres. More complicated complex curves are characterized by their genus - the number of holes the surface has. In general one is interested in numbers $N(d,g)$ - the number of complex projective curves of degree d and of genus g . The opening sequence corresponds to the case $g=0$, there are, however, analogous sequences for each value of g . So how are these higher genus sequences calculated?

The key idea is that of a topological recursion relation. It was shown above that a complicated surface could be decomposed into a number of c-objects and a number of self-gluing operations. Thus high genus diagrams - diagrams with a number of holes - can be constructed from the c-object alone, and it is this c-object - a genus zero object - that defines a Frobenius manifold. Thus one should be able to calculate the numbers $N(d,g)$ just from knowledge of the genus 0 numbers $N(d)$ which are encoded in the associated Frobenius manifold.

This simple fact brushes over an enormous amount of recent advances in mathematics. Even in one dimension the problems are hard. It was conjectured by Witten that for the simplest 1-dimensional theories everything was constrained by an equation known as the Korteweg-deVries equation, and this was proved by Kontsevich (a result that contributed to the award of a Fields Medal to him in 1998). This equation is *very* special and has an intriguing history of its own, starting with an observation made on the banks of a Scottish canal.

5. From topology to nonlinear system and solitons

Regarding Gromov-Witten invariants are correlators of a Topological Quantum Field theory was a brilliant observation - but in a sense it just replaces one hard problem with another hard problem. Why could these object be calculated so easily? The answer lies in the theory of solitons and integrable systems. It turns out that the KdV equation is the prime example of an integrable system - systems despite their nonlinearities can be solved exactly without recourse to numerical approximations.

The soliton was first discovered by accident by the naval architect, John Scott Russell, in August 1834 on the Glasgow to Edinburgh canal. In his own words

> *I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of the water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now very generally bears.*

He immediately noticed that this was a new phenomenon, and a major part of his subsequent work was on its many properties. Russell's work was slow to be accepted; Sir John Herchel and George Stokes - both prominent fellows of the Royal Society - gave other, incorrect, explanations. However, in the 1870's both Boussinesq and Lord Rayleigh found the basic solution to the problem. This work was rederived in 1895 by Korteweg and deVries, and the basic equation

$$\boxed{u_t = u_{xxx} + uu_x}$$

still bears their names (though often shortened to just the KdV equation). The 'great wave of translation' is essentially a nonlinear phenomenon and is now known as a solitary wave or soliton.

The history of integrable systems is a long and convoluted one, with many missed opportunities. Many of today's key ideas were introduced in the 19th century and the early 20th century but were forgotten or not developed. The modern theory originates with the work of Kruskal and Zabusky in 1965. They were the first to call Russell's solitary wave a soliton. Since then the theory has developed immensely, though paradoxically, a precise definition of integrability remains illusive.

The whole theory of Frobenius manifolds, with its application to enumerative geometry, topological quantum field theory and singularity theory works because, at its heart, are systems of integrable equations. The work of Witten and Kontsevich is related to properties of the KdV equation - the paradigm example of an integrable system - so Gromov-Witten invariants are governed by properties of an integrable system, and this provides an explanation of why turning a hard problem in enumerative geometry into a hard problem in quantum field theory was so successful: the problem was integrable and hence could be solved exactly.

Some of the main problems in this area are to do with understanding higher genus invariants and how they are constructed - the Vision of the Future is that the theory of integrable systems must be involved. My own research has involved the study of genus-one terms for certain important families of Frobenius manifolds. The success of this research has resulted from a fusion of ideas from integrable systems and singularity theory - the idea being to extract information on caustics - surfaces inside Frobenius manifolds where things break down - to help to find the solutions.

This interaction between integrable systems and Frobenius manifolds is not a one-way phenomenon: there has been a feedback from Frobenius manifolds into integrable systems theory, and many ideas are still waiting to be developed. There are interesting times ahead.

Acknowledgment: I would like to thank Joachim Kock for his help in drawing the pictures. Financial support was provided by the EPSRC, grant GR/R05093.

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Recent developments, particularly on the link between soliton theory and topology, may be found in:

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Biographical details of many of the mathematicians mentioned in this article may be found at the University of St. Andrew's History of Mathematics web site:

http://www-groups.dcs.st-and.ac.uk/∼history/index.html

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