Holonomy of braids and its 2-category extension

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Braid groups were introduced by E. Artin in the 1920’s.

The isotopy classes of geometric braids as above form a group by composition. This is the braid group with $n$ strands denoted by $B_n$. 
Braid relations

$B_n$ is generated by $\sigma_i$, $1 \leq i \leq n - 1$ with relations

$$
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1
$$
Braid cobordisms

branched covering with simple branched points

![Diagram of a braid cobordism]

surface braid (Kamda, Carter and Saito)

braid cobordism category $\mathcal{BC}_n$:
- objects: geometric braids with $n$ strands
- morphisms: relative isotopy classes of cobordisms between braids
Plan

- Monodromy representations of logarithmic connections
- Knizhnik-Zamolodchikov (KZ) connection
- Homological representations and hypergeometric integrals
- 2-categories
- Higher holonomy
- Representations of braid cobordism category
$\mathcal{F}_n(X)$: configuration space of ordered distinct $n$ points in $X$.

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n; x_i \neq x_j \text{ if } i \neq j\},$$

$$\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$$
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\]

\[
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\]

Suppose \( X = \mathbb{C} \).

\[
\pi_1(\mathcal{F}_n(\mathbb{C})) = P_n, \quad \pi_1(\mathcal{C}_n(\mathbb{C})) = B_n
\]

We set \( X_n = \mathcal{F}_n(\mathbb{C}) \)
We set
\[ \omega_{ij} = d \log(z_i - z_j), \quad 1 \leq i \neq j \leq n. \]

Consider a total differential equation of the form \[ d\phi = \omega \phi \] for a logarithmic form
\[ \omega = \sum_{i<j} A_{ij} \omega_{ij} \]

with \( A_{ij} \in M_m(\mathbb{C}) \).
As the flatness condition we infinitesimal pure braid relations

\[[A_{ik}, A_{ij} + A_{jk}] = 0, \quad (i, j, k \text{ distinct}),\]
\[[A_{ij}, A_{k\ell}] = 0, \quad (i, j, k, \ell \text{ distinct})\]

Algebra of horizontal chord diagrams:

\[
\begin{align*}
X_{ik}X_{ij} & \quad X_{ij}X_{ik} & \quad X_{ik}X_{jk} & \quad X_{jk}X_{ik}
\end{align*}
\]
$\omega_1, \cdots, \omega_k$: differential forms on $M$

$\Omega M$: loop space $M$

$$\Delta_k = \{(t_1, \cdots, t_k) \in \mathbb{R}^k; 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}$$

$$\varphi: \Delta_k \times \Omega M \rightarrow \overbrace{M \times \cdots \times M}^{k}$$

defined by $\varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k))$
K. T. Chen’s iterated integrals of differential forms

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\( \Omega M : \) loop space \( M \)

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defined by \( \varphi(t_1, \cdots, t_k; \gamma) = (\gamma(t_1), \cdots, \gamma(t_k)) \)

The \textit{iterated integral} of \( \omega_1, \cdots, \omega_k \) is defined as

\[ \int \omega_1 \cdots \omega_k = \int_{\Delta_k} \varphi^* (\omega_1 \times \cdots \times \omega_k) \]
The expression
\[ \int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k) \]
is the integration along fiber with respect to the projection
\[ p : \Delta_k \times \Omega M \to \Omega M. \]
differential form on the loop space \( \Omega M \)
with degree \( p_1 + \cdots + p_k - k \), where \( p_j = \deg \omega_j \).
As a differential form on the loop space $d \int \omega_1 \cdots \omega_k$ is

$$\sum_{j=1}^{k} (-1)^{\nu_{j-1}+1} \int \omega_1 \cdots \omega_{j-1} d\omega_j \omega_{j+1} \cdots \omega_k$$

$$+ \sum_{j=1}^{k-1} (-1)^{\nu_j+1} \int \omega_1 \cdots \omega_{j-1}(\omega_j \wedge \omega_{j+1}) \omega_{j+2} \cdots \omega_k$$

where $\nu_j = \text{deg} \omega_1 + \cdots + \text{deg} \omega_j - j$. 
Universal finite type invariants for braids

We put

$$\omega = \sum_{i<j} \omega_{ij} X_{ij}.$$  

Then there is a universal holonomy map

$$\Theta_0 : \pi_1(X_n, x_0) \longrightarrow \mathbb{C}\langle\langle X_{ij} \rangle\rangle/\alpha$$

declared by

$$\Theta_0(\gamma) = 1 + \sum_{k=1}^{\infty} \int_\gamma \omega \cdots \omega$$

$$\alpha : \text{ideal generated by infinitesimal pure braid relations}$$

$$\mathbb{C}\langle\langle X_{ij} \rangle\rangle/\alpha : \text{algebra of horizontal chord diagrams}$$
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\[ \alpha : \text{ideal generated by infinitesimal pure braid relations} \]
\[ \mathbb{C} \langle \langle X_{ij} \rangle \rangle / \alpha : \text{algebra of horizontal chord diagrams} \]
This induces an isomorphism
\[ \mathbb{C} \hat{P}_n \cong \mathbb{C} \langle \langle X_{ij} \rangle \rangle / \alpha \]
\[ \mathbb{C} \hat{P}_n : \text{Malcev completion} \]
The space of conformal blocks

Conformal field theory

\((\Sigma, p_1, \ldots, p_n) : \) Riemann surface with marked points

\(\mapsto\)

\(\mathcal{H}_\Sigma : \) complex vector space - the space of conformal blocks

The mapping class group \(\Gamma_{g,n}\) acts on \(\mathcal{H}_\Sigma\):

Quantum representations
\( \mathfrak{g} \) : complex semi-simple Lie algebra
\( \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}(\xi) \oplus \mathbb{C}c \) : affine Lie algebra with commutation relation

\[
[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \text{Res}_{\xi=0} df \ g \langle X, Y \rangle c
\]

\( \hat{\mathfrak{g}} = \mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_- \) (triangular decomposition)

For the Lie algebra \( \mathfrak{g} \) we take
\( \alpha_1, \cdots, \alpha_r \) : simple roots
\( \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \) : half sum of positive roots
$K$ : a positive integer (level)
$\theta$ : the longest root normalized as $\langle \theta, \theta \rangle = 2$
$\lambda$ : dominant integral weight s. t. $\langle \lambda, \theta \rangle \leq K$ (level $K$ weight)
$V_{\lambda}$ : irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$

**Construction of representations of $\hat{\mathfrak{g}}$**
$\mathcal{M}_{\lambda} = U(N_-)V_{\lambda}$, $N_+ V_{\lambda} = 0$

- $V_{\lambda}$ : irreducible $\mathfrak{g}$-module with highest weight $\lambda$
- $c$ acts as $K \cdot \text{id.}$

$\mathcal{H}_{\lambda}$ : irreducible quotient of $\mathcal{M}_{\lambda}$ called the integrable highest weight modules.
Suppose $\lambda_1, \cdots, \lambda_n$ are level $K$ weights.  
$p_1, \cdots, p_n \in \Sigma$ (Riemann surface of genus $g$)  
Assign highest weights $\lambda_1, \cdots, \lambda_n$ to $p_1, \cdots, p_n$.  
$\mathcal{H}_{\lambda_j}$ : irreducible representations of $\hat{g}$ with highest weight $\lambda_j$ at level $K$.  

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$H_{\lambda_j}$: irreducible representations of $\widehat{g}$ with highest weight $\lambda_j$ at level $K$.

$M_p$ denotes the set of meromorphic functions on $\Sigma$ with poles at most at $p_1, \cdots, p_n$. 

The space of conformal blocks - definition -
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$\mathcal{M}_p$ denotes the set of meromorphic functions on $\Sigma$ with poles at most at $p_1, \cdots, p_n$.

The space of conformal blocks is defined as

$$\mathcal{H}_\Sigma(p, \lambda) = H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} / (g \otimes \mathcal{M}_p)$$

where $g \otimes \mathcal{M}_p$ acts diagonally via Laurent expansions at $p_1, \cdots, p_n$. 
The space of conformal blocks determines a vector bundle over the moduli space $\mathcal{M}_{g,n}$ with a projectively flat connection.

We focus on the case of genus 0. We assign level $K$ weights $\lambda_1, \cdots \lambda_n$ and $\lambda_{\infty}$ at infinity. The space of conformal block is a quotient space of

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \otimes V_{\lambda_{n+1}}^* / \mathfrak{g}$$

In this case the above connection is the KZ connection.
$\mathfrak{g}$ : complex semi-simple Lie algebra.
$\{I_\mu\}$ : orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to \text{End}(V_i), \; 1 \leq i \leq n$ representations.
KZ connections

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\( \{I_\mu\} \): orthonormal basis of \( \mathfrak{g} \) w.r.t. Killing form.
\( \Omega = \sum_\mu I_\mu \otimes I_\mu \)
\( r_i : \mathfrak{g} \to \text{End}(V_i), \ 1 \leq i \leq n \) representations.

\( \Omega_{ij} \): the action of \( \Omega \) on the \( i \)-th and \( j \)-th components of \( V_1 \otimes \cdots \otimes V_n \).

\[
\omega = \frac{1}{\kappa} \sum_{i<j} \Omega_{ij} d \log(z_i - z_j), \quad \kappa \in \mathbb{C} \setminus \{0\}
\]

\( \omega \) defines a flat connection for a trivial vector bundle over the configuration space \( X_n = \mathcal{F}_n(\mathbb{C}) \) with fiber \( V_1 \otimes \cdots \otimes V_n \) since we have

\[
\omega \wedge \omega = 0
\]
As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

$$\theta_\kappa : B_n \to GL(V \otimes^n).$$
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$$\theta_\kappa : B_n \to GL(V^\otimes n).$$
Local system over the configuration space

We write
\[ \sum_{i=1}^{n} \lambda_i - \lambda_\infty = \sum_{j=1}^{r} k_j \alpha_j \]
and put \( m = \sum_{j=1}^{r} k_j \).

\( \pi : X_{n+m} \to X_n \) : projection defined by
\( (z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n) \).
\( X_{n,m} \) : fiber of \( \pi \).

\[ \Phi = \prod_{1 \leq i < j \leq n} (z_i - z_j) \frac{\langle \lambda_i, \lambda_j \rangle}{\kappa} \prod_{1 \leq i \leq m, 1 \leq \ell \leq n} (t_i - z_\ell) \frac{\langle \alpha_i, \lambda_\ell \rangle}{\kappa} \times \prod_{1 \leq i < j \leq m} (t_i - t_j) \frac{\langle \alpha_i, \alpha_j \rangle}{\kappa} \]

Consider the local system \( \mathcal{L} \) associated with \( \Phi \).
Put

\[ Y_{n,m} = \frac{X_{n,m}}{(\mathcal{S}_{k_1} \times \cdots \times \mathcal{S}_{k_r})} \]
Put

\[ Y_{n,m} = \frac{X_{n,m}}{(\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_r})} \]

According to Schechtman-Varchenko and others, one can construct horizontal sections of the KZ connections by means of hypergeometric integrals of the form

\[ \int_{\Delta} \Phi R(z, t) dt_1 \wedge \cdots \wedge dt_m \]

with some rational function \( R(z, t) \).
\( \Delta \) is a cycle in \( H_m(Y_{n,m}, \mathcal{L}^*) \).
We can construct a period map

\[ \phi : H_m(Y_{n,m}, \mathcal{L}^*) \to \mathcal{H}^*(p, \lambda) \]
The period map

\[ \phi : H_m(Y_{n,m}, \mathcal{L}^*) \to \mathcal{H}^*(p, \lambda) \]

is surjective and is equivariant with respect to the action of the pure braid group \( P_n \). If \( K \) is sufficiently large relative to \( \lambda_1, \ldots, \lambda_n \) the period map \( \phi \) gives an isomorphism. In particular, the linear representation

\[ \rho_{n,m} : P_n \to \text{Aut} \ \mathcal{H}^*(p, \lambda) \]

and the monodromy representation of the KZ equation

\[ \overline{\vartheta}_{k,m} : P_n \to \text{Aut} \ \mathcal{H}^*(p, \lambda) \]

are equivalent.
2-categories

objects, morphisms, 2-morphisms

```
y \bullet \quad \alpha \quad \bullet x
```

vertical composition

```
y \bullet \quad \alpha \quad \bullet x = y \bullet \quad \alpha' \cdot \alpha \quad \bullet x
```

horizontal composition

```
z \bullet \quad \alpha_1 \quad \bullet y \quad \alpha_2 \quad \bullet x = z \bullet \quad \alpha_1 \circ \alpha_2 \quad \bullet x
```
There is a work in progress to construct 2-holonomy of KZ connection for braid cobordism based on the 2-connection investigated by L. Cirio and J. Martins of the form

\[ A = \sum_{i<j} \omega_{ij} \Omega_{ij} \]

\[ B = \sum_{i<j<k} (\omega_{ij} \wedge \omega_{ik} P_{jik} + \omega_{ij} \wedge \omega_{jk} P_{ijk}), \]

where \( A \) has values in the algebra of 2-chord diagrams, a categorification of the algebra of horizontal chord diagrams and

\[ \delta B = dA + \frac{1}{2} A \wedge A. \]
Consider Chen’s formal homology connection

\[ \omega \in \Omega^*(M) \otimes \widetilde{TH_+(M)} \]

with the following properties.

1. \[ \omega = \sum x^i \otimes x_i + \cdots \], \( \{x_i\} \): basis of \( TH_+(M) \)
   \[ \text{deg } x_i = p_1 - 1 \text{ for } x_i \in H_{p_i}(M) \]
2. \[ \delta \omega + \kappa = 0 \]
3. \[ \kappa = d\omega + \epsilon(\omega)\omega \wedge \omega, \ \epsilon(\omega) = \pm 1 \]
4. \( \delta \) is a derivation of degree \(-1\)
Theorem

There is a representation of the homotopy 2-groupoid modulo isotopy

\[ Hol : \Pi_2(M)/\sim \longrightarrow \hat{TH}_+(M)_{\leq 2}/\mathcal{J} \]

where \( \mathcal{J} \) is the ideal generated by the image of

\[ \delta_3 : \hat{TH}_+(M)_3 \longrightarrow \hat{TH}_+(M)_2 \]

The ideal \( \mathcal{J} \) corresponds to the 2-flatness condition.
Representations of braid cobordism category

Consider the case \( M = X_n \).
Universal holonomy map from the homotopy path groupoid

\[
\Theta_0 : \Pi_1(X_n) \longrightarrow C\langle\langle X_{ij}\rangle\rangle
\]
given by iterated integrals.

\[
\widetilde{TH}_+^1(M) \cong C\langle\langle X_{ij}\rangle\rangle
\]

Theorem

The universal holonomy map \( \Theta_0 \) can be lifted to a representation of the braid cobordism category

\[
Hol : BC_n \longrightarrow \widetilde{TH}_+^1(M)_{\leq 2}/J
\]
Categorification and related problems

\[ C : \text{cobordism between links } L_1 \text{ and } L_2 \]
\[ \text{Kh}(C) : \text{Kh}(L_1) \to \text{Kh}(L_2) \]
invariants of 2-knots (Khovanov, Jacobson)

**Braid group action on categories**

- Khovanov-Rouquier-Lauda algebra
- Derived categories of coherent sheaves on Calabi-Yau manifolds
- Fukaya-Seidel category

**Problem**: Extend the above actions to the braid cobordism category \( \mathcal{B}C_n \).