Topics in calculus and linear algebra
Summaries from a term in Maths 2D
Introduction

This course represents a loosely related collection of essential topics in lower-division mathematics; certainly required reading for any maths student but also of relevance to a wide range of other fields of study. Aptly named topics in calculus and linear algebra, a central theme in this course is the interdependence — one that is often not highlighted to students — between calculus and linear algebra. These notes are intended to serve as a guide-for-revision and will hopefully complement the course notes, tutorial exercises and lecture notes for the course. The lecture summaries originally appeared on the course’s moodle page, typically the day the lecture was given. It is an extremely enjoyable course to teach; I hope it is equally enjoyable to learn this material.

Liam Watson, Glasgow, Spring 2014.
This course combines objects/ideas from calculus and linear algebra. As a point of departure for the course, this lecture reviewed two things: (1) Linear, first order differential equations and some tricks for solving them (namely, separation of variables and integrating factors); and (2) diagonalization of matrices (working a $2 \times 2$ example) by finding eigenvalues and eigenvectors. As an application of the latter, we saw that it is easy to take large powers of a diagonalizable matrix, once the diagonal matrix $D$ and conjugating matrix $P$ have been found.
In this lecture we saw another application of matrix diagonalization. Namely, we used matrix notation to express a system of linear differential equations (that is, we wrote $\dot{x} = Ax$), and then applied diagonalization ($PAP^{-1} = D$) to simplify — and solve — the system using a change of variables of the form $x = Py$. 
Non-diagonalizable $2 \times 2$ and $3 \times 3$ matrices can be “almost” diagonalized: There exists an invertible matrix $P$ and an “almost diagonal” matrix $J$ such that $P^{-1}AP = J$. We then showed how this matrix decomposition can be used to solve non-diagonalizable systems of ordinary differential equations $\dot{x} = Ax$. Changing variables to $y = P^{-1}x$ gives an uncoupled system $\dot{y} = Jy$ that can be solved using the methods from Lecture 1.
Lecture 4
Diagonalizing real symmetric matrices

We made two important observations about real symmetric matrices: (1) Any eigenvalue must be a real number; and (2) given distinct eigenvalues the corresponding eigenvectors must be orthogonal. These are key observations towards the important fact that a real-valued symmetric matrix $A$ is diagonalizable with $Q^T A Q = D$ where $Q$ is an orthogonal matrix.
Today we looked at some examples illustrating how to diagonalize a real symmetric matrix $A$. In particular, we saw how to construct an orthogonal matrix $Q$ so that $Q^T AQ$ is a diagonal matrix. In a $3 \times 3$ example, we illustrated a key step of the proof of the theorem (from last time) that this form of diagonalization is always possible for real symmetric matrices: We saw that if some part of the matrix $A$ is already in diagonal form, then the orthogonal matrix $Q$ can be chosen so that diagonalizing leaves that part of $A$ unchanged.
Today we saw the definition of a real quadratic form of order $n$. In summary, this is a real-valued, multi-variable function $q : \mathbb{R}^n \to \mathbb{R}$ with a special property: if $\mathbf{x}$ is a vector in $\mathbb{R}^n$ then $q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where $A$ is an $n \times n$ real symmetric matrix.
Lecture 7
Quadratic forms, continued: completing the square

From last time, we know that a real symmetric matrix \( A \) determines a real quadratic form \( q = x^T A x \). Interpreting this as a quadratic function in variables \( (x_1, x_2, \ldots, x_n) \) we saw that, by the process of completing the square, it is possible to find a change of variables \( y = Px \) so that \( P^T A P = D \) is a diagonal matrix and \( q = y^T D y \) is expressed only in terms of squares of the variables \( (y_1, y_2, \ldots, y_n) \). The key observation is that this gives another method for diagonalizing a symmetric matrix \( A \); this method does not make reference to eigenvalues (namely, the diagonal entries need not be eigenvalues of \( A \)).
We now have a theorem that guarantees any quadratic form $q = x^T A x$ (of order $n$) may be expressed (via non-singular change of variables) as

$$q = \alpha_1 y_1^2 + \alpha_2 y_2^2 + \cdots + \alpha_p y_p^2 - \alpha_{p+1} y_{p+1}^2 - \cdots - \alpha_r y_r^2$$

where $p \leq r \leq n$ and each $\alpha_i$ is a positive real number. Naively, this is obtained by completing the square ($n$ times); today we saw what to do in the case when completing the square breaks down. The key idea is to introduce a change of variables of the form $x_i x_j = (u + v)(u - v) = u^2 - v^2$, effectively producing new squares to work with from a cross term $x_i x_j$. Along the way we saw that the $\alpha_i$ in the expression [♠] are not uniquely determined (you and a friend could have different changes of variables!). However, as we will see next time, Sylvester’s Law of Inertia tells us that the integers $p$ and $r$ are uniquely determined by the quadratic form $q$ (that is, they are the same regardless of choices for changes of variables, etc.).
Sylvester’s Law of Inertia

Using Sylvester’s Law of Inertia (see Lecture 8) we can define the rank $r$ and signature $r - 2p$ of a quadratic form. We also defined what it means for quadratic forms and symmetric matrices to be positive-definite (positive semi-definite, etc.) and we showed how to check positive-definiteness by diagonalization.
Lecture 10
Positive definite matrices

We defined a real symmetric matrix $A$ to be positive definite if the associated quadratic form $x^T A x$ is positive definite. By appealing to orthogonal diagonalization, we proved that $A$ is positive definite if and only if $A$ has all positive eigenvalues; this results in a quick check for positive definiteness. We saw another characterization for positive definite matrices in terms of leading principal minors. By example, this method can be much simpler than certifying that all of the roots of the characteristic polynomial $\chi_A(\lambda)$ are positive and real.

Positive definite matrices are a special sub-class of a class (symmetric matrices) that we have seen have nice properties. For example, if $A, B$ are symmetric and $A$ is positive definite then it is possible to change variables via some matrix $P$ so that

$$ P^T A P = I \quad (1) $$

$$ P^T B P = D \quad (2) $$

where $D$ is a diagonal matrix. The assertion (1) should not come as a complete surprise (if it does then please ask!) but the assertion (2), achieved simultaneously, is a little surprising.
Today we saw the definition of an inner product on a vector space $V$ defined over the field of real or complex numbers; this is a function taking a pair of vectors $\mathbf{u}, \mathbf{v}$ in $V$ that returns an element of the ground field $\langle \mathbf{u}, \mathbf{v} \rangle$ satisfying a list of five conditions. Using this we defined the norm $\|\mathbf{u}\|$ of a vector $\mathbf{u}$ in $V$ to be the real number $\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$ and proved the Cauchy-Schwartz inequality for real inner product spaces:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$
Lecture 12

The Gram-Schmidt method

Given a collection of linearly independent vectors \( \{v_1, \ldots, v_n\} \) in an inner product space \( V \), the Gram-Schmidt method produces an orthonormal basis \( \{u_1, \ldots, u_n\} \) so that \( \text{span}\{u_1, \ldots, u_n\} \cong \text{span}\{v_1, \ldots, v_n\} \) (that is, these describe the same subspaces of \( V \)). Here is the recipe:

**Step 1:** Let \( x_1 = v_1 \) and \( x_k = v_k - \sum_{0<i<k} \text{proj}_{x_i}(v_k) \)

where \( \text{proj}_{x_i}(v_k) = c_{i,k} x_i \) and the coefficient (a scalar) is \( c_{i,k} = \frac{\langle x_i, v_k \rangle}{\langle x_i, x_i \rangle} \);

**Step 2:** set \( u_i = \frac{1}{|x_i|} x_i \) where \( i = 1, \ldots, n \).

We checked that \( \langle x_1, x_2 \rangle = 0 \) using the properties of the inner product; you should verify that \( \langle x_1, x_2 \rangle = 0 \) and \( \langle x_2, x_3 \rangle = 0 \) as well (it's messy but completely mechanical and follows from the properties of the inner product).
Today we returned to the proof of the following theorem (see Lecture 5), as an application of the Gram-Schmidt process:

**Theorem.** Every real symmetric matrix $A$ can be diagonalized via $Q^T A Q = D$ where $Q$ is orthogonal and $D$ consists of eigenvalues of $A$ on the diagonal.

Here is an overview of the key steps, consisting entirely of things we’ve seen in this course:

**Ingredient 1:** $A = A_1$ (an $n \times n$, real symmetric matrix) has a real eigenvalue $\lambda_1$ for which we can choose $v_1 \neq 0$ with $A_1 v_1 = \lambda_1 v_1$.

**Ingredient 2:** Given any basis $\{v_1, v_2, \cdots, v_n\}$ for $\mathbb{R}^n$ ($v_1$ is from the previous step) we can apply Gram-Schmidt to obtain an orthonormal basis $\{u_1, u_2, \cdots, u_n\}$ (and $u_1$ is still an eigenvector with eigenvalue $\lambda_1$).

**Ingredient 3:** Let $S_1$ be the orthogonal matrix whose columns are the $u_i$. Then $S_1^T A_1 S_1$ is symmetric with first column $\lambda_1 e_1$.

**Ingredient 4:** This process is repeated $n$ times so that

$$D = (S_1 S_2 \cdots S_{n-1} S_n)^T A S_1 S_2 \cdots S_{n-1} S_n$$

and $Q = S_1 S_2 \cdots S_{n-1} S_n$ is the desired orthogonal matrix.
Today I presented a quick-trip through first-year calculus. The key point (that we will wish to build on in this course) was the tangent line approximation to a function \( f : \mathbb{R} \to \mathbb{R} \) at a point \( a \), given by
\[
f(x) \approx f(a) + f'(a)(x - a),
\]
which we looked at from a few different vantage points. Starting from the **Fundamental Theorem of Calculus** we saw that
\[
f(x) = f(a) + \int_a^x f'(t) dt
\]
and by applying integration by parts this could be re-written as
\[
f(x) = f(a) + f'(a)(x - a) + \int_a^x tf''(t) dt.
\]
Next we applied the **Extreme Value Theorem** and the **Intermediate Value Theorem** (I proved the precise theorem that we needed in class) to get rid of this integral:
\[
f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(\xi)(x - a)^2
\]
where \( a \leq \xi \leq x \) (this \( \xi \) is the greek letter \( xi \)). These last two (equivalent) identities are known as **Taylor’s theorem**.

Now suppose that \( a \) is a critical point so that \( f'(a) = 0 \). If we want to understand the local behaviour of \( y = f(x) \) near the point \( a \) then we should consider points \( x = a + \epsilon \) for which \( |\epsilon| \) is very small. Then
\[
f(a + \epsilon) = f(a) + f'(a)(\epsilon) + \frac{1}{2}f''(\xi)\epsilon^2
\]
becomes
\[
f(a) + \frac{1}{2}f''(\xi)(\epsilon)^2
\]
for \( \xi \) between \( a \) and \( a + \epsilon \) and (applying continuity of \( f'' \)):
\begin{itemize}
  \item \( f''(a) > 0 \implies \frac{1}{2}f''(\xi)\epsilon^2 > 0 \implies f(a + \epsilon) > f(a) \) (so \( a \) is a local minimum)
  \item \( f''(a) < 0 \implies \frac{1}{2}f''(\xi)\epsilon^2 < 0 \implies f(a + \epsilon) < f(a) \) (so \( a \) is a local maximum)
\end{itemize}
That is, we established the second derivative test using Taylor’s Theorem!
Lecture 15
The Hessian

Today we moved from the single-variable \( f : \mathbb{R} \to \mathbb{R} \) to the multi-variable \( f : \mathbb{R}^n \to \mathbb{R} \) setting (we focussed on the case \( n = 2 \), but though arbitrary \( n \) is no more difficult). Our approach was to look at the multi-variable Taylor Theorem, by analogy with the single-variable statement. We have:

\[
f(a + p) = (p) + [\nabla f(a)]^T p + \frac{1}{2} p^T [\nabla^2 f(\xi)] p \quad [\clubsuit]
\]

where \( a \) is a point (vector!) in \( \mathbb{R}^n \) and \( p \) is a small vector, in the sense that \( ||p|| \) is a small positive number; this means that \( a + p \) describers a point near \( a \) in \( \mathbb{R}^n \). Here, you might think of \( \xi \) as some vector “between” \( a \) and \( a + p \) in the sense that \( \xi = a + tp \) for \( 0 \leq t \leq 1 \). Finally, \( \nabla f(a) \) is the Jacobian evaluated at \( a \) (this is a vector!) and \( \nabla^2 f(\xi) \) is the Hessian evaluated at \( \xi \) (this is a real symmetric matrix!). Everything in sight should be compared with Lecture 14.

We could interpret our work from last time as saying that \( f''(a) \) determines a real quadratic form of order 1; in this more general setting we have that \( \nabla^2 f(a) \) determines a real quadratic form \( q \) of order \( n \). Now if \( a \) is a critical point then \( \nabla f(a) = 0 \) and as before:

\[
q \text{ is positive definite } \implies p^T [\nabla^2 f(a)] p > 0
\]

\[
\implies \frac{1}{2} p^T [\nabla^2 f(\xi)] p > 0 \quad \text{(by continuity; } \xi \text{ is a vector near } a)\\
\implies f(a + t p) > f(a) \quad \text{(apply } [\clubsuit] \text{)}\\
\implies a \text{ is a local minimum}
\]

As before, this gives rise to the second derivative test. Fortunately, we know lots of tricks for determining when a symmetric matrix is definite...
Lecture 16

The second derivative test

From previous lectures we know that the second derivative(s) give some control of the behaviour of a function near a critical point. More precisely, given a function $f: \mathbb{R}^n \to \mathbb{R}$ and a critical point $a$ in $\mathbb{R}^n$, the Hessian $\nabla^2 f$ gives rise to a real symmetric matrix $\nabla^2 f(a)$. The associated quadratic form controls the behaviour of $f(x)$ at the critical point $a$.

The case $n = 2$ is quite illustrative. In this case $A = \nabla^2 f(a)$ gives rise to a real quadratic form of order 2

$$q(x, y) = ax^2 + 2bxy + cy^2$$

where $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. By completing the square we get

$$q(x, y) = a\left(x - \frac{b}{a}y\right)^2 + \frac{1}{a} \det(A)y^2$$

so that

- $a > 0$ & $\det(A) > 0 \implies q(x, y)$ positive definite and $a$ is a local minimum
- $a < 0$ & $\det(A) > 0 \implies q(x, y)$ negative definite and $a$ is a local minimum
- $\det(A) < 0 \implies q(x, y)$ is indefinite and $a$ is a saddle point
- $\det(A) = 0 \implies q(x, y)$ is degenerate and the test is inconclusive
A typical optimisation problem takes the form

“maximise \( f(x, y) \) subject to the constraint \( g(x, y) = 0 \).”

Sometimes it is possible to use the constraint to get a \( y \) as a function of \( x \); \( y = h(x) \), say. Then we can define the single variable, unconstrained, reduced function \( \varphi(x) = f(x, h(x)) \) and use methods from level 1 maths to solve the problem.

It’s not always possible to write \( y = h(x) \) when given the constraint \( g(x, y) = 0 \), so when this happens there is a more complicated trick: the method of Lagrange multipliers. This defines the Lagrange function, with Lagrange multiplier \( \lambda \),

\[
\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y)
\]

and we search for the critical points \( \nabla \mathcal{L} = 0 \). Notice that this is equivalent to requiring that \( \nabla f = -\lambda \nabla g \) and \( g = 0 \). A nice interpretation of the Lagrange multiplier in terms of level sets considers

\[
\mathcal{L}(x, y, \lambda) = (x, y) + \lambda [g(x, y) - c]
\]

that is, our constraint is the level set \( g(x, y) = 0 \) and nearby level sets are of the form \( g(x, y) = c \) for small values \( |c| \). Then \( \lambda \) is the rate of change of \( \mathcal{L} \) with respect to \( c \), that is, the Lagrange multiplier governs how the problem changes for small variations in the constraint.
Lecture 18

Lagrange multipliers, continued

Given a function \( f : \mathbb{R}^2 \to \mathbb{R} \) and a constraint \( g : \mathbb{R}^2 \to \mathbb{R} \) the associated Lagrange function is

\[
L(x, y, \lambda) = (x, y) + \lambda g(x, y)
\]

and the method of Lagrange multipliers suggests that we should search for critical points \( \nabla L = 0 \). We have seen that this is equivalent to asking that \( \nabla f = -\lambda \nabla g \) and \( g = 0 \) simultaneously. The latter is just the original constraint, but where does the former come from?

This is an observation about how the (single) level set \( g(x, y) = 0 \) corresponding to the constraint interacts with the (varying) level sets of the function \( f(x, y) \). More precisely, you have seen in previous courses that the vector \( \nabla g(x, y) \) is orthogonal to the curve \( g(x, y) = 0 \). On the other hand, to study a critical value \( P = (a, b) \) of the constrained function we should consider a path \( \gamma(t) = (x(t), y(t)) \) confined to the level set \( g(x, y) = 0 \) for which \( \gamma(0) = (x(0), y(0)) = P \). Then letting \( h(t) = f(x(t), y(t)) \) (the composition of \( \gamma \) and \( f \)) we know, by construction, that 0 is a critical point for \( h \). Now using the chain rule

\[
h'(t) = \langle \nabla f(x(t), y(t)), \gamma'(t) \rangle
\]

so that since \( h'(0) = 0 \) we conclude that \( \langle \nabla f(x(0), y(0)), \gamma'(0) \rangle = 0 \). This says that \( \nabla f \) must be orthogonal to the level set \( g(x, y) = 0 \) at the point \( P \). So we have shown that, at a critical point, \( \nabla f \) and \( \nabla g \) are parallel vectors. If we pick the scaling factor to be \(-\lambda\) we get the key ingredient for the Lagrange multiplier method: \( \nabla f = -\lambda \nabla g \).

It’s worth mentioning that there is something much more general going on here. If \( f : \mathbb{R}^n \to \mathbb{R} \) is constrained by \( g_i : \mathbb{R}^n \to \mathbb{R} \) for \( i = 1, \ldots, m \) (\( m \) different constraints!) then setting \( g^T = (g_1, \ldots, g_m) \) we can define

\[
L(x, \lambda) = f(x) + \langle \lambda, g(x) \rangle
\]

where \( x^T = (x_1, \ldots, x_n) \) and \( \lambda^T = (\lambda_1, \ldots, \lambda_m) \) then \( \nabla L = 0 \) will give rise to a system of \( n + m \) equations in \( n + m \) unknowns.
Recall that the MacLauren series of a function $y = f(x)$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

which gives a generalization of a polynomial (called a power series) and a new, equivalent, expression of the function for points near $x = 0$. An important observation is that the terms of this series are only defined if the function is differentiable. Fourier series give another form of series-expression for a function that relies on integrability instead of differentiability. This has certain advantages, namely, the area under a function can often be described even when derivatives don’t make sense. The series in question takes the form

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

where $a_0, a_n$ and $b_n$ are all obtained from integrals involving the given function $y = f(x)$ over the interval $[-\pi, \pi]$. The function must satisfy certain conditions (differentiability isn’t one of them though!), but we’ll look at this more carefully next time.
Fourier series: Finding the coefficients

The goal of Fourier series is to write

\[ f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \]

given a function \( f: [-\pi, \pi] \to \mathbb{R} \). The process of finding the values of the real numbers \( a_0, a_n \) and \( b_n \) involves integrating both sides of \( \bullet \) in various ways, and should remind you of all sorts of useful identities in trigonometry. For example, to obtain \( a_0 \) consider

\[
\int_{-\pi}^{\pi} f(x) dx = \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos(nx) \, dx + b_n \int_{-\pi}^{\pi} \sin(nx) \, dx \right)
\]

and notice that \( \int_{-\pi}^{\pi} \cos(nx) \, dx = \int_{-\pi}^{\pi} \sin(nx) \, dx = 0 \). An important point here is the definition of an integral as the area under a function, rather than the Fundamental Theorem of Calculus which allows you to calculate such areas by appealing to antiderivatives. As a result we found that

\[
\int_{-\pi}^{\pi} f(x) dx = \frac{a_0}{2} \int_{-\pi}^{\pi} dx \quad \text{or} \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.
\]

More generally, to find the coefficient \( b_m \) for a positive integer \( m \), we multiply \( \bullet \) through by \( \sin(mx) \) and integrate both sides to get:

\[
\int_{-\pi}^{\pi} f(x) \sin(mx) \, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin(mx) \, dx
\]

\[
+ \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos(nx) \sin(mx) \, dx + b_n \int_{-\pi}^{\pi} \sin(nx) \sin(mx) \, dx \right)
\]

Now appeal to integration under an odd function to get most of the terms to vanish, and use the angle sum formula \( \sin(A + B) \) (and \( \sin(A - B) \) too!) in what remains to take care of all but one term:

\[
\int_{-\pi}^{\pi} f(x) \sin(mx) \, dx = b_m \int_{-\pi}^{\pi} \sin(mx) \sin(mx) \, dx
\]

Another application of the angle sum formula will allow you to handle this single integral and get

\[
b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) \, dx.
\]
Lecture 21

Fourier series as linear combinations

Today we reviewed everything from last lecture from a different point of view: Consider the vector space $V$ consisting of continuous functions $f : (−\pi, \pi) \to \mathbb{R}$. Recall that $V$ has an inner product given by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$ 

Then we have shown that the infinite set $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots\}$ consists of orthogonal vectors in $V$ with respect to the inner product $\langle \cdot, \cdot \rangle$. Now the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx),$$

for real coefficients $a_0, a_n, b_n$, shows that $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots\}$ is an infinite, orthogonal basis for $V$. 
For a continuous function $f$ defined on $(-\pi, \pi)$ we have seen that the Fourier series $S_f$ may be interpreted as a linear combination of continuous functions (sin and cos) in an infinite dimensional real vector space. But we’ve also observed that continuity is not required of $f$ in order to define $S_f$. In this case one needs to address the behaviour of the Fourier series at a point of discontinuity $-\pi < c < \pi$ of $f$. It turns out that

$$2S_f(c) = f(c^-) + f(c^+)$$

(where $f(c^\pm) = \lim_{x\to c^\pm} f(x)$) so that $S_f$ takes an average value at points of discontinuity. One way to make sense of this is to remember that $f(c^-) = f(c^+)$ if and only if $f$ is continuous at the point $c$. That is, $S_f(c) = f(c)$ when $f$ is continuous at $c$, as we have already seen.
When $f$ is a function defined on $(0, \pi)$ there are various tricks for building a new function and using the associated Fourier series in order to study $f$. We saw two ways of doing this; both should be thought of as applications of the general machinery of Fourier series.

The even trick constructs an even function $f_{\text{even}}$ satisfying $f_{\text{even}}(x) = f(x)$ for $x > 0$. Then the half-range cosine series of $f$ is just the Fourier series $S_{f_{\text{even}}}(c)$.

The odd trick constructs an odd function $f_{\text{odd}}$ satisfying $f_{\text{odd}}(x) = f(x)$ for $x > 0$. Then the half-range sine series of $f$ is just the Fourier series $S_{f_{\text{odd}}}(c)$. 