NOETHERIAN HOPF ALGEBRAS

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http://www.maths.gla.ac.uk/ kab/
**DEFINITION:** Throughout, $H$ will denote a noetherian Hopf algebra over an algebraically closed field $k$. This means that $H$ is an associative $k$–algebra, with 3 maps, satisfying a number of axioms:

- the *coproduct*, an algebra homomorphism

  \[ \Delta : H \rightarrow H \otimes_k H : h \mapsto \sum h_1 \otimes h_2; \]

- the *counit*, an algebra homomorphism

  \[ \varepsilon : H \rightarrow k; \]

- the *antipode*, an algebra anti-homomorphism

  \[ S : H \rightarrow H. \]
**EXAMPLES:** 1. If $G$ is a group, the *group algebra* $kG$ is a Hopf algebra, with, for $g \in G$,

$$
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}.
$$

2. If $\mathfrak{g}$ is a Lie algebra, the *enveloping algebra* $U(\mathfrak{g})$ is a Hopf algebra, with, for $x \in \mathfrak{g}$,

$$
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x.
$$

3. If $G$ is an algebraic group, its *coordinate ring* $O(G)$ is a Hopf algebra, with, for $x, y \in G$,

$$
\Delta(f)(x, y) = f(xy), \quad \varepsilon(f) = f(1_G), \quad S(f)(x) = f(x^{-1}).
$$
HISTORY

- defined by H. Hopf in 1941.


- key examples, quantum groups $U_q(\mathfrak{g})$, and quantised function algebras $O_q(G)$, discovered by Drinfeld and Jimbo in 1980s.

- has led to huge upsurge of interest over past 25 years - connections with algebra, noncommutative geometry, physics, integrable systems,.....
Let $k = \mathbb{C}, 0 \neq q \in \mathbb{C}$. The quantised enveloping algebra $U_q(\mathfrak{sl}(2, \mathbb{C}))$ is the algebra

$$\mathbb{C}\langle K^\pm, E, F : KEK^{-1} = q^2E;$$

$$KFK^{-1} = q^{-2}F;$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \rangle,$$

$$\Delta(E) = E \otimes 1 + K \otimes E,$$

$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\Delta(K) = K \otimes K;$$

$$\varepsilon(K) = 1, \text{ and } S(K) = K^{-1}$$

$$\varepsilon(E) = 0 \text{ and } S(E) = -K^{-1}E,$$

$$\varepsilon(F) = 0 \text{ and } S(F) = -FK.$$
Let $k = \mathbb{C}, 0 \neq q \in \mathbb{C}$. The quantised function algebra $\mathcal{O}_q(SL(2, \mathbb{C}))$ is the algebra

$$\mathbb{C}\langle a, b, c, d : \quad ab = qba; \ ac = qca; \ bc = cb; \ bd = qdb; \ cd = qdc; \ ad - da = (q - q^{-1})bc; \ ad - qbc = 1 \rangle,$$

$$\Delta(a) = a \otimes a + b \otimes c,$$

$$\Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c,$$

$$\Delta(d) = c \otimes b + d \otimes d,$$

$$\varepsilon(a) = \varepsilon(d) = 1; \ \varepsilon(b) = \varepsilon(c) = 0,$$

$$S(a) = d; \ S(b) = -q^{-1}b; \ S(c) = -qc; \ S(d) = a.$$
Why are Hopf algebras interesting?

- Because they have a very rich representation theory!

- If $V$ and $W$ are (say, left) $H$–modules, then so are

$$V \otimes_k W,$$

$$\text{Hom}_k(V, W),$$

$$k,$$

$$V^*.$$

In general $V \otimes W \nleq W \otimes V$, which is part of the fun.....
**QUESTION:** Let $H$ be a noetherian Hopf algebra. What can be said about $H$ as a ring? Focus here on homological properties. Recall that [Larson, Sweedler, 1969]

if $H$ is a finite dimensional Hopf algebra, then $H$ is self-injective.

**CONJECTURE:** [Brown-Goodarel, 1997] Suppose that $H$ is a noetherian Hopf algebra. Then $H$ has finite injective dimension:

$$\text{l.inj.dim.}(H) = \text{r.inj.dim.}(H) = d < \infty.$$ 

The evidence was minimal.... basically, we couldn’t think of any counterexamples....
In fact we made a stronger conjecture: say that $H$ is AS-Gorenstein if

$$\text{l.inj.dim.}(H) = d < \infty,$$

with

$$\text{Ext}_H^i(k, H) = \delta_{id} k;$$

and the same holds on the right.

Consider for example what happens when $H$ is finite dimensional....

We proposed that every noetherian Hopf algebra is AS-Gorenstein.

Ten years later, still the case that all known noetherian Hopf algebras are AS-Gorenstein.
Specifically, the following classes are AS-Gorenstein:

1. $kG$ for $G$ polycyclic-by-finite, $d = h(G)$;

2. $U(g)$, $g$ a fin.dim. Lie algebra, $d = \dim(g)$;

3. $U_q(g)$, $g$ f.d.semisimple Lie alg., $d = \dim(g)$;

4. $O_q(G)$, $G$ a semisimple alg. gp., $d = \dim(G)$;

5. $H$ an affine noetherian Hopf algebra satisfying a polynomial identity, $d = GK - \dim(H)$.

Of these, the most difficult is the last: this is a 2003 theorem of Wu and Zhang.
WHAT USE IS THE AS-GORENSTEIN PROPERTY?

- There are many structural consequences: existence of Artinian quotient rings; possession of “good” dimension function; finite global dimension implies $H$ is a direct sum of prime algebras....

- Focus on one application - “duality”.

- Consider first what happens when $H$ is finite dimensional.... 
When $H$ is finite dimensional,

\[ H \text{ is a Frobenius algebra.} \]

So

\[ H^* = \text{Hom}_k(H, k) \cong \nu H^1 \]

for a certain algebra automorphism of $H$, called the \textit{Nakayama automorphism}. And then

\[ V \mapsto \text{Hom}_H(V, \nu H^1) \]

defines a contravariant equivalence of categories

\[ \{\text{left } H - \text{modules}\} \longrightarrow \{\text{right } H - \text{modules}\}. \]
To generalise the duality of Frobenius algebras to an infinite dimensional setting, we have to work at the level of the bounded derived category $\mathcal{D}^b(\text{mod}-H)$. So, in slogans:

- replace modules by *complexes of modules*;

- identify *quasi-isomorphic* complexes, (so that, e.g., a module gets identified with all of its projective or injective resolutions).
**Definition:** [Yekutieli] A *rigid dualizing complex* $R$ for an algebra $H$ is a complex of $H$–bimodules in the derived category which, via the functor

$$\text{RHom}_H(-, R),$$
defines a duality ($\equiv$ a contravariant equivalence) between $\mathcal{D}^b(\text{left } H \text{–modules})$ and $\mathcal{D}^b(\text{right } H \text{–modules})$. If $R$ exists, it’s unique.

**Example:** Let $F$ be a finite dimensional algebra. Then $R$ exists and is $F^* := \text{Hom}_k(F, k)$. So if $F$ is in addition a *Frobenius algebra*

$$R = F^* \cong \nu F^1,$$

for the Nakayama automorphism $\nu$ of $F$. 

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Theorem (B-Zhang, 2006). 1. Let $H$ be an AS-Gorenstein noetherian Hopf algebra of injective dimension $d$, with bijective antipode $S$. Then $H$ has a rigid dualizing complex, namely

$$R \cong \nu A^1[d],$$

where $\nu$ is a certain algebra automorphism of $H$, called the Nakayama automorphism.

2. $\nu$ is uniquely determined by $H$ (up to an inner automorphism). Namely,

$$\nu = S^2 \tau,$$

where $\tau$ is the left winding automorphism determined by the right structure of $\text{Ext}_H^d(k, H)$. 
CONSEQUENCES: 1. Twisted Poincaré duality: Let $H$ be as in the theorem, and assume that $H$ has finite global dimension (which is then necessarily $d$). Then for every $H$–bimodule $M$ and for all $i$,

$$H^i(A, M) = H_{d-i}(H, \nu^{-1} M).$$

Applied in the setting e.g. of group algebras, we retrieve the fact [Bieri, 1972] that polycyclic-by-finite groups are “Poincaré duality groups”. 
CONSEQUENCES: 2. The antipode: Let $H$ be as in the theorem. Then $(H, \Delta^{op}, S^{-1}, \varepsilon)$ is also a Hopf algebra (with the same algebra structure as $H$). Equating the resulting 2 answers for $\nu$, we get

$$S^4 = \gamma \circ \rho \circ \tau^{-1}$$

where $\gamma$ is inner and $\rho$ is the right winding automorphism got from $\text{Ext}^d_H(k, H)$.

Note: For $H$ finite dimensional, this is a 1976 result of Radford, with an explicit $\gamma$.

QUESTION: What is $\gamma$ when $H$ is infinite dimensional?