Elliptic families of solutions of the KP equation and compact cycles in the moduli spaces of algebraic curves

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Vanishing properties of $\mathcal{M}_{g,k}$

The moduli spaces $\mathcal{M}_{g,k}$ of smooth genus $g$ Riemann surfaces with punctures have curious vanishing properties.

- **Diaz’ theorem (1986):**
  
  *There does not exist a complete (complex) cycle in $\mathcal{M}_g$ of dimension greater than $g - 2$*

  Note, that is the upper bound. The know constructions give complete cycles of dimension of order $\log_3 g$, only.

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  *The tautological ring $R^* (\mathcal{M}_{g,k})$ vanishes in dimensions greater then $g - 2 + k$*

  The tautological ring $R^* (\mathcal{M}_{g,k})$ is generated by classes

  $$\psi_i = c_1(L_i), \quad \kappa_i = p_*(\psi_i^{i+1}) \in H^*(\mathcal{M}_g).$$

  Here $L_j$ are canonical line bundles over $\mathcal{M}_{g,k}$. 
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Here $L_j$ are canonical line bundles over $\mathcal{M}_{g,k}$. 
Faber conjectured (1999) that:
\[ R^*(\mathcal{M}_{g,k}) \text{ looks "like" the cohomology ring of a compact complex variety of dimension } g - 2 + k \]
Widely accepted by experts "geometric explanation" of vanishing properties of $\mathcal{M}_{g,k}$ is the existence of its stratification by certain number of affine strata or the existence of a cover of $\mathcal{M}_{g,k}$ by certain number of open affine sets.

Historically, Arbarello first realized that a stratification of $\mathcal{M}_g$ could be useful for a study of its geometrical properties. He studied the stratification (known already for Rauch)

$$\mathcal{W}_2 \subset \mathcal{W}_3 \subset \cdots \subset \mathcal{W}_{g-1} \subset \mathcal{W}_g = \mathcal{M}_g,$$

where $\mathcal{W}_n$ if the locus of curves having a Weierstrass point of order at most $n$, and then conjectured that $\mathcal{W}_n \setminus \mathcal{W}_{n-1}$ is affine.
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Relatively recently, the author jointly with S. Grushevsky proposed an alternative approach for geometrical explanation of the vanishing properties of \( M_{g,k} \) motivated by certain constructions of the Whitham perturbation theory of integrable systems. The key elements of the new approach are:

- the moduli space \( \mathcal{M}^{(n)}_{g,k}, \, n = (n_1, \ldots, n_k) \) of smooth genus \( g \) Riemann surfaces with the fixed \( n_\alpha \)-jets of local coordinates in the neighborhoods of labeled points is the total space of a \textit{real-analytic} foliation, whose leaves \( \mathcal{L} \) are locally smooth \textit{complex subvarieties} of real codimension \( 2g \);

- on \( \mathcal{M}^{(n)}_{g,k} \) there is an ordered set of \( (\dim_{\mathbb{R}} \mathcal{L}) \) continuous functions, which restricted onto the leaves of the foliation are piecewise harmonic. Moreover, the first of these function restricted onto \( \mathcal{L} \) is a \textit{subharmonic} function.
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Proof of Arbarello’s conjecture

**Theorem**

*Any compact complex cycle in $\mathcal{M}_g$ of dimension $g - n$ must intersect $\mathcal{W}_n$.***
New upper bound for dimensions of complete (complex) cycles in the moduli space $\mathcal{M}_{ct}^g$ of stable curves of compact type.

**Theorem**

There do not exist complete complex subvarieties of $\mathcal{M}_{ct}^g$ having non empty intersection with $\mathcal{M}_g$ of dimension greater than $g - 1$.

For $g \geq 2$ the maximum dimension of complete complex subvarieties in $\mathcal{M}_{ct}^g$ is $\frac{3}{2} g - 2$. 

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New upper bound for dimensions of complete (complex) cycles in the moduli space $\mathcal{M}^c_{g}$ of stable curves of compact type.

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Previously known bounds

- **Diaz:**
  
  *there is no compact cycle in* \( \mathcal{M}^c_g \) *of dimension greater that* \( 2g - 3 \).

- **Keel and Sadun:**
  
  *for* \( g \geq 3 \) *there do not exist complete complex subvarieties of* \( \mathcal{M}^c_g \) *of dimension greater than* \( 2g - 4 \).

The proof is by easy induction arguments starting from the base \( g = 3 \). The proof of the base statement is a corollary of remarkable vanishing result:

- *there do not exist a complete complex subvarieties of the moduli space* \( A_g \) *of principally polarized abelian varieties of codimension* \( g \).
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- *there do not exist a complete complex subvarieties of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of codimension $g$.*
The foliation structure arises through identification of $\mathcal{M}_{g,k}^{(n)}$ with the moduli space of curves with fixed real-normalized meromorphic differential. By definition a real normalized meromorphic differential is a differential whose periods over any cycle on the curve are real.

**Lemma**

For any fixed singular parts of poles with pure imaginary residues, there exists a unique meromorphic differential $\Psi$, having prescribed singular part at $p_\alpha$ and such that all its periods on $\Gamma$ are real, i.e.

$$\text{Im} \left( \oint_c \Psi \right) = 0, \quad \forall \ c \in H^1(\Gamma, \mathbb{Z}).$$
A notion of real normalized differentials is "almost equivalent" to a notion of harmonic functions.

- Indeed, the real normalization implies that the imaginary part $y(p) = \text{Im } F(p)$ of the abelian integral $F(p) := \int^p \psi$ is single-valued, and hence is a *harmonic function* on $\Gamma \setminus \{p_\alpha\}$.

- Conversely, for a given harmonic function $y(p)$ on $\Gamma \setminus \{p_\alpha\}$ there exists a unique up to a constant conjugated harmonic function $x(p)$, i.e. a function $x(p)$ such that $F(p) = x(p) + i y(p)$ is *holomorphic*. Hence $\psi = dF$ is a real normalized holomorphic differential on $\Gamma \setminus \{p_\alpha\}$.

Conditions on order of poles of $\psi$ at the marked points is equivalent to certain boundary conditions on harmonic functions.
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**Definition**

A leaf $\mathcal{L}$ of the foliation on $\mathcal{M}^{(n)}_{g,k}$ defined to be the locus along which the periods of the corresponding differentials remain (covariantly) constant.

The leaves $\mathcal{L}$ of the foliation can be regarded as a generalization of the Hurwitz spaces of $\mathbb{P}^1$ covers.

It is basic fact of the Whitham theory:

**Theorem (Kr-Phong 1995)**

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Integrable systems and algebraic geometry
A set of holomorphic coordinates on $\mathcal{M}_{g, k}^{(n)}$ are "critical" values of the corresponding abelian integral $F(p) = c + \int_p \Psi$, $p \in \Gamma$:

At the generic point, where zeros $q_s$ of $\Psi$ are distinct, the coordinates on $\mathcal{L}$ are the evaluation of $F$ at these critical points:

$$\varphi_s = F(q_s), \quad \Psi(q_s) = 0, \quad s = 0, \ldots, d = \dim \mathcal{L},$$  \hspace{1cm} (1)

normalized by the condition $\sum_s \varphi_s = 0$.  

A direct corollary of the real normalization is the statement that:

- *imaginary parts* \( f_s = \text{Im}\varphi_s \) *of the critical values depend only on labeling of the critical points*

They can be arranged into decreasing order

\[
f_0 \geq f_1 \geq \cdots \geq f_{d-1} \geq f_d.
\]

After that \( f_j \) can be seen as a well-defined continuous function on \( \mathcal{M}_{g,k}^{(n)} \), which restricted onto \( \mathcal{L} \) is a piecewise harmonic function. Moreover, \( f_0 \) restricted onto \( \mathcal{L} \) is a *subharmonic function*, i.e, \( f_0 \) *has no local maximum on \( \mathcal{L} \) unless it is a constant.*
Let $X$ be a complete cycle in $\mathcal{M}_g$ and $Z$ be its preimage under the forgetful map: $\mathcal{M}_{g,2} \subset \mathcal{C}^2_g \rightarrow \mathcal{M}_g$.

→ On $Z$ the function $f_0$ (defined by critical values of real-normalized differential with two simple poles) must achieve its maximum at some point.

→ At this point the function $f_0$ achieves its maximum on $Z \cap \mathcal{L}$.

→ Hence, it is a constant on $Z \cap \mathcal{L}$.

→ If $f_0$ is a constant then (inductively) all the other functions $f_j$ are constants.

→ Then, $Z \cap \mathcal{L}$ is at most zero-dimensional, i.e. $Z$ intersects $\mathcal{L}$ transversally.

→ $\dim X \leq g - 2$
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The elliptic CM system is a system of \( N \) particles with the Hamiltonian

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H_2 = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - 2 \sum_{i \neq j} \wp(q_i - q_j),
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where \( \wp(q) \) is the Weierstrass \( \wp \)-function. It is completely integrable and admits the Lax representation \( \dot{L} = [L, M] \), where \( L = L(t, z) \) and \( M = M(t, z) \) are \((N \times N)\) matrices depending on a spectral parameter \( z \).

The spectral curves of the CM system are defined by the characteristic equation for the Lax matrix

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\Gamma_{N, \tau}^{spec} \subset \mathbb{C} \times E_\tau : R(k, z) = \det(k \cdot I - L(t, z)) = 0
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They form a \( N \)-parametric family. The parameters are the commuting Hamiltonians \( H_i \). For generic values of the parameters the spectral curves are smooth curves of genus \( N \).
Calogero-Moser curves revisited

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Integrable systems and algebraic geometry
For particular values of the parameters the spectral curve are singular.

Let $\mathcal{K}_{g,N,\tau} \subset M_g$ be a family of smooth genus $g$ algebraic curves $\Gamma$ that are the normalization $\Gamma \mapsto \Gamma_{N,\tau}^{spec}$ of $N$-particle CM system. It can be shown that:

- $\mathcal{K}_{g,N,\tau}$ is $g - 1$-dimensional affine subvariety of $M_g$.
- The closer of $\mathcal{K}_{g,N,\tau}$ as $N \to \infty$ is the whole moduli space $M_g$. 
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In terms of the real normalized differentials the locus $\mathcal{K}_g = \bigcup_{N,\tau} \mathcal{K}_{g,N,\tau}$ is characterized as:

- the locus of curves on which there exists a pair of meromorphic differentials $\Psi_1, \Psi_2$ with the only pole of the second order at a puncture $p_0$ and with integer periods

$$\oint_\gamma \Psi_i \in \mathbb{Z}$$

For $\Gamma \in \mathcal{K}_g$ the parameters $N$ and $\tau$ are recovered by the formulae

$$N = \langle \oint \Psi_1, \oint \Psi_2 \rangle, \quad \tau = \frac{\Psi_1}{\Psi_2}(p_0)$$
In terms of the real normalized differentials the locus $\mathcal{K}_g = \bigcup_{N,\tau} \mathcal{K}_{g,N,\tau}$ is characterized as:

- the locus of curves on which there exists a pair of meromorphic differentials $\Psi_1, \Psi_2$ with the only pole of the second order at a puncture $p_0$ and with integer periods

$$\int_\gamma \Psi_i \in \mathbb{Z}$$

For $\Gamma \in \mathcal{K}_g$ the parameters $N$ and $\tau$ are recovered by the formulae

$$N = \langle \int \Psi_1, \int \Psi_2 \rangle, \quad \tau = \frac{\Psi_1}{\Psi_2}(p_0)$$
Elliptic families of the KP equation solutions

The theory of the elliptic CM system is isomorphic to the theory of elliptic solutions of the KP equation. Namely, a function \( u(x, y, t) \) which is an elliptic function with respect to the variable \( x \) satisfies the KP equation if and only if it has the form

\[
 u(x, y, t) = -2 \sum_{i=1}^{N} \wp(x - q_i(y, t)) + c, \tag{2}
\]

and its poles \( q_i \) as functions of \( y \) satisfy the equations of motion of the elliptic CM system.

It can be shown directly that if \( u(x, y, t, \lambda) \) is an elliptic family of solutions of the KP equation, i.e. for fixed \( (x, y, t) \) the function \( u \) is an elliptic function of the variable \( \lambda \in E_\tau \), then it has the form

\[
 u = -2 \sum_{i=1}^{N} \left[ \lambda_i^2 \wp(\lambda - \lambda_i) - \lambda_i x x \zeta(\lambda - \lambda_i) \right] + c(x, y, t), \quad \lambda_i = \lambda_i(x, y, t). \tag{3}
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The sum of the residues vanishes for an elliptic function \( u \). Therefore, \( \sum_i \lambda_{ixx} = 0 \). We say that the poles \( \lambda_i, i = 1, \ldots, N \) are balanced if they can be presented in the form

\[
\lambda_i(x, y, t) = q_i(x, y, t) - hx, \quad \sum_{i=1}^{N} q_i(x, y, t) = \text{const},
\]

where \( h \) is an arbitrary non-zero constant.

As it was shown by Akmetshin, Volvovskiy and Kr, the balance poles of \( u \) satisfy equations that are the equations of motion of a Hamiltonian system with the phase space that is the space of functions \((q_1(x), \ldots, q_N(x), p_1(x), \ldots, p_N(x))\) with the Poisson brackets

\[
\{q_i(x), p_j(\tilde{x})\} = \delta_{ij} \delta(x - \tilde{x})
\]

and with the Hamiltonian \( \hat{H} = \int H(x) \, dx \),

\[
H = \sum_{i=1}^{N} p_i^2(h - q_i x) - \frac{1}{Nh} \left( \sum_{i=1}^{N} p_i(h - q_i x) \right)^2 - U(q), \quad (5)
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The sum of the residues vanishes for an elliptic function $u$. Therefore, $\sum \lambda_{i\alpha} = 0$. We say that the poles $\lambda_i$, $i = 1, \ldots, N$ are *balanced* if they can be presented in the form

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where $h$ is an arbitrary non-zero constant.

As it was shown by Akmetshin, Volvovskiy and Kr, the balance poles of $u$ satisfy equations that are the equations of motion of a hamiltonian system with the phase space that is the space of functions $(q_1(x), \ldots, q_N(x), p_1(x), \ldots, p_N(x))$ with the Poisson brackets

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where

\[ U(q) = \sum_{i=1}^{N} \frac{q_{i,xx}^2}{4(h - q_{i,x})} - \frac{1}{2} \sum_{j \neq i} \left[ (h - q_{j,x})q_{i,xx} - (h - q_{i,x})q_{j,xx} \right] \zeta(q_i - q_j) \]

\[ + \frac{1}{2} \sum_{j \neq i} \left[ (h - q_{j,x})^2(h - q_{i,x}) + (h - q_{j,x})(h - q_{i,x})^2 \right] \varphi(q_i - q_j) \]

\[ + \frac{\partial}{\partial x} \left( \frac{h}{2} \sum_{i \neq j} (q_{i,x} - q_{j,x}) \zeta(q_i - q_j) \right). \]
Example.

For $N = 2$ this system is a hamiltonian system on the space of two functions $q(x), p(x)$, where we set

$$q = q_1 = -q_2, \quad \frac{1}{\hbar} p(h^2 - q_x^2) = p_1(h - q_x) = -p_2(h - q_x),$$

The Poisson brackets are canonical, i.e.

$$\{ q(x), p(\tilde{x}) \} = \delta(x - \tilde{x}).$$

The Hamiltonian density $H$ in the coordinates $\{ p, q \}$ is equal to

$$H = \frac{2}{\hbar} p^2(h^2 - q_x^2) - h \frac{q_{xx}^2}{2(h^2 - q_x^2)} - 2h(h^2 - 3q_x^2) \wp(2q).$$

It was noticed by A. Shabat that the equations of motion given by this Hamiltonian are equivalent to Landau–Lifshitz equation.
The spectral curves \( \Gamma \) giving elliptic families of the KP solutions can be characterized in two equivalent ways:

- they are curves whose Jacobian contains an elliptic curve, i.e. \( E_\tau \subset J(G) \)

This is a nontrivial constraint and the space of corresponding algebraic curves has codimension \( g - 1 \) in the moduli space of all the curves.

- the locus \( \hat{\mathcal{K}}_g \) of curves on which there exists a pair of meromorphic differentials \( \Psi_1, \Psi_2 \) with the only pole of order at most \( g \) at a puncture \( p_0 \), with integer periods \( \oint_\gamma \Psi_i \in \mathbb{Z} \), and such that \( dz = \tau \Psi_1 - \Psi_2 \) is a holomorphic differential.

Note, that the holomorphic differential \( dz \) defines a map \( \Gamma \rightarrow E_\tau \).
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Integrable systems and algebraic geometry
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Integrable systems and algebraic geometry
Let $X$ be a compact cycle in $\mathcal{M}_g^{ct}$ and let $Y$ be the preimage $Y$ of $X$ under the forgetful map $\mathcal{M}_g^{ct,(g)} \to \mathcal{M}_g^{ct}$. If the dimension of $X$ is at least $g$ then:

- the intersection of $Y$ with $\hat{\mathcal{K}}_g$ is at least one-dimensional
- the compactness of $X$ implies that the value of $\tau$ along this intersection is constant.
- the subvariety $\hat{\mathcal{K}}_{g,N,\tau}$ is affine. Hence, $Y \cap \hat{\mathcal{K}}_{g,N,\tau}$ intersects the boundary.

Then simply induction arguments lead to a contradiction.
Let $X$ be a compact cycle in $M_g^{ct}$ and let $Y$ be the preimage $Y$ of $X$ under the forgetful map $M_{g,1}^{ct,(g)} \to M_g^{ct}$. If the dimension of $X$ is at least $g$ then:

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Idea of the proof of the main theorem

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HAPPY BIRTHDAY SASHA!