Classical integrable defects as quasi Bäcklund transformations

Anastasia Doikou

Heriot-Watt University

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General frame

- Integrable defects (quantum level) impose severe constraints on relevant algebraic and physical quantities (e.g. scattering amplitudes) (*Delfino, Mussardo, Simonetti, Konic, LeClair,* ....)
- In discrete integrable systems there is a systematic description of local defects based on QISM (*Faddeev, Takhtajan, Sklyanin...*)
- In integrable field theories a defect is introduced as discontinuity plus gluing conditions (also defects as "frozen" BTs, (*Bowcock, Corrigan, Zambon,...*)), integrability issue not systematically addressed; other attempts (*Caudrelier, Kundu, Habibulin,...*)
- We developed a systematic *algebraic* means to investigate integrable theories with point like defects. Integrability is ensured by construction. A systematic connections with the Bäcklund transformation is also provided.

Outline

1. The general frame
   - The defect $\tilde{L}$-matrix
   - The classical quadratic algebra
2. Local integrals of motion, and relevant Lax pairs for the sine-Gordon model
3. The discrete case, integrals of motion, the Toda chain
4. "Dual description", local equations of motion: *Quasi Bäcklund transformation*
5. Discussion and future perspectives

The defect

The problem, point-like defect at x_0 (continuum) or at n^{th} site (discrete):

$$\mathcal{H} = \int dx\ H^+(x) + \int dx\ H^-(x) + \mathcal{D}(x_0)$$

$$\mathcal{H} = \sum_j H_j^+ + \sum_j H_j^- + \mathcal{D}_{n,n\pm 1}$$

$H^\pm$ the left right bulk Hamiltonian (densities). $\mathcal{D}$ is the defect contribution, such that integrability is preserved: *non-trivial task*. However, both at classical and quantum level a systematic algebraic means exist.

Classical Integrability

The Lax pair $\mathbb{U}$, $\mathbb{V}$; the linear auxiliary problem (e.g. *Faddeev-Takhtajan*):

$$\frac{\partial \Psi(x,t)}{\partial x} = \mathbb{U}(x,t)\ \Psi(x,t)$$
$$\frac{\partial \Psi(x,t)}{\partial t} = \mathbb{V}(x,t)\ \Psi(x,t)$$

Compatibility condition leads to

Zero curvature condition

$$\dot{\mathbb{U}}(x,t) - \mathbb{V}'(x,t) + \Big[\mathbb{U}(x,t), \mathbb{V}(x,t)\Big] = 0$$

Gives rise to the equations of motion of the system.

The monodromy matrix

The continuum monodromy matrix

$$T(x_0, y_0, \lambda) = P \exp \Big\{ \int_{x_0}^{y_0} dx \ \mathbb{U}(x) \Big\}$$

Solution of the differential equation

$$\frac{\partial T(x,y)}{\partial x} = \mathbb{U}(x,t) \ T(x,y)$$

$\mathbb{U}$ obeys linear classical algebra, T satisfies the:

Classical algebra

$$\Big\{ T_a(\lambda), \ T_b(\mu) \Big\} = \Big[r_{ab}(\lambda - \mu), \ T_a(\lambda) \ T_b(\mu) \Big]$$

The classical r-matrix satisfies the CYBE (*Sklyanin, Semenov-Tian-Shansky*)

$$\boxed{[r_{12}, \ r_{13}] + [r_{12}, \ r_{23}] + [r_{13}, \ r_{23}] = 0.}$$

Classical integrability

The monodromy matrix T satisfies the classical algebra, thus

The transfer matrix

$$t(\lambda) = Tr\ T(\lambda)$$

provides the charges in involution;

$$\Big\{t(\lambda),\ t(\mu)\Big\} = 0, \qquad \ln t(\lambda) = \sum_m \frac{\mathcal{I}^{(m)}}{\lambda^m}$$

integrability ensured by construction. $\ln t(\lambda) \ \to$ *local* integrals of motion

The defect frame

The key object, modified monodromy:

Defect monodromy matrix

$$T(L,-L,\lambda) = T^{+}(L,x_0,\lambda)\ \tilde{L}(x_0,\lambda)\ T^{-}(x_0,-L,\lambda)$$

where we define

$$T^{\pm} = P\exp\Big\{\int dx\ \mathbb{U}^{\pm}(x)\Big\}$$

The L defect matrix obeys

$$\Big\{\tilde{L}_a(\lambda_1),\ \tilde{L}_b(\lambda_2)\Big\} = \Big[r_{ab}(\lambda_1-\lambda_2),\ \tilde{L}_a(\lambda_1)\ \tilde{L}_b(\lambda_2)\Big]$$

$T^{\pm}$ satisfy the classical algebra, thus T obeys the same algebra, integrability also ensured

The defect frame

Auxiliary linear problem for $\mathbb{U}^{\pm}$, $\mathbb{V}^{\pm}$ for the defect theory:

$$\frac{\partial \Psi^{\pm}(x,t)}{\partial x} = \mathbb{U}^{\pm}\ \Psi^{\pm}(x,t)$$
$$\frac{\partial \Psi^{\pm}(x,t)}{\partial t} = \mathbb{V}^{\pm}\ \Psi^{\pm}(x,t)$$

The corresponding

Zero curvature condition

$$\dot{\mathbb{U}}^{\pm}(x,t) - \mathbb{V}^{\pm\prime}(x,t) + \Big[\mathbb{U}^{\pm}(x,t), \mathbb{V}^{\pm}(x,t)\Big] = 0 \qquad x \neq x_0$$

On the defect point

Defect zero curvature condition

$$\Psi^{+}(x) = \tilde{L}(x)\ \Psi^{-}(x)$$

$$\frac{d\tilde{L}(x_0)}{dt} = \tilde{\mathbb{V}}^{+}(x_0)L(x_0) - L(x_0)\tilde{\mathbb{V}}^{-}(x_0)$$

local IM

First recall that:

$$\frac{\partial T^{\pm}(x,y,t)}{\partial x} = \mathbb{U}^{\pm}(x,t)\ T^{\pm}(x,y,t)$$

Based on the latter consider the decomposition ansatz:

$$T^{\pm}(x,y;\lambda) = (1 + W^{\pm}(x))e^{Z^{\pm}(x,y)}(1 + W^{\pm}(y))^{-1}$$

W anti-diagonal, Z diagonal. Also,

$$W^{\pm} = \sum_{n=0}^{\infty} \frac{W^{\pm(n)}}{u^n}, \qquad Z^{\pm} = \sum_{n=-1}^{\infty} \frac{Z^{\pm(n)}}{u^n}$$

in the sine-Gordon case $u = e^{\lambda}$. Substituting the ansatz to the differential equation above identify $W^{\pm(n)}$, $Z^{\pm(n)}$ matrices.

Substitution leads to Riccati-type:

Differential equations

$$\frac{dW^{\pm}}{dx} + W^{\pm}\mathbb{U}_d - \mathbb{U}_d W^{\pm} + W^{\pm}\mathbb{U}_a^{\pm} W^{\pm} - \mathbb{U}_a^{\pm} = 0$$

$$\frac{dZ^{\pm}}{dx} = \mathbb{U}_d + \mathbb{U}_a^{\pm} W^{\pm}$$

Solving the latter one identifies the $W^{(n)}$, $Z^{(n)}$, hence the charges in involution.

The sine-Gordon model with defect

The $\mathbb{U}$-operator for the sine-Gordon model:

$$\boxed{\mathbb{U}(x,t,u) = \frac{\beta}{4i}\pi(x,t)\sigma^z + \frac{mu}{4i} e^{\frac{i\beta}{4}\phi\sigma^z}\sigma^y e^{-\frac{i\beta}{4}\phi\sigma^z} - \frac{mu^{-1}}{4i} e^{-\frac{i\beta}{4}\phi\sigma^z}\sigma^y e^{\frac{i\beta}{4}\phi\sigma^z}}$$

$u \equiv e^{\lambda}$, $\sigma^{x,y,z}$ Pauli matrices. The r-matrix (*Faddeev-Takhtajan, Sklyanin*):

$$r(\lambda) = \frac{\beta^2}{8\sinh\lambda}\begin{pmatrix} \frac{\sigma^z+1}{2}\cosh\lambda & \sigma^- \\ \sigma^+ & \frac{-\sigma^z+1}{2}\cosh\lambda \end{pmatrix}.$$

$\mathbb{U}$ satisfies the linear Poisson algebra leads:

$$\Big\{\phi(x),\ \pi(y)\Big\} = \delta(x-y)$$

The sine-Gordon model with defect

The relevant defect matrix (type II) (*Avan-Doikou*)

$$\tilde{L}(\lambda) = \begin{pmatrix} e^{\lambda} V - e^{-\lambda} V^{-1} & \bar{a} \\ a & e^{\lambda} V^{-1} - e^{-\lambda} V \end{pmatrix}.$$

$\tilde{L}$ satisfies the classical algebra, hence:

$$\Big\{V,\ \bar{a}\Big\} = \frac{\beta^2}{8} V\ \bar{a},$$

$$\Big\{V,\ a\Big\} = -\frac{\beta^2}{8} Va,$$

$$\Big\{\bar{a},\ a\Big\} = \frac{\beta^2}{4}(V^2 - V^{-2})$$

- Relevant studies: (*Bowcock-Corrigan-Zambon, Caudrelier, Habibulin-Kundu, Aguirre etal.*)

The sine-Gordon: local IM

Recall the generating function of the local IM

$$\mathcal{G}(\lambda) = \ln tr(T^{+}\ \tilde{L}\ T^{-})$$

Generating function

$$\mathcal{G}(\lambda) = Z_{11}^{+} + Z_{11}^{-} + \ln \Big[(1+W^{+})^{-1}(\Omega^{+}(x_0))^{-1}\tilde{L}(x_0)\Omega^{-}(x_0)(1+W^{-}) \Big]_{11}$$

$\Omega^{\pm} = e^{\frac{i\beta}{4}\phi^{\pm}\sigma^{z}}$.

Expanding the latter expression in powers of u^{-1} we obtain the following:

$$\mathcal{G}(\lambda) = \sum_{m=0}^{\infty} \frac{\mathcal{I}^{(m)}}{u^{m}}$$

The sine-Gordon: local IM

The first charge (the u^{-1}-expansion) leads to $\mathcal{I}^{(1)}$, the u-expansion leads to $\mathcal{I}^{(-1)}$:

$$\mathcal{I}^{(-1)}(\phi,\ \pi,\ V,\ a, \bar{a}) = \mathcal{I}^{(1)}(-\phi,\ \pi,\ V^{-1},\ a, \bar{a})$$

Define the

Hamiltonian

$$\begin{aligned}
\mathcal{H} &= \frac{2im}{\beta^2}(\mathcal{I}^{(1)} - \mathcal{I}^{(-1)}) \\
&= \int_{-L}^{x_0^-} dx\ \Big(\frac{1}{2}(\pi^{-2}(x) + \phi^{-\prime 2}(x)) - \frac{m^2}{\beta^2}\cos(\beta\phi^-(x))\Big) \\
&+ \int_{x_0^+}^{L} dx\ \Big(\frac{1}{2}(\pi^{+2}(x) + \phi^{+\prime 2}(x)) - \frac{m^2}{\beta^2}\cos(\beta\phi^+(x))\Big) \\
&+ \frac{4m}{\beta^2\mathcal{D}}\cos\frac{\beta}{4}(\phi^+ + \phi^-)\ \Big(\bar{a} - a\Big) + \frac{2i}{\beta\mathcal{D}}\Big(\phi^{+\prime} + \phi^{-\prime}\Big)\mathcal{A}
\end{aligned}$$

The sine-Gordon: local IM

Also derive the (*Avan-Doikou*):

Momentum

$$\mathcal{P} = \frac{2im}{\beta^2}\Big(\mathcal{I}^{(1)} + \mathcal{I}^{(-1)}\Big)$$
$$= \int_{-L}^{x_0^-} dx\ \phi^{-'}(x)\pi^-(x) + \int_{x_0^+}^{L} dx\ \phi^{+'}(x)\pi^+(x)$$
$$- \frac{4mi}{\beta^2\mathcal{D}} \sin\frac{\beta}{4}(\phi^+ + \phi^-)\ \Big(\bar{a} + a\Big) + \frac{2i}{\beta\mathcal{D}}\Big(\pi^+ + \pi^-\Big)\mathcal{A}$$

$$\mathcal{D} = e^{-\frac{i\beta}{4}(\phi^+ - \phi^-)}V + e^{\frac{i\beta}{4}(\phi^+ - \phi^-)}V^{-1}$$
$$\mathcal{A} = e^{-\frac{i\beta}{4}(\phi^+ - \phi^-)}V - e^{\frac{i\beta}{4}(\phi^+ - \phi^-)}V^{-1}$$

- Commutativity among the IM will be discussed later.

Lax pair

Next step, derive time component of the Lax pair $\mathbb{V}$, and sewing conditions. Explicit expressions (*Semenov Tian Shansky, Faddeev-Takhtajan, Avan-Doikou*):

$$\mathbb{V}^{+}(x,\lambda,\mu) = t^{-1} tr_a\Big(T_a^{+}(L,x) r_{ab}(\lambda-\mu) T_a^{+}(x,x_0) \tilde{L}_a(x_0) T_a^{-}(x_0,-L)\Big)$$

$$\mathbb{V}^{-}(x,\lambda,\mu) = t^{-1} tr_a\Big(T_a^{+}(L,x_0) \tilde{L}_a(x_0) T_a^{-}(x_0,x) r_{ab}(\lambda-\mu) T_a^{-}(x,-L)\Big)$$

$$\tilde{\mathbb{V}}^{+}(x_0,\lambda,\mu) = t^{-1} tr_a\Big(T_a^{+}(L,x_0) r_{ab}(\lambda-\mu) \tilde{L}_a(x_0) T_a^{-}(x_0,-L)\Big)$$

$$\tilde{\mathbb{V}}^{-}(x_0,\lambda,\mu) = t^{-1} tr_a\Big(T_a^{+}(L,x_0) \tilde{L}_a(x_0) r_{ab}(\lambda-\mu) T_a^{-}(x_0,-L)\Big).$$

The sine-Gordon: the Lax pair

From the explicit expression for the $\mathbb{V}$ operators we find or the left and right bulk theories:

$$\mathbb{V}^{\pm}_{\mathcal{H}} = \frac{\beta}{4i}\phi^{\pm'}\sigma^z + \frac{vm}{4i}\Omega^{\pm}\sigma^y(\Omega^{\pm})^{-1} + \frac{v^{-1}m}{4i}(\Omega^{\pm})^{-1}\sigma^y\Omega^{\pm}$$

$$\mathbb{V}^{\pm}_{\mathcal{P}} = \frac{\beta}{4i}\pi^{\pm}\sigma^z + \frac{vm}{4i}\Omega^{\pm}\sigma^y(\Omega^{\pm})^{-1} - \frac{v^{-1}m}{4i}(\Omega^{\pm})^{-1}\sigma^y\Omega^{\pm}$$

Computation of the $\mathbb{V}$ operators on the defect point leads to

$$\tilde{\mathbb{V}}^{\pm}_{\mathcal{H}} = \mathbb{V}^{\pm}_{\mathcal{H}} + \delta_{\mathcal{H}}$$
$$\tilde{\mathbb{V}}^{\pm}_{\mathcal{P}} = \mathbb{V}^{\pm}_{\mathcal{P}} + \delta_{\mathcal{P}}$$

The sine-Gordon: sewing conditions

Continuity requirements around the defect point

$$\delta_{\mathcal{H}} \to 0, \qquad \delta_{\mathcal{P}} \to 0$$

lead to:

Sewing conditions

$$S_1 : \qquad V = e^{\frac{i\beta}{4}(\phi^+ - \phi^-)}$$

$$S_2 : \qquad \pi^+(x_0) - \pi^-(x_0) = \frac{im}{\beta} \cos \frac{\beta}{4}(\phi^+(x_0) + \phi^-(x_0)) \left(a + \bar{a}\right)$$

$$S_2' : \qquad \phi^{+'}(x_0) - \phi^{-'}(x_0) = \frac{m}{\beta} \sin \frac{\beta}{4}(\phi^+(x_0) + \phi^-(x_0)) \left(\bar{a} - a\right)$$

- *Jump* across the defect point!

The sine-Gordon: commutativity

Commutativity among the IM, explicitly checked, formally guaranteed

Commutativity

$$\Big\{\mathcal{H},\ \mathcal{P}\Big\} = 0$$

- The latter is proven using the sewing conditions, i.e. Dirac (not Poisson) commutativity! *On-shell* integrability.
- In NLS *off-shell* integrability

$$\Big\{\mathcal{I}_1,\ \mathcal{I}_2\Big\} = 0$$

no use of constraints. Issue related to suitable continuum limits!

Sewing conditions

Main proposition: rational case only (*Avan-Doikou*)! (e.g. NLS model)

Compatibility

$$\left\{\mathcal{H}^{(k)},\ \mathcal{C}_{\pm}^{(m,l)}\right\} = \sum_{i=0}^{k-1}\left[\mathcal{C}_{\pm}^{(k,i)},\ \mathbb{V}^{\pm(m+i,l)}(x_0^{\pm})\right] + \sum_{i=0}^{k-1}\left[\tilde{\mathbb{V}}^{\pm(k,i)}(x_0),\ \mathcal{C}_{\pm}^{(m+i,l)}\right]$$

- $\mathcal{C}_{\pm}^{(p,l)}$ matrices with entries the constraints. Proof based on the form of $\mathbb{V}$, and the underlying algebra.
- Sub-manifold of sewing conditions (dynamical constraints) invariant under the Hamiltonian action!

The discrete case

The associated auxiliary linear problem; Lax pair $\Big(L_n(t;\lambda),\ V_n(t;\lambda)\Big)$:

$$\Psi_{n+1}(t;\lambda) = L_n(t;\lambda)\ \Psi_n(t;\lambda)$$
$$\frac{\partial \Psi_n(t;\lambda)}{\partial t} = V_n(t;\lambda)\ \Psi_n(t;\lambda)$$

On the defect point:

$$\Psi_{n+1}(t;\lambda) = \tilde{L}_n(t;\lambda)\ \Psi_n(t;\lambda)$$
$$\frac{\partial \Psi_n}{\partial t} = V_n(t;\lambda)\Psi_n(t;\lambda)$$

where $\tilde{L}_n$ is the defect matrix.

Compatibility

$$\frac{\partial \tilde{L}_n(t;\lambda)}{\partial t} = V_{n+1}(t;\lambda)\ \tilde{L}_n(t;\lambda) - \tilde{L}_n(t;\lambda)\ V_n(t;\lambda).$$

- Resemblance with the t-part of the BT!

The discrete case

Define the modified monodromy N-site matrix as

$$T(\lambda) = L_N(\lambda)\ L_{N-1}(\lambda) \dots \tilde{L}_n(\lambda - \Theta) \dots L_1(\lambda)$$

L the defect matrix satisfies:

$$\Big\{\tilde{L}_{1n}(\lambda),\ \tilde{L}_{2n}(\lambda')\Big\} = \Big[r_{12}(\lambda - \lambda'),\ \tilde{L}_{1n}(\lambda)\tilde{L}_{2n}(\lambda_2)\Big]$$

Moreover,

$$t(\lambda) = tr T(\lambda), \qquad \Big\{t(\lambda),\ t(\lambda')\Big\} = 0,$$

provides the charges in involution.

The Toda chain

- The bulk Lax pair of the model is given by:

$$L_j(\lambda) = \begin{pmatrix} \lambda - p_j & e^{q_j} \\ -e^{-q_j} & 0 \end{pmatrix}, \qquad j \neq n.$$

$$r(\lambda) = \frac{\mathrm{P}}{\lambda}, \quad \mathrm{P} \qquad \text{the permutation operator.}$$

- Then $q_i, \; p_i$ are canonical variables:

$$\Big\{ q_i, p_j \Big\} = \delta_{ij}.$$

The Toda chain

- The defect Lax matrix type II:

$$\tilde{L}_n^{(II)}(\lambda) = \begin{pmatrix} \lambda - \Theta + \alpha_n & \beta_n \\ \gamma_n & \lambda - \Theta - \alpha_n \end{pmatrix}.$$

- From the quadratic algebra (classical $\mathfrak{sl}_2$):

$$\Big\{\alpha_n,\ \beta_n\Big\} = \beta_n$$
$$\Big\{\alpha_n\, \gamma_n\Big\} = -\gamma_n$$
$$\Big\{\beta_n\, \gamma_n\Big\} = 2\alpha_n.$$

Momentum & Hamiltonian

- Expansion of $\ln trT(\lambda)$ gives the IM, the first two momentum and Hamiltonian (*Doikou*):

$$P = -\sum_{j\neq n} p_j + \alpha_n$$

$$H = -\frac{1}{2}\sum_{j\neq j} p_j^2 - \sum_{j\neq n,n-1} e^{q_{j+1}-q_j} - e^{q_{n+1}-q_{n-1}} - \beta_n e^{-q_{n-1}} - \gamma_n e^{q_{n+1}} - \frac{{\alpha_n}^2}{2}.$$

- As in the continuum case given $L, \tilde{L}$ we may derive V's.

$$V_{2n}(t;\lambda,\mu) = t^{-1}\ tr\Big[T_{1S}(N,n+1;\lambda)\tilde{L}_n(\lambda)r_{12}(\lambda-\mu)T_{1S}(n-1,1;\lambda)\Big]$$
$$V_{2n+1}(t;\lambda,\mu) = t^{-1}\ tr\Big[T_{1S}(N,n+1;\lambda)r_{12}(\lambda-\mu)\tilde{L}_n(\lambda)T_{1S}(n-1,1;\lambda)\Big].$$

The V operators

- The V operator around the defect point:

$$\tilde{V}_n = \begin{pmatrix} \lambda & e^{q_{n+1}} + \beta_n \\ -e^{-q_{n-1}} & 0 \end{pmatrix}$$
$$\tilde{V}_{n+1} = \begin{pmatrix} \lambda & e^{q_{n+1}} \\ \gamma_n - e^{-q_{n-1}} & 0 \end{pmatrix}.$$

- The bulk V-matrix is given in as

$$V_j(\lambda) \begin{pmatrix} \lambda & e^{q_j} \\ -e^{-q_{j-1}} & 0 \end{pmatrix} \qquad j \neq n,\ n+1.$$

Note: $V_{n,n+1}$ around the defect point "deformed" compared to the bulk!

Defects as *quasi* Bäcklund transformations

Recall the BT as gauge transformations; Darboux matrices.

- The continuous case Let M be a matrix that transforms the auxiliary function

$$\tilde{\Psi}(x,t;\lambda) = M(x,t;\lambda,\Theta)\ \Psi(x,t;\lambda)$$

one obtains the equations for the Bäcklund transformation

Bäcklund transformation

$$\frac{\partial M(x,t;\lambda,\Theta)}{\partial x} = \tilde{U}(x,t;\lambda)\ M(x,t;\lambda,\Theta) - M(x,t;\lambda,\Theta)\ U(x,t;\lambda)$$
$$\frac{\partial M(x,t;\lambda,\Theta)}{\partial t} = \tilde{V}(x,t;\lambda)\ M(x,t;\lambda,\Theta) - M(x,t;\lambda,\Theta)\ V(x,t;\lambda).$$

- Solution $\rightarrow$ explicit form of the Bäcklund transformation for the system under consideration.

Defects as *quasi* Bäcklund transformations

- Compare the t part of the BT

$$\frac{\partial M(x,t;\lambda,\Theta)}{\partial t} = \tilde{V}(x,t;\lambda)\ M(x,t;\lambda,\Theta) - M(x,t;\lambda,\Theta)\ V(x,t;\lambda).$$

with the time evolution equ. on the defect point:

$$\frac{\partial \tilde{L}(x_0,t;\lambda)}{\partial t} = \tilde{V}^{+}(x_0,t;\lambda)\ \tilde{L}(x_0,t;\lambda) - \tilde{L}(x_0,t;\lambda)\ \tilde{V}^{-}(x_0,t;\lambda)$$

- $\tilde{V}^{\pm}$ are the Lax pair time components computed around the defect point

$$\tilde{V}^{\pm}(x_0) \to V^{\pm}(x_0^{\pm}).$$

The discrete case

- The discrete Darboux matrix:

$$\tilde{\Psi}_n(t;\lambda) = M_n(t;\lambda,\Theta)\ \Psi_n(t;\lambda)$$

- The conditions of the discrete Bäcklund transformations:

Discrete Bäcklund transformation

$$M_{n+1}(t;\lambda,\Theta)\ L_n(t;\lambda) = \tilde{L}_n(t;\lambda)\ M_n(t;\lambda,\Theta)$$
$$\frac{\partial M_n(t;\lambda,\Theta)}{\partial t} = \tilde{V}_n(t;\lambda)\ M_n(t;\lambda,\Theta) - M_n(t;\lambda)\ V_n(t;\lambda).$$

- Compare now the t part of the discrete BT

$$\frac{\partial M_n(t;\lambda,\Theta)}{\partial t} = \tilde{V}_n(t;\lambda)\ M_n(t;\lambda,\Theta) - M_n(t;\lambda)\ V_n(t;\lambda)$$

with the time evolution equ. on the defect point

$$\frac{\partial \tilde{L}_n(t;\lambda)}{\partial t} = V_{n+1}(t;\lambda)\ \tilde{L}_n(t;\lambda) - \tilde{L}_n(t;\lambda)\ V_n(t;\lambda).$$

- The similarity is obvious. What about the x part of BT?

The x-part of BT: "duality"

- "Dual" description. Discrete case & continuous; time like monodromies (*Avan-Caudrelier-Doikou-Kundu*)

$$T_{\rm T}(n, t_1, t_2; \lambda) = \mathcal{P} \exp\Big\{ \int_{t_2}^{t_1} V_n(t; \lambda)\ dt \Big\}, \qquad t_1 > t_2.$$

- Assume that V satisfies equal times Poisson structure:

$$\Big\{ V_{1n}(t;\lambda),\ V_{2n}(t';\mu) \Big\}_{\rm T} = \Big[r_{12}(\lambda-\mu),\ V_{1n}(t;\lambda) + V_{2n}(t';\mu) \Big] \delta(t-t')$$

r same classical r-matrix appearing in algebra for L; t Poisson commutation relations:

$$\Big\{ Tr(T_T(\lambda)),\ Tr(T_T(\mu)) \Big\}_T = 0.$$

A time-like integrable hierarchy.

The x-part of BT: "duality"

- In the presence of defects, auxiliary linear problem (discrete) (*Doikou*):

$$\Psi^{\pm}_{j+1}(t;\lambda) = L^{\pm}_j(t;\lambda)\ \Psi^{\pm}_j(t;\lambda)$$
$$\frac{\partial \Psi^{\pm}_j(t;\lambda)}{\partial t} = V^{\pm}_j(t;\lambda)\ \Psi^{\pm}_j(t;\lambda), \qquad t \neq t_0.$$

F^+, for $t > t_0$ and F^-, for $t < t_0$. On the defect point we have:

$$\Psi^+_j(t_0;\lambda) = \tilde{A}_j(t_0;\lambda)\ \Psi^-_j(t_0;\lambda)$$

- Compatibility condition

Discrete space part of BT

$$L^+_j(t_0;\lambda)\ \tilde{A}_j(t_0;\lambda) = \tilde{A}_{j+1}(t_0)\ L^-_j(t_0;\lambda)$$

- Similar to the space part of the Bäcklund transformation.

The x-part of BT: "duality"

- The continuous case: local defect at $t = t_0$, then

$$\psi^{\pm}(x, t_0; \lambda) = \tilde{A}(x, t_0; \lambda)\ \Psi^{-}(x, t_0; \lambda).$$

$\tilde{A}$ and $\tilde{L}$ structurally the same, choose them to coincide.

- Compatibility of equations on the defect point provides:

Space part of BT

$$\frac{\partial \tilde{A}(x, t_0; \lambda)}{\partial x} = U^{+}(x, t_0; \lambda)\ \tilde{A}(x, t_0; \lambda) - \tilde{A}(x, t_0; \lambda)\ U^{-}(x, t_0; \lambda).$$

The U operator from V

- From the t Poisson algebra that V satisfies the t obtain the "dual" Semenov Tian Shansky formula (*Avan-Caudrelier-Doikou-Kundu*):

$$U_2(x,t;\lambda,\mu) = t_{\mathrm{T}}^{-1}(\lambda)\ tr\Big[T_{1\mathrm{T}}(x,\mathrm{T},t;\lambda)\ r_{12}(\lambda-\mu)\ T_{1\mathrm{T}}(x,t,-\mathrm{T};\lambda)\Big],$$
$$t_{\mathrm{T}}(\lambda) = tr(T_{\mathrm{T}}(\lambda)).$$

- Around the defect

$$\tilde{U}_2^-(x,t_0;\lambda) = t_{\mathrm{T}}^{-1}\ tr\Big[T_{1\mathrm{T}}(x,\mathrm{T},t_0;\lambda)\tilde{A}(x,t_0,\lambda)r_{12}(\lambda-\mu)T_{1\mathrm{T}}(x,t_0,-\mathrm{T};\lambda)\Big]$$
$$\tilde{U}_2^+(x,t_0;\lambda) = t_{\mathrm{T}}^{-1}\ tr\Big[T_{1\mathrm{T}}(x,\mathrm{T},t_0;\lambda)r_{12}(\lambda-\mu)\tilde{A}(x,t_0,\lambda)T_{1\mathrm{T}}(x,t_0,-\mathrm{T};\lambda)\Big]$$

Example: the Toda chain

- The space-like description of Toda chain with defect:

$$\frac{\partial \tilde{L}_n}{\partial t} = \tilde{V}_{n+1}\ L_n(\lambda) - L_n(\lambda)\ \tilde{V}_n(\lambda)$$

- Solve the latter equation set of relations for time evolution of degrees of freedom of the defect:

$$\begin{aligned}\dot{\alpha}_n &= e^{q_{n+1}}\ \gamma_n + e^{-q_{n-1}}\ \beta_n \\ \dot{\beta}_n &= -2\alpha_n e^{q_{n+1}} - (\alpha_n - \Theta)\beta_n \\ \dot{\gamma}_n &= -2\alpha_n e^{-q_{n-1}} + (\alpha_n - \Theta)\gamma_n\end{aligned}$$

The Toda chain

$$\frac{\partial L_{n+1}(\lambda)}{\partial t} = V_{n+2}(\lambda)\ L_{n+1}(\lambda) - L_{n+1}(\lambda)\ \tilde{V}_{n+1}(\lambda)$$
$$\frac{\partial L_{n-1}(\lambda)}{\partial t} = \tilde{V}_{n}(\lambda)\ L_{n-1}(\lambda) - L_{n-1}(\lambda)\ V_{n-1}(\lambda),$$

lead to conditions among the relevant fields

$$\dot{q}_{n+1} = p_{n+1} - \Theta$$
$$\dot{p}_{n+1} = e^{q_{n+2}-q_{n+1}} - e^{q_{n+1}}\left(\gamma_n - e^{-q_{n-1}}\right)$$

$$\dot{q}_{n-1} = p_{n-1} - \Theta$$
$$\dot{p}_{n-1} = -e^{q_{n-1}-q_{n-2}} + e^{-q_{n-1}}\left(\beta_n + e^{q_{n+1}}\right)$$

"Deformed" compared to the bulk:

$$\dot{q}_j = p_j$$
$$\dot{p}_j = e^{q_{j+1}-q_j} - e^{q_j-q_{j-1}}.$$

Examples: the sine-Gordon model

- The Lax pair for the sine-Gordon model:

$$U(x,t;\lambda) = \frac{1}{2}\begin{pmatrix} -W_t & \sinh(\lambda+W) \\ \sinh(\lambda-W) & W_t \end{pmatrix},$$
$$V(x,t;\lambda) = \frac{1}{2}\begin{pmatrix} -W_x & \cosh(\lambda+W) \\ \cosh(\lambda-W) & W_x \end{pmatrix}$$

$W = \frac{i\beta}{2}\phi$.

- Type II defect matrix

$$\tilde{L}_n^{(II)}(\lambda) = \begin{pmatrix} e^{\lambda}\mathcal{V} - e^{-\lambda}\mathcal{V}^{-1} & \bar{a} \\ a & e^{\lambda}\mathcal{V}^{-1} - e^{-\lambda}\mathcal{V} \end{pmatrix}.$$

The t part (space-like defect):

$$2\bar{a}_t = -\bar{a}(W_x^+ + W_x^-) - 2\cosh(\frac{W^+ + W^-}{2} - \Theta)\ \sinh(W^+ - W^-)$$
$$2a_t = a(W_x^+ + W_x^-) + 2\cosh(\frac{W^+ + W^-}{2} + \Theta)\ \sinh(W^+ - W^-).$$

$$2(\ln\mathcal{V})_t = W_x^- - W_x^+ + \frac{e^{-\Theta}}{2}(ae^{\frac{w^+ + w^-}{2}} - \bar{a}e^{\frac{-w^+ - w^-}{2}})$$
$$2(\ln\mathcal{V})_t = W_x^+ - W_x^- + \frac{e^{\Theta}}{2}(ae^{\frac{-w^+ - w^-}{2}} - \bar{a}e^{\frac{w^+ + w^-}{2}}).$$

Compatibility conditions of the latter equations lead to:

$$W_x^+ - W_x^- = \frac{a}{2}\sinh(\frac{W^+ + W^-}{2} - \Theta) + \frac{\bar{a}}{2}\sinh(\frac{W^+ + W^-}{2} + \Theta).$$

The sine-Gordon model

The x part (time-like defect):

$$2\bar{a}_x = -\bar{a}(W_t^+ + W_t^-) - 2\sinh(\frac{W^+ + W^-}{2} - \Theta)\ \sinh(W^+ - W^-)$$
$$2a_x = a(W_t^+ + W_t^-) - 2\sinh(\frac{W^+ + W^-}{2} + \Theta)\ \sinh(W^+ - W^-).$$

$$2(\ln \mathcal{V})_x = W_t^- - W_t^+ + \frac{e^{-\Theta}}{2}(ae^{\frac{W^+ + W^-}{2}} - \bar{a}e^{\frac{-W^+ - W^-}{2}})$$
$$2(\ln \mathcal{V})_x = W_t^+ - W_t^- + \frac{e^{\Theta}}{2}(-ae^{\frac{-W^+ - W^-}{2}} + \bar{a}e^{\frac{W^+ + W^-}{2}}).$$

Compatibility conditions of the latter equations lead to:

$$W_t^+ - W_t^- = \frac{a}{2}\cosh(\frac{W^+ + W^-}{2} - \Theta) - \frac{\bar{a}}{2}\cosh(\frac{W^+ + W^-}{2} + \Theta).$$

Comments

- Solve the latter equations in the presence on zero one or two solitons. Provide the time (space) evolution of the degrees of freedom of the defect
- In the presence of defects the x & t part of the Bäcklund transformation are note satisfied simultaneously
- Some of the BT conditions coincide with the analyticity conditions arising from $\tilde{V}^{\pm} \to V^{\pm}$
- Type II defects may be seen as products of two type I defects (*Corrigan etal*). Implications on the solutions of BT (x, t separately)?
- Action of BT on the auxiliary linear problem, info on classical scattering, transmission...

Generalizations

- Classical level: similar analysis in the case of NLS and sigma models (*Avan-Doikou, Doikou-Karaiskos*)
- Quantum level: extended the study to other integrable models with defects e.g. (an)isotropic Heisenberg chains, and higher rank generalizations for both type-I and type-II defects (*Doikou*).
- (Quasi) Bäcklund transformations associated to higher rank algebras.

Discussion

- Deeper understanding of the *off-shell* vs *on-shell* integrability; related to suitable continuum limits.
- Study of extended (not point like) defects, and defects associated to *non-ultra-local* algebras.
- Defects in face (RSOS) models, and dynamical algebras; vertex-face transformation.