NOETHERIAN HOPF ALGEBRAS

Ken Brown, University of Glasgow

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DEFINITION: Throughout, H will denote a *noetherian Hopf algebra over an algebraically closed field* k. This means that H is an associative $k-$algebra, with 3 maps, satisfying a number of axioms:

- the *coproduct*, an algebra homomorphism

$$\Delta : H \longrightarrow H \otimes_k H : h \mapsto \sum h_1 \otimes h_2;$$

- the *counit*, an algebra homomorphism

$$\varepsilon : H \longrightarrow k;$$

- the *antipode*, an algebra anti-homomorphism

$$S : H \longrightarrow H.$$

EXAMPLES: 1. If G is a group, the *group algebra* kG is a Hopf algebra, with, for $g \in G$,

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}.$$

2. If $\mathfrak{g}$ is a Lie algebra, the *enveloping algebra* $U(\mathfrak{g})$ is a Hopf algebra, with, for $x \in \mathfrak{g}$,

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x.$$

3. If G is an algebraic group, its *coordinate ring* $\mathcal{O}(G)$ is a Hopf algebra, with, for $x, y \in G$,

$$\begin{aligned} \Delta(f)(x,y) &= f(xy), \quad \varepsilon(f) = f(1_G), \\ & \qquad\qquad S(f)(x) = f(x^{-1}). \end{aligned}$$

HISTORY

- defined by H. Hopf in 1941.

- first monograph by Sweedler, 1969.

- key examples, quantum groups $U_q(\mathfrak{g})$, and quantised function algebras $\mathcal{O}_q(G)$, discovered by Drinfeld and Jimbo in 1980s.

- has led to huge upsurge of interest over past 25 years - connectionswith algebra, noncommutative geometry, physics, integrable systems,.....

Let $k = \mathbb{C}, 0 \neq q \in \mathbb{C}$. The *quantised enveloping algebra* $U_q(\mathfrak{sl}(2,\mathbb{C}))$ is the algebra

$$\begin{aligned} \mathbb{C}\langle\, K^{\pm 1}, E, F : \qquad KEK^{-1} &= q^2 E; \\ KFK^{-1} &= q^{-2} F; \\ EF - FE &= \tfrac{K - K^{-1}}{q - q^{-1}} \,\rangle, \end{aligned}$$

$$\begin{aligned} \Delta(E) = &\quad E \otimes 1 + K \otimes E, \\ \Delta(F) = &\; F \otimes K^{-1} + 1 \otimes F, \\ \Delta(K) = &\qquad K \otimes K; \end{aligned}$$

$$\begin{aligned} \varepsilon(K) = \;& 1, \textit{ and } S(K) = \; K^{-1} \\ \varepsilon(E) = \;& 0 \textit{ and } S(E) = \; -K^{-1}E, \\ \varepsilon(F) = \;& 0 \textit{ and } S(F) = \; -FK. \end{aligned}$$

Let $k = \mathbb{C}, 0 \neq q \in \mathbb{C}$. The *quantised function algebra* $\mathcal{O}_q(SL(2,\mathbb{C}))$ is the algebra

$$
\begin{aligned}
\mathbb{C}\langle a,b,c,d : \quad & ab = qba;\ ac = qca;\ bc = cb; \\
& bd = qdb;\ cd = qdc; \\
& ad - da = (q - q^{-1})bc;\ ad - qbc = 1\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\Delta(a) &= a \otimes a + b \otimes c, \\
\Delta(b) &= a \otimes b + b \otimes d, \\
\Delta(c) &= c \otimes a + d \otimes c, \\
\Delta(d) &= c \otimes b + d \otimes d,
\end{aligned}
$$

$$\varepsilon(a) = \varepsilon(d) = 1;\ \varepsilon(b) = \varepsilon(c) = 0,$$

$$S(a) = d;\ S(b) = -q^{-1}b;\ S(c) = -qc;\ S(d) = a.$$

Why are Hopf algebras interesting?

- Because they have a very rich representation theory!

- If V and W are (say, left) $H-$modules, then so are

$$V \otimes_k W,$$

$$\text{Hom}_k(V, W),$$

$$k,$$

$$V^*.$$

In general $V \otimes W \not\cong W \otimes V$, which is part of the fun.....

QUESTION: Let H be a *noetherian* Hopf algebra. What can be said about H *as a ring*?

Focus here on homological properties. Recall that [Larson, Sweedler, 1969]

if H is a finite dimensional Hopf algebra,
then H is self-injective.

CONJECTURE: [Brown-Goodearl, 1997] Suppose that H is a noetherian Hopf algebra. Then H has finite injective dimension:

$$\text{l.inj.dim.}(H) = \text{r.inj.dim.}(H) = d < \infty.$$

The evidence was minimal.... basically, we couldn't think of any counterexamples....

In fact we made a stronger conjecture: say that H is *AS-Gorenstein* if

$$\text{l.inj.dim.}(H) = d < \infty,$$

with

$$\text{Ext}^i_H(k, H) = \delta_{id}k;$$

and the same holds on the right.

Consider for example what happens when H is *finite dimensional*....

We proposed that every noetherian Hopf algebra is AS-Gorenstein.

Ten years later, still the case that all known noetherian Hopf algebras are AS-Gorenstein.

Specifically, the following classes are AS-Gorenstein:

1. kG for G polycyclic-by-finite, $d = h(G)$;
2. $U(\mathfrak{g})$, $\mathfrak{g}$ a fin.dim. Lie algebra, $d = \dim(\mathfrak{g})$;
3. $U_q(\mathfrak{g})$, $\mathfrak{g}$ f.d.semisimple Lie alg., $d = \dim(\mathfrak{g})$;
4. $\mathcal{O}_q(G)$, G a semisimple alg. gp., $d = \dim(G)$;
5. H an affine noetherian Hopf algebra satisfying a polynomial identity, $d = \mathrm{GK} - \dim(H)$.

Of these, the most difficult is the last: this is a 2003 theorem of Wu and Zhang.

WHAT USE IS THE AS-GORENSTEIN PROPERTY?

- There are many structural consequences: existence of Artinian quotient rings; possession of “good” dimension function; finite global dimension implies H is a direct sum of prime algebras....

- Focus on one application - “duality”.

- Consider first what happens when H is finite dimensional....

When H is finite dimensional,

H is a Frobenius algebra.

So

$$H^* = \mathrm{Hom}_k(H, k) \;\cong\; {}^{\nu}H^1$$

for a certain algebra automorphism of H, called the *Nakayama automorphism.* And then

$$V \mapsto \mathrm{Hom}_H(V, \, {}^{\nu}H^1)$$

defines a contravariant equivalence of categories

$$\{\textit{left } H - \textit{modules}\} \longrightarrow \{\textit{right } H - \textit{modules}\}.$$

To generalise the duality of Frobenius algebras to an infinite dimensional setting, we have to work at the level of the *bounded derived category* $\mathcal{D}^b(\text{mod}-H)$.... so, in slogans:

- replace modules by *complexes of modules*;

- identify *quasi-isomorphic* complexes, (so that, e.g., a module gets identified with all of its projective or injective resolutions).

Definition: [Yekutieli] A *rigid dualizing complex* R for an algebra H is a complex of $H-$bimodules in the derived category which, via the functor

$$\mathrm{RHom}_H(-, R),$$

defines a duality ($\equiv$ a contravariant equivalence) between $\mathcal{D}^b(\textit{left } H-\textit{modules})$ and $\mathcal{D}^b(\textit{right } H-\textit{modules})$. If R exists, it's unique.

Example: Let F be a finite dimensional algebra. Then R exists and is $F^* := \mathrm{Hom}_k(F, k)$. So if F is in addition a *Frobenius algebra*

$$R = F^* \cong {}^{\nu}F^1,$$

for the Nakayama automorphism ν of F.

Theorem (B-Zhang, 2006). *1. Let H be an AS-Gorenstein noetherian Hopf algebra of injective dimension d, with bijective antipode S. Then H has a rigid dualizing complex, namely*

$$R \cong {}^{\nu}A^{1}[d],$$

where ν is a certain algebra automorphism of H, called the <u>Nakayama automorphism</u>*.*

2. ν is uniquely determined by H (up to an inner automorphism). Namely,

$$\nu = S^2\tau,$$

where τ is the left winding automorphism determined by the right structure of $\mathrm{Ext}_H^d(k, H)$.

CONSEQUENCES: 1. **Twisted Poincaré duality:** Let H be as in the theorem, and assume that H has finite global dimension (which is then necessarily d). Then for every $H-$bimodule M and for all i,

$$H^i(A, M) = H_{d-i}(H, {}^{\nu^{-1}}M).$$

Applied in the setting e.g. of group algebras, we retrieve the fact [Bieri, 1972] that polycyclic-by-finite groups are "Poincaré duality groups".

CONSEQUENCES: 2. **The antipode:** Let H be as in the theorem. Then $(H, \Delta^{op}, S^{-1}, \varepsilon)$ is also a Hopf algebra (with the *same* algebra structure as H). Equating the resulting 2 answers for ν, we get

$$S^4 \quad = \quad \gamma \circ \rho \circ \tau^{-1}$$

where γ is inner and ρ is the *right* winding automorphism got from $\mathrm{Ext}^d_H(k, H)$.

Note: For H finite dimensional, this is a 1976 result of Radford, with an *explicit* γ.

QUESTION: What is γ when H is infinite dimensional?