# Integrability in Grassmann geometries

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# Plan

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  (c) Geometry 'on solutions': Einstein-Weyl geometry.

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- Linearisable systems.
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- Integrability in 4D.
- Monge-Ampère systems in higher dimensions.

## General construction

Let $\mathbf{Gr}(d, n)$ be the Grassmannian of $d$-dimensional linear subspaces of an $n$-dimensional vector space $V^n$. A submanifold $X \subset \mathbf{Gr}(d, n)$ gives rise to a differential system $\Sigma(X)$ that governs $d$-dimensional submanifolds of $V^n$ whose Gaussian image is contained in $X$. Since $d$-dimensional submanifolds of $V^n$ are parametrised by $n - d$ functions of $d$ variables, we will assume that the codimension of $X$ in $\mathbf{Gr}(d, n)$ also equals $n - d$: in this case $\Sigma(X)$ will be a determined system of $n - d$ first-order PDEs for $n - d$ unknown functions of $d$ independent variables.

Based on:

*B. Doubrov, E.V. Ferapontov, B. Kruglikov and and V.S. Novikov, On the integrability in Grassmann geometries: integrable systems associated with fourfolds in* $\mathbf{Gr}(3, 5)$*, arXiv:1503.02274.*

## Systems associated with fourfolds $X \subset \mathbf{Gr}(3,5)$

Introducing in $V^5$ coordinates $x^1, x^2, x^3, u, v$ one can parametrise three-dimensional submanifolds of $V^5$ in the form $u = u(x^1, x^2, x^3)$, $v = v(x^1, x^2, x^3)$. The corresponding system $\Sigma(X)$ reduces to a pair of first-order PDEs for $u$ and $v$,

$$F(u_1, u_2, u_3, v_1, v_2, v_3) = 0, \quad G(u_1, u_2, u_3, v_1, v_2, v_3) = 0, \tag{1}$$

$u_i = \partial u/\partial x^i,\ v_i = \partial v/\partial x^i$. Here the Grassmannian $\mathbf{Gr}(3,5)$ is identified with the space of $2 \times 3$ matrices,

$$U = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix},$$

and equations (1) specify a fourfold $X \subset \mathbf{Gr}(3,5)$.

# Example 1: dKP equation

The system

$$v_y - \frac{1}{2}v_x^2 - u_x = 0, \quad v_t - \frac{1}{3}v_x^3 - v_x u_x - u_y = 0,$$

defines a Bäcklund transformation between the dKP equation,

$$u_{xt} - u_x u_{xx} - u_{yy} = 0,$$

and the mdKP equation,

$$v_{xt} - (v_y - \frac{1}{2}v_x^2)v_{xx} - v_{yy} = 0.$$

## Example 2: Veronese web equation

Let $a_1 + a_2 + a_3 = 0$ and $\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 = 0$ be constants. The system

$$a_1\tilde{a}_2 u_x v_y - a_2\tilde{a}_1 u_y v_x = 0, \quad a_1\tilde{a}_3 u_x v_t - a_3\tilde{a}_1 u_t v_x = 0,$$

defines a Bäcklund transformation between the equation for $u$,

$$a_1 u_x u_{yt} + a_2 u_y u_{xt} + a_3 u_t u_{xy} = 0,$$

and the analogous equation for $v$,

$$\tilde{a}_1 v_x v_{yt} + \tilde{a}_2 v_y v_{xt} + \tilde{a}_3 v_t v_{xy} = 0.$$

## Equivalence group $\mathbf{SL}(5)$

The linear action of $\mathbf{SL}(5)$ on the variables $x^1, x^2, x^3, u, v$ naturally extends to $\mathbf{Gr}(3,5)$, identified with $2 \times 3$ matrices $U$ of partial derivatives $u_i, v_i$:

$$U \to (AU + B)(CU + D)^{-1};$$

note that the extended action is no longer linear. These transformations preserve the class of equations (1), indeed, first-order derivatives transform through first-order derivatives only. Moreover, they preserve the integrability. Two $\mathbf{SL}(5)$-related equations should be regarded as ‘the same’.

# Four equivalent approaches to the integrability in 3D

(a) The method of hydrodynamic reductions.

(b) Dispersionless Lax pairs.

(c) Geometry ‘on solutions’: Einstein-Weyl geometry.

(c) Geometry ‘on equation’: $\mathbf{GL}(2)$ geometry.

# The method of hydrodynamic reductions

Applies to quasilinear equations

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0.$$

Consists of seeking N-phase solutions

$$\mathbf{u} = \mathbf{u}(R^1, ..., R^N).$$

The phases $R^i(x, y, t)$ are required to satisfy a pair of commuting equations

$$R^i_y = \mu^i(R)R^i_x, \qquad R^i_t = \lambda^i(R)R^i_x,$$

(called hydrodynamic reductions). Commutativity conditions: $\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}$.

**Definition** A 2+1 quasilinear system is said to be integrable if, for any N, it possesses infinitely many N-component reductions parametrized by N arbitrary functions of one variable.

# Hydrodynamic reductions: continuation

First we represent system (1) in evolutionary form,

$$u_t = f(u_x, u_y, v_x, v_y), \quad v_t = g(u_x, u_y, v_x, v_y).$$

Next, we bring it into quasilinear form by choosing first-order derivatives of $u$ and $v$ as the new dependent variables, and writing out all possible consistency conditions among them. Applying the method of hydrodynamic reductions, one can write down the integrability conditions in symbolic form,

$$d^3 f = R(df, dg, d^2 f, d^2 g), \quad d^3 g = S(df, dg, d^2 f, d^2 g),$$

$40$ equations altogether (in involution!).

Theorem 1. The moduli space of non-degenerate integrable systems (1) is $30$-dimensional.

## Dispersionless Lax pairs

System

$$v_y - \frac{1}{2}v_x^2 - u_x = 0, \quad v_t - \frac{1}{3}v_x^3 - v_x u_x - u_y = 0,$$

possesses the Lax pair

$$S_y = S_x^2 + v_x S_x, \quad S_t = \frac{4}{3}S_x^3 + 2v_x S_x^2 + (u_x + v_x^2)S_x;$$

the compatibility condition $S_{yt} = S_{ty}$ is satisfied identically.

In general:

$$S_y = P(S_x, u_i, v_i), \quad S_t = Q(S_x, u_i, v_i).$$

Theorem 2. Every non-degenerate integrable system (1) possesses a dispersionless Lax pair.

Generic case is given by Odesskii-Sokolov construction.

# Geometry 'on solutions': conformal structure

Formal linearisation of system (1) results upon setting $u \to u + \epsilon p$, $v \to v + \epsilon q$, and keeping terms of order $\epsilon$. This gives linear system for $p, q$:

$$\sum_i \begin{pmatrix} F_{u_i} & F_{v_i} \\ G_{u_i} & G_{v_i} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_{x^i} = 0.$$

Dispersion relation/principal symbol is defined as

$$\det \left( \sum_i \xi_i \begin{pmatrix} F_{u_i} & F_{v_i} \\ G_{u_i} & G_{v_i} \end{pmatrix} \right) = 0.$$

This gives a conic $g^{ij}\xi_i\xi_j = 0$. For nonlinear systems, the corresponding conformal structure $g^{ij}$ depends on a solution.

# Geometry 'on solutions': Einstein-Weyl geometry

Given conformal structure $g^{ij}$, introduce the covector $\omega$,

$$\omega_k = 2g_{kj}\mathcal{D}_{x^s}(g^{js}) + \mathcal{D}_{x^k}(\ln\det g_{ij}),$$

and the symmetric Weyl connection $\mathbb{D}$ such that $\mathbb{D}_k g_{ij} = \omega_k g_{ij}$.

Theorem 3. System (1) is integrable if and only if on every solution the triple $\mathbb{D}, g, \omega$ satisfies the Einstein-Weyl equations,

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}.$$

Here $R_{(ij)}$ is the symmetrised Ricci tensor of $\mathbb{D}$, and $\Lambda$ is some function.

Einstein-Weyl geometry is integrable (Cartan, Hitchin). Thus, solutions to integrable equations carry integrable geometry.

# Geometry ‘on equation’: $\mathrm{GL}(2)$ geometry

The tangent bundle to the Grassmannian $\mathbf{Gr}(3,5)$ carries canonical generalised conformal structure defined by the family of Segre cones $du_i dv_j - du_j dv_i = 0$. Given a fourfold $X \subset \mathbf{Gr}(3,5)$, the intersection of its tangent space $\mathrm{T}X$ with the Segre cone is a two-dimensional rational cone of degree three; its projectivisation is a rational normal curve of degree three (twisted cubic). This is known as a $\mathbf{GL}(2)$ structure on $X$. It was demonstrated by Bryant that every four-dimensional $\mathbf{GL}(2)$ structure defines on $X$ a canonical affine connection (with torsion).

Theorem 4. System (1) is integrable if and only if $X$ possesses infinitely many holonomic 3-folds. This is equivalent to the condition that the curvature $R$ and the covariant derivative $\nabla T$ of the torsion $T$ of the Bryant connection are certain invariant quadratic expressions in $T$,

$$R = f(T^2), \quad \nabla T = g(T^2).$$

## Linearisable systems

Systems of Monge-Ampère type are linear combinations of minors of the $2 \times 3$ matrix $U$:

$$a^{ij}(u_i v_j - u_j v_i) + b^i u_i + c^i v_i + m = 0,$$
$$\alpha^{ij}(u_i v_j - u_j v_i) + \beta^i u_i + \gamma^i v_i + \mu = 0.$$

**Proposition.** For non-degenerate system (1), the following conditions are equivalent:

(a) System is linearisable by a transformation from the equivalence group $\mathbf{SL}(5)$.

(b) System belongs to the Monge-Ampère class.

(c) System is invariant under an $8$-dimensional subgroup of $\mathbf{SL}(5)$.

(d) The principal symbol defines conformal structure which is conformally flat on every solution.

# Linearly degenerate systems: definition

The definition is inductive. Start with a 2D system,

$$F(u_x, u_t, v_x, v_t) = 0, \quad G(u_x, u_t, v_x, v_t) = 0.$$

Writing it in evolutionary form, $u_t = f(u_x, v_x),\ v_t = g(u_x, v_x)$, differentiating by $x$ and setting $u_x = a, v_x = b$, we obtain a 2-component system of hydrodynamic type, $a_t = f(a, b)_x,\ b_t = g(a, b)_y$. The system is said to be linearly degenerate if the corresponding characteristic speeds $\lambda^i$ are constant in the direction of the associated eigenvectors $\xi_i$: $L_{\xi_i} \lambda^i = 0$.

In 3D, system (1) is said to be linearly degenerate if every its travelling wave reduction to 2D is linearly degenerate in the above sense.

## Linearly degenerate integrable systems: the Chasles construction

Let $V^5$ be a vector space, and $A \in \mathbf{SL}(5)$ a projective automorphism of $V^5$. Chasles considered a fourfold in $\mathbf{Gr}(2,5)$ spanned by 2-dimensional subspaces $\langle \xi, A\xi \rangle$ where $\xi \in V^5$. By duality, this gives a fourfold $X \subset \mathbf{Gr}(3,5)$.

**Proposition.** System (1) is linearly degenerate and integrable if and only if the associated fourfold $X$ comes from the Chasles construction.

These fourfolds can also be characterised as images of quadratic maps $\mathbb{P}^4 \dashrightarrow \mathbf{Gr}(3,5)$.

Different canonical forms are labelled by Jordan normal forms of $A$. Thus, the generic case of semisimple $A$ gives the Veronese web equation.

# Integrability in 4D

Consider 4D systems,

$$F(u_i, v_i) = 0, \quad G(u_i, v_i) = 0.$$

Theorem 5 The moduli space of non-degenerate integrable systems in 4D is 36-dimensional. Any such system is necessarily linearly degenerate. Furthermore, the following conditions are equivalent:

(a) System is integrable by the method of hydrodynamic reductions.

(b) Conformal structure $g$ defined by the principal symbol is anti-self-dual on every solution.

No explicit description yet.

Particular integrable examples are provided by systems of Monge-Ampère type.

# Monge-Ampère systems in higher dimensions

Any such system is specified by a pair of differential $d$-forms in a $(d+2)$-dimensional vector space $V$ with coordinates $x^1, \ldots, x^d, u, v$. Utilising the isomorphism between $\Lambda^d$ and $\Lambda^2$, we can reduce the theory of normal forms of Monge-Ampère systems to the classification of pencils of skew-symmetric 2-forms.

Proposition. In four dimensions, any non-degenerate system of Monge-Ampère type is $\mathbf{SL}(6)$-equivalent to one of the following normal forms:

1. $u_2 - v_1 = 0, \quad u_3 + v_4 = 0,$
2. $u_2 - v_1 = 0, \quad u_3 + v_4 + u_1v_2 - u_2v_1 = 0,$
3. $u_2 - v_1 = 0, \quad u_3v_4 - u_4v_3 - 1 = 0,$
4. $u_2 - v_1 = 0, \quad u_1 + v_2 + u_3v_4 - u_4v_3 = 0.$

All these systems are integrable by the method of hydrodynamic reductions.

All of them are equivalent to various heavenly-type equations.

## Some open problems

- For $d = 3$, the moduli space of non-degenerate integrable systems $\Sigma(X)$ associated with submanifolds of codimension $n - 3 \geq 2$ in $\mathbf{Gr}(3, n)$ is finite-dimensional. Submanifolds $X$ corresponding to 'generic' integrable systems are not algebraic.

- In higher dimensions $d \geq 4$, any non-degenerate integrable system $\Sigma(X)$ associated with a submanifold of codimension $n - d \geq 2$ in $\mathbf{Gr}(d, n)$ is necessarily linearly degenerate. Submanifolds $X$ corresponding to linearly degenerate integrable systems are rational (generally, singular).

- It would be challenging to classify integrable systems that correspond to *algebraic* fourfolds $X \subset \mathbf{Gr}(3, 5)$. The homology class of any such $X$ can be represented as $a\sigma + b\eta$ where $a, b$ are nonnegative integers, and $\sigma, \eta$ are the standard four-dimensional Schubert cycles. Which values of $a$ and $b$ are compatible with the requirement of integrability?