Deformed quantum CMS systems and symmetric Lie superalgebras

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Algebra of integrals

Deformed trigonometric CMS operator of the type A has the form

$$\mathcal{L}_2 = \sum_{i=1}^{n}\left(x_i\frac{\partial}{\partial x_i}\right)^2 + k\sum_{j=1}^{m}\left(y_j\frac{\partial}{\partial y_j}\right)^2 -$$

$$k\sum_{i<j}^{n}\frac{x_i+x_j}{x_i-x_j}\left(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\right) -$$

$$\sum_{i<j}^{m}\frac{y_i+y_j}{y_i-y_j}\left(y_i\frac{\partial}{\partial y_i} - y_j\frac{\partial}{\partial y_j}\right) -$$

*joint with A.P. Veselov, arXiv:1412.8768

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{x_i+y_j}{x_i-y_j}\left(x_i\frac{\partial}{\partial x_i}-ky_j\frac{\partial}{\partial y_j}\right)$$

It will be convenient to denote $x_{n+j}:=y_j,\ j=1,\ldots,m$ and to introduce parity function $p(i)=0,\ i=1,\ldots,n,\ p(i)=1,\ i=n+1,\ldots,n+m$. We also introduce the notation

$$\partial_j=x_j\frac{\partial}{\partial x_j},\ f_{ij}=\frac{x_i}{x_i-x_j}\ j=1,\ldots,n+m.$$

and denoted the algebra generated by these elements as $\mathfrak{A}_{n,m}$.

Theorem 1. *Algebra of the KMC problem can be described in terms of generators and relations in the following way*

$$f_{ij}+f_{ji}=1,\ \ f_{ij}f_{il}+f_{ji}f_{jl}+f_{li}f_{lj}=1,\ \ i<j<l$$

$$\partial_l f_{ij}-f_{ij}\partial_l=(\delta_{li}-\delta_{lj})f_{ij}f_{ji}$$

Let us denote by $C(\mathcal{L}_2)$ the centraliser of $\mathcal{L}_2$ in $\mathfrak{A}_{n,m}$. It is an important problem to describe the centraliser.

Definition 1. *Harish-Chandra homomorphism is the homomorphism* $\varphi : \mathfrak{A}_{n,m} \to \mathbb{C}[\xi_1, \dots, \xi_{n+m}]$ *such that*

$$\varphi(\partial_i) = \xi_i, \, , i = 1, \dots, n+m,$$

$$\varphi(f_{ij}) = 1, \, i < j, i, j = 1, \dots, n+m$$

Let $\rho(k) \in \mathbb{C}^{n+m}$ be the following deformed analogue of the Weyl vector

$$\rho(k) = \frac{1}{2}\sum_{i=1}^{n}(k(2i-n-1)-m)e_i +$$

$$\frac{1}{2}\sum_{j=1}^{m}(k^{-1}(2j-m-1)+n)e_{j+n}$$

and consider the bilinear form $(,)$ on $\mathbb{C}^{n+m}$ defined in the basis $e_1, \dots, e_{n+m}$ by

$$(e_i, e_i) = 1, \, i = 1, \dots, n,$$

$$(e_j, e_j) = k, \, j = n+1, \dots, n+m.$$

Theorem 2. *When restricted to* $C(\mathcal{L}_2)$ *Harish-Chandra homomorphism is injective and if* k *is not rational number its image is the subalgebra* $\Lambda^*_{n,m}(k) \subset \mathbb{C}[\xi_1, \ldots, \xi_{n+m}]$ *consisting of polynomials with the following properties:*

$$f(w(\xi + \rho(k))) = f(\xi + \rho(k)), \quad w \in S_n \times S_m$$

and for every $i \in \{1, \ldots, n\},\ j \in \{n+1, \ldots, n+m\}$

$$f(\xi - e_i + e_j) = f(\xi) \tag{1}$$

on the hyperplane $(\xi + \rho(k), e_i - e_j) = \frac{1}{2}(1+k)$.

Therefore this centraliser is always commutative and quite big. How to construct integrals? In other words if we know $P \in \Lambda^*_{n,m}(k)$ how can one construct $\mathcal{L} \in C(\mathcal{L}_2)$ such that

$$\varphi(\mathcal{L}) = P \ ?$$

In the non deformed situation it is possible to construct inverse map by means of Lax pair or

Dunkl operators. In our situation as far as we know it is possible to use only quantum Lax pair.

Introduce a version of quantum Moser $(n+m)\times(n+m)$-matrices L, M by

$$L_{ii} = k^{p(i)}\partial_i - \sum_{j\neq i} k^{1-p(j)}\frac{x_i}{x_i - x_j},$$

$$L_{ij} = k^{1-p(j)}\frac{x_i}{x_i - x_j},\ i \neq j$$

$$M_{ii} = -\sum_{j\neq i}\frac{2k^{1-p(j)}x_i x_j}{(x_i - x_j)^2},\ M_{ij} = \frac{2k^{1-p(j)}x_i x_j}{(x_i - x_j)^2},\ i \neq j.$$

Note that matrix M satisfies the relations

$$Me = e^*M = 0,\ e = (\underbrace{1,\ldots,1}_{n+m})^t,$$

$$e^* = (\underbrace{1\ldots,1}_{n},\underbrace{1/k,\ldots,1/k}_{m}).$$

Define also matrix Hamiltonian H by

$$H = \mathcal{L}_2 E_{n+m},$$

where E_{n+m} is a unit matrix of the size $(n+m)\times(n+m)$. Then it can be checked that these matrices satisfy Lax relation

$$[L,H]=[L,M].$$

This implies that the "deformed total trace"

$$\mathcal{L}_s=\sum_{i,j}k^{-p(i)}(L^s)_{ij}$$

commute with $\mathcal{L}_2$.

Corollary. *Operators $\mathcal{L}_s$ commute with each other.*

Let us denote by $\mathcal{D}_{n,m}$ the algebra generated by the above operators. This algebra does not always coincide with the centraliser.

Spectral decomposition

Let now $\Lambda_{n,m}$ be the algebra consisting of $S_n\times S_m$-invariant Laurent polynomials

$$f\in\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1},y_1^{\pm1}\dots,y_m^{\pm1}]^{S_n\times S_m}$$

satisfying the quasi-invariance conditions

$$x_i \frac{\partial f}{\partial x_i} - k y_j \frac{\partial f}{\partial y_j} \equiv 0 \tag{2}$$

on the hyperplane $x_i = y_j$ for all $i = 1, \dots, n$, $j = 1, \dots, m$ with k being arbitrary and the same as in the definition of $\mathcal{D}_{n,m}$. We claim that the algebra $\mathcal{D}_{n,m}$ preserves it.

Theorem 3. *The operators $\mathcal{L}_s$ for all $s = 1, 2, \dots$ map the algebra $\Lambda_{n,m}$ to itself.*

So we can consider $\Lambda_{n,m}$ as a module over algebra $\mathcal{D}_{n,m}$.

Let $\chi : \mathcal{D}_{n,m} \to \mathbb{C}$ be a homomorphism and define the corresponding *generalised eigenspace* $\mathfrak{A}_{n,m}(\chi)$ as the set of all $f \in \mathfrak{A}_{n,m}$ such that for every $D \in \mathcal{D}_{n,m}$ there exists $N \in \mathbb{N}$ such that $(D - \chi(D))^N(f) = 0$. If the dimension of $\mathfrak{A}_{n,m}(\chi)$ is finite then such N can be chosen independent on f.

We say that the integral weight $\lambda \in X_{n,m} = \mathbb{Z}^{n+m}$ *dominant* if

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \ \lambda_{n+1} \geq \lambda_{n+2} \geq \cdots \geq \lambda_{n+m}.$
The set of dominant weights is denoted $X_{n,m}^{+}$.

For every $\lambda \in X_{n,m}^{+}$ we define the homomorphism $\chi_\lambda : \mathfrak{D}_{n,m} \to \mathbb{C}$ by

$$\chi_\lambda(D) = \varphi(D)(\lambda), \ D \in \mathfrak{D}_{n,m}$$

where φ is the Harish-Chandra homomorphism.

Proposition 1. *Algebra $\mathfrak{A}_{n,m}$ as a module over the algebra $\mathfrak{D}_{n,m}$ can be decomposed into direct sum of generalised eigenspaces*

$$\mathfrak{A}_{n,m} = \oplus_\chi \mathfrak{A}_{n,m}(\chi), \tag{3}$$

where the sum is taken over the set of the homomorphisms χ_λ, $\lambda \in X_{n,m}^{+}$.

Example 1. *Let $n = m = 1$ and let $k \notin \mathbb{Q}$.*

$$\Lambda_{1,1} = \{f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \mid \partial_x f - k\partial_y f \in (x-y)\}$$

Set for integers (λ, μ) :

$$P_{\lambda,\mu} = x^\lambda y^\mu - \frac{\lambda - k\mu}{\lambda - 1 - k(\mu+1)} x^{\lambda-1} y^{\mu+1},$$

$$(\lambda, \mu) \neq (0,0),\ (1,-1)$$

$$P_{0,0} = 1,\ P_{1,-1} = \frac{x}{y} + \frac{y}{x}$$

Lemma 1. *We have the decomposition into generalised eigenspaces*

$$\mathfrak{A}_{1,1} = \bigoplus_{(\lambda,\mu)\neq(0,0),(1,1)} < P_{\lambda,\mu} > \oplus < P_{(0,0)},\ P_{(1-1)} > .$$

The image of the algebra of the integrals in the generalised eigen-space $< P_{(0,0)}, P_{(1-1)} >$ *coincides with the algebra of dual numbers* $\mathbb{C}[\varepsilon]$, $\varepsilon^2 = 0$.

If $k \in \mathbb{Q}$ the decomposition may be different. Let us consider two most important examples.

Example 2. *Let $k = -1$. Then*

$$\mathfrak{A}_{1,1} = \{f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \mid \partial_x f + \partial_y f \in (x - y)\}$$

This is so called algebra of supersymmetric polynomials. In this case we have

$$\mathfrak{A}_{1,1} = \bigoplus_{\lambda+\mu\neq 0} < P_{\lambda,\mu} > \oplus < \frac{x^a}{y^a}, a \in \mathbb{Z} >$$

and all generalised eigen-spaces are just eigenspaces and one of them is infinite dimensional one.

Example 3. *Let $k = -1/2$. Then*

$$\mathfrak{A}_{1,1} = \{f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \mid \partial_x f + 1/2\partial_y f \in (x - y)\}$$

In this case we have

$$\mathfrak{A}_{1,1} = \bigoplus_{2\lambda+\mu\neq 0,1} < P_{\lambda,\mu} > \bigoplus_{a\in\mathbb{Z}} < \frac{x^a}{y^{2a}}, \frac{x^a}{y^{2a}}\left(\frac{x}{y} + \frac{y}{x}\right) >$$

and all generalised eigen-spaces are one dimensional or two dimensional.

Representations of the Lie superalgebra $\mathfrak{gl}(n,m)$

In this section we suppose that $k=-1$.

According to the PBW theorem any Lie algebra can considered as a subspace in the accociative algebra A with the bracket

$$[a,b]=ab-ba,\ \ a,b\in A$$

Let now $A=A_0\oplus A_1$ be a $\mathbb{Z}_2$ graded algebra, with parity function $p(a)=0$ if $a\in A_0$ and $p(a)=1$ if $a\in A_1$. Then in the similar way we can define a bracket in A for homogeneous elements by the formula

$$[a,b]=ab-(-1)^{p(a)p(b)}ba$$

In this section we will make connections with representation theory of Lie superalgebra $\mathfrak{gl}(n,m)$.

Consider $n+m$ dimensional space

$$V=<e_1,\dots,e_{n+m}>$$

with the a parity function $p(i) = 0, 1 \leq i \leq n$, $p(i) = 1$, $n < i \leq n + m$.

Let $L(V)$ be the corresponding algebra of linear operators. Define a parity on $L(V)$ by the rule $p(E_{ij}) = p(i) + p(j)$.

This is a $\mathbb{Z}/2\mathbb{Z}$ grading and we can define supercommutator as before. This algebra is called the general liner Lie superalgebra $\mathfrak{gl}(n, m)$. Consider the diagonal matrices

$$\mathfrak{h} =< E_{11}, \ldots, E_{n+m,n+m} > .$$

This is a Cartan subalgebra. Let us denote by $\mathcal{F}$ the category of finite dimensional modules with semi-simple action of $\mathfrak{h}$ and with integers eigenvalues of E_{ii}.

Definition 2. *Grothendieck algebra $K(\mathcal{F})$ over $\mathbb{C}$ of the category $\mathcal{F}$ is the algebra generated by* *isomorphism classes of finite dimensional representations and defining relations*

$$[V] = [V_1] + [V_2] \text{ if } 0 \to V_1 \to V \to V_2 \to 0$$

and

$$[V_1 \otimes V_2] = [V_1][V_2]$$

It is easy to see that $K(\mathcal{F})$ has a basis consisting of the classes of irreducible representations.

Definition 3. *Let V be a finite dimensional $\mathfrak{gl}(n,m)$ module. Let us represent it as a direct sum of irreducible $\mathfrak{h}$ modules*

$$V = \bigoplus_{\lambda \in (\mathfrak{f})^*} V(\lambda)$$

Then the function

$$Sch(V) = \sum_{\lambda} sdim(V(\lambda))x^{\lambda}$$

is called the super-character of V, where $sdim V(\lambda.\mu)$ $\dim V(\lambda.\mu)_0 - \dim V(\lambda.\mu)_1$.

We also can define in the usual way the universal enveloping algebra $U(\mathfrak{gl}(n,m))$ and its center $Z(\mathfrak{gl}(n,m))$. It can be describe explicitly in terms of recurrent relations (goes back to Gelfand)

$$E_{ij}^{(s)} = \sum_{l} (-1)^{p(l)} E_{il} E_{lj}^{(s-1)}$$

with $E_{ij}^{(1)} = E_{ij}$. One can check that these elements satisfy the following commutation relations

$$[E_{ij}, E_{st}^{(l)}] = \delta_{js} E_{it}^{(l)} - (-1)^{(p(i)+p(j))(p(s)+p(t))} \delta_{it} E_{sj}^{(l)},$$

which imply that the elements $Z_s = \sum_i E_{ii}^{(s)}$ are central.

Every element from the centre acts as a scalar operator in every irreducible module. Therefore we have a natural action $Z(\mathfrak{gl}(n,m))$ on the $K(\mathcal{F})$.

The main result can be stated as following.

Theorem 4. *The supercharacter map*

$$sch : K(\mathfrak{gl}(n,m)) \longrightarrow \Lambda_{n,m}$$

is an isomorphism of algebras and transforms the action of $Z(\mathfrak{gl}(n,m))$ *to the action of* $\mathcal{D}_{n,m}$*. In particular the elements* Z_s *goes to elements* $\mathcal{L}_s$*.*

Representations of the Lie superalgebra $\mathfrak{gl}(n,2m)$

In this section we suppose that $k = -1/2$.

Symmetric Lie superalgebra is a pair $(\mathfrak{g},\theta)$, where θ is an involutional automorphism. So we have the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces of θ :

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p},\ [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}.$$

In order to define automorphism we need to define its fixed subalgebra $\mathfrak{k}$. Let us take $\mathfrak{g} =$

$\mathfrak{gl}(n, 2m)$ and $\mathfrak{p}$ be the subalgerba which preserves the bilinear form

$$(e_i, e_i) = 1,\ 1 \le i \le n,\ (e_{n+2j-1}, e_{n+2j}) = 1,$$

$$(e_{n+2j}, e_{n+2j-1}) = -1,\ 1 \le j \le m$$

and automorphism θ can be recovered from the equality

$$(\theta(x)v, w) + (-1)^{p(x)p(v)}(v, xw) = 0$$

There is a natural surjective homomorphism

$$\psi : Z(\mathfrak{gl}(n, 2m)) \longrightarrow \mathcal{D}_{n,m}$$

which is called the radial part homomorphism. It can be shown that in particularly

$$\psi(Z_s) = 2^s \mathcal{L}_s$$

Therefore we can consider $\Lambda_{n,m}$ as a module over the algebra $Z(\mathfrak{gl}(n, 2m))$.

Theorem 5. *Let $\mathfrak{D}_{n,m}$ be the algebra deformed CMS operators with parameter $k = -\frac{1}{2}$*

which naturally acts on the algebra $\Lambda_{n,m}$ *deformed Laurent polynomials with the parameter* $k = -\frac{1}{2}$ *and*

$$\Lambda_{n,m} = \bigoplus_{\chi} \Lambda_{n,m}(\chi)$$

be the decomposition into direct sum of the generalised eigenspaces.

Then for any finite dimensional generalised eigenspace $\Lambda_{n,m}(\chi)$ *their exists a unique projective indecomposable module* P *over* $\mathfrak{gl}(n, 2m)$ *and a natural map*

$$\psi : (P^*)^{\mathfrak{b}} \longrightarrow \Lambda_{n,m}(\chi)$$

which is an isomorphism of the vector spaces and $Z(\mathfrak{gl}(n, 2m))$ *modules.*