Darboux transformations for the vector sine-Gordon and related structures

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Abstract

We construct a Darboux transformation for the vector sine-Gordon (vSG) equation group and derive its Bäcklund transformation. The construction is purely algebraic and makes use of the reduction group of the Lax representation of the vSG. Then using the Darboux transformation we find a related Yang-Baxter map on the sphere, an integrable discrete vector sine-Gordon equation (dvSG) and two integrable differential difference equations which are related to each other by a Miura transformation.

Plan of the talk

- The vmKdV-vSG hierarchy
- Darboux transformations for the vSG
- A related Yang-Baxter map
- The dvSG
- Two integrable differential difference equations on the sphere

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The vmKdV-vSG hierarchy

Consider the Lax operators

$$\mathcal{L}(\lambda) = D_x - \mathcal{U}(\lambda), \quad \mathcal{A}(\lambda) = D_t - \mathcal{V}(\lambda)$$

where $\mathcal{U}(\lambda)$ and $\mathcal{V}(\lambda)$ belong in $\mathfrak{gl}_{N+2}(\mathbb{C})[\lambda, \lambda^{-1}]$.

In particular we assume

$$\mathcal{U}(\lambda) = \lambda J + U, \quad \mathcal{V}(\lambda) = \sum_{i=-l}^{m} \lambda^i V_i$$

with $l, m \in \mathbb{N}$.

Consider the $\mathfrak{gl}_{N+2}$ automorphisms

$$\begin{array}{rccl} r: & A(\lambda) & \mapsto & -A(\lambda)^T \\ h: & A(\lambda) & \mapsto & \overline{A(\bar{\lambda})} \\ s: & A(\lambda) & \mapsto & QA(-\lambda)Q^{-1} \end{array}$$

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We assume

$$\begin{array}{lll} r(\mathcal{U}(\lambda)) = \mathcal{U}(\lambda), & h(\mathcal{U}(\lambda)) = \mathcal{U}(\lambda), & s(\mathcal{U}(\lambda)) = \mathcal{U}(\lambda), \\ r(\mathcal{V}(\lambda)) = \mathcal{V}(\lambda), & h(\mathcal{V}(\lambda)) = \mathcal{V}(\lambda), & s(\mathcal{V}(\lambda)) = \mathcal{V}(\lambda). \end{array}$$

The invariance of $\mathcal{U}(\lambda) = \lambda J + U$ and $\mathcal{V}(\lambda) = \sum_i \lambda^i V_i$ under the automorphisms r and h implies that

$$J, \;\; U, \;\; V_i \in \mathfrak{so}_{N+2}(\mathbb{R}).$$

while their invariance under r implies the $\mathbb{Z}_2-$gradation

$$\mathfrak{so}_{N+2}(\mathbb{R}) = E_0 \oplus E_1$$

where

$$E_j = \{a \in \mathfrak{so}_{N+2}(\mathbb{R}); QaQ^{-1} = (-1)^j a\}, \quad j \in \mathbb{Z}_2$$

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It follows that

$$J,\ V_{2k+1} \in E_1 \quad \text{and} \quad U,\ V_{2k} \in E_0.$$

Elements of E_0 are skew-symmetric matrices of the form

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

while of E_1 have the following form

$$\begin{pmatrix} 0 & * & \cdots & * \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix}$$

The vmKdV equation

The compatibility condition of the Lax operators $\mathcal{L}$ and $\mathcal{A}$ with

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \vec{\gamma}^T \\ 0 & -\vec{\gamma} & 0 \end{pmatrix}$$

reads

$$\mathcal{U}_t - \mathcal{V}_x + [\mathcal{U}, \mathcal{V}] = 0$$

and is equivalent to the vmKdV equation

$$\vec{\gamma}_t + \frac{3}{2}||\vec{\gamma}||^2\vec{\gamma}_x + \vec{\gamma}_{xxx} = 0$$

where $\vec{\gamma}^T = (\gamma_1, \ldots, \gamma_N)$ and

$$\mathcal{V} = \lambda^3 J + \lambda^2 U + \lambda\,(kJ + [U_x, J]) + (kU + [U_x, U] - U_{xx})$$

where $k = -||\vec{\gamma}||^2/2$.

The vSG equation

The vSG equation for $\vec{\alpha}^T = (\alpha_1, \ldots, \alpha_N)$ and β scalar is given by

$$D_t(\beta^{-1}\vec{\alpha}_x) = \vec{\alpha}, \quad ||\vec{\alpha}||^2 + \beta^2 = 1$$

and it admits the following Lax pair

$$\mathcal{L} = D_x - \lambda J - U, \ \mathcal{A} = D_t - \lambda^{-1} V,$$

where

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{\vec{\alpha}_x^T}{\beta} \\ 0 & \frac{\vec{\alpha}_x}{\beta} & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0 & -\beta & -\vec{\alpha}^T \\ \beta & 0 & 0 \\ \vec{\alpha} & 0 & 0 \end{pmatrix}.$$

When $N = 1$ in polar coordinates $(\alpha, \beta) = (\sin\phi, \cos\phi)$ we obtain the classical SG equation.

1. K. Pohlmeyer and K.H. Rehren. *Reduction of the two-dimensional O(n) nonlinear σ-model*. Journal of Mathematical Physics, 20(12):2628–2632, 1979.
2. H. Eichenherr and K. Pohlmeyer. *Lax pairs for certain generalizations of the sine-Gordon equation*. Physics Letters B, 89(1):76–78, 1979.
3. Ioannis Bakas, Q-Han Park, and Hyun-Jong Shin. *Lagrangian formulation of symmetric space sine-Gordon models*. Physics Letters B, 372(12):45–52, 1996.
4. Jing Ping Wang. *Generalized Hasimoto transformation and vector Sine-Gordon equation*. In S. Abenda, G. Gaeta, and S. Walcher, editors, Symmetry and Perturbation Theory, SPT 2002. World Scientific, 2003.

Darboux transformations

A Darboux transformation for the linear system

$$\mathcal{L}(\vec{\alpha}, \beta, \lambda)\Psi = 0, \qquad \mathcal{A}(\vec{\alpha}, \beta, \lambda)\Psi = 0$$

is a linear transformation acting on the fundamental solution of the linear problem

$$\Psi \mapsto \Psi_1 = M\Psi, \quad \det(M) \neq 0$$

such that Ψ_1 satisfies

$$\mathcal{L}(\vec{\alpha}_1, \beta_1, \lambda)\Psi_1 = 0, \qquad \mathcal{A}(\vec{\alpha}_1, \beta_1, \lambda)\Psi_1 = 0.$$

It follows that the Darboux matrix M satisfies

$$M_x = \mathcal{U}_1 M - M\mathcal{U}, \quad M_t = \mathcal{V}_1 M - M\mathcal{V}$$

where $\mathcal{U}_1 = \mathcal{U}(\vec{\alpha}_1, \beta_1)$ and $\mathcal{V}_1 = \mathcal{V}(\vec{\alpha}_1, \beta_1)$.

A Darboux transformation can be inverted and iterated and defines a shift operator $\mathcal{S}$ such that

$$\mathcal{S} : (\Psi_i, \vec{\alpha}_i, \beta_i) \mapsto (\Psi_{i+1}, \vec{\alpha}_{i+1}, \beta_{i+1})$$

If the Darboux matrix contains a free parameter μ then we denote it by M_μ and the corresponding shift by $\mathcal{S}_\mu$. Commutativity of the shifts $\mathcal{S}_\mu$ and $\mathcal{S}_\nu$ implies the discrete Lax representation

$$\mathcal{S}_\nu(M_\mu)M_\nu = \mathcal{S}_\mu(M_\nu)M_\mu$$

which is equivalent to a system of partial difference equations. The shifts $\mathcal{S}_\mu$ and $\mathcal{S}_\nu$ act on the $\mathbb{Z}^2$ lattice where on each vertex (m, n) we attach the variables

$$\vec{\alpha}_{m,n} = \mathcal{S}_\mu^m \mathcal{S}_\nu^n \vec{\alpha}, \quad \beta_{m,n} = \mathcal{S}_\mu^m \mathcal{S}_\nu^n \beta.$$

We assume that the Darboux matrix $M(\lambda)$ for the vSG is rational in λ (without loss of generality) and invariant under the lifted action of h, r and s, i.e.

$$\begin{array}{rcll} r: & M(\lambda) & \mapsto & \underline{M(\lambda)}^{-T} = M(\lambda) \\ h: & M(\lambda) & \mapsto & \overline{M(\bar{\lambda})} = M(\lambda) \\ s: & M(\lambda) & \mapsto & QM(-\lambda)Q^{-1} = M(\lambda) \end{array}$$

Proposition: Assume that the Darboux matrix M of the vSG is invariant under the action of r, h and s and independent of λ. Then M is a constant matrix of the form

$$M = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Omega \end{pmatrix}$$

where $\Omega \in O_N(\mathbb{R})$. Moreover,

$$\beta_1 = \beta, \quad \vec{\alpha}_1 = \Omega\vec{\alpha}.$$

We are interested in the simplest Darboux matrix $M(\lambda)$, rational in λ, with a pole at $\lambda = \mu$ and invariant under the action of s, r and h.
First we average over the $G_s \simeq \mathbb{Z}_2$ subgroup generated by s

$$M_\mu(\lambda) = \left\langle M_0 + \frac{M_1}{\lambda - \mu} \right\rangle_{G_s} = A_\infty + \frac{A}{\lambda - \mu} - \frac{QAQ^{-1}}{\lambda + \mu}$$

where

$$A_\infty = \langle M_0 \rangle_{G_s} = \frac{1}{2}(M_0 + QM_0 Q^{-1}) \text{ and } A = \frac{1}{2} M_1.$$

Since $A_\infty \in GL_{N+2}^{G_s}$ and from equations

$$D_x M_\mu = \mathcal{U}_1 M_\mu - M_\mu \mathcal{U} \text{ and } D_t M_\mu = \mathcal{V}_1 M_\mu - M_\mu \mathcal{V}$$

follows that A_∞ is a constant matrix and of the form

$$A_\infty = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \Omega \end{pmatrix}$$

$$M_\mu(\lambda) M_\mu(\lambda)^T = \mathbb{1}, \quad \overline{M_\mu(\overline{\lambda})} = M_\mu(\lambda) \Rightarrow A_\infty \in O_{N+2}(\mathbb{R}).$$

Conclusion: A_∞ is a constant Darboux matrix and thus without loss of generality I can assume that

$$M_\mu(\lambda) = \mathbb{1} + \frac{A}{\lambda - \mu} - \frac{QAQ^{-1}}{\lambda + \mu}$$

The orthogonality condition $M_\mu(\lambda)M_\mu(\lambda)^T = \mathbb{1}$ implies also that

$$AA^T = 0 \text{ and } A\left(\mathbb{1} - \frac{1}{2\mu}QA^TQ\right) + \left(\mathbb{1} - \frac{1}{2\mu}QAQ\right)A^T = 0$$

Assumption: A is a rank one matrix i.e.

$$A = |b\rangle\langle a|\,, \quad \langle a| = (p, q, a_1, \dots, a_N), \quad |b\rangle = (b_1, \dots, b_{N+2})^T$$

It follows that

$$\langle a|\, a\rangle = 0 \quad \text{and} \quad |b\rangle = \frac{2\mu}{\langle a|\, Q\, |a\rangle} Q\, |a\rangle$$

$$M_\mu(\lambda) = \mathbb{1} + \frac{2\mu}{\lambda - \mu}P - \frac{2\mu}{\lambda + \mu}QPQ, \quad P = \frac{Q\,|a\rangle\langle a|}{\langle a|\, Q\, |a\rangle}$$

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Remark1: P is a projector, i.e. $P^2 = P$
Remark2: $|a\rangle \mapsto c\,|a\rangle$ does not change P
Remark3: $PP^T = PQP = 0$

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This assumption implies that $|a\rangle \in \mathbb{R}^{N+2}$ and since $\langle a| \; a\rangle = 0$ it follows that $|a\rangle = 0$. In this case we have a trivial Darboux matrix $M_\mu(\lambda) = \mathbb{1}$.

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where (since $\langle a|\, a\rangle = 0$) $||\mathbf{u}||^2 = 1$.
Remark: It follows that $\langle a|\, Q\, |a\rangle = 2$ and thus

$$P = \frac{1}{2} Q\, |a\rangle \langle a|$$

We have proven

Proposition: The matrix

$$M_\mu(\lambda) = \mathbb{1} + \frac{\mathrm{i}\mu}{\lambda - \mathrm{i}\mu} Q \left|a\right\rangle \left\langle a\right| - \frac{\mathrm{i}\mu}{\lambda + \mathrm{i}\mu} \left|a\right\rangle \left\langle a\right| Q$$

with $\mu \in \mathbb{R}$ and

$$\left|a\right\rangle = \begin{pmatrix} \mathrm{i} \\ \mathbf{u} \end{pmatrix}, \quad \mathbf{u} \in \mathbb{S}^N$$

satisfies the following relations

$$M(\lambda)^{-1} = M(\lambda)^T, \quad \overline{M(\overline{\lambda})} = M(\lambda), \quad QM(-\lambda)Q^{-1} = M(\lambda).$$

The Darboux matrix M_μ has to satisfy the semi-discrete zero curvature conditions

$$D_t M_\mu = \mathcal{V}_1 M_\mu - M_\mu \mathcal{V}, \;\; D_x M_\mu = \mathcal{U}_1 M_\mu - M_\mu \mathcal{U}$$

from where we obtain that

$$\mathbf{u} = \frac{\mathbf{v}_1 + \mathbf{v}}{||\mathbf{v}_1 + \mathbf{v}||}, \quad \mathbf{v} = \begin{pmatrix} \beta \\ \vec{\alpha} \end{pmatrix}$$

and also the Bäcklund transformation

$$D_t \left(\frac{\mathbf{v}_1 + \mathbf{v}}{||\mathbf{v}_1 + \mathbf{v}||} \right) = -\frac{1}{2\mu}(\mathbf{v}_1 + \mathbf{v})$$

$$\frac{D_x \vec{\alpha}_1}{\beta_1} - \frac{D_x \vec{\alpha}}{\beta} = -\frac{2\mu}{||\mathbf{v}_1 + \mathbf{v}||}(\vec{\alpha}_1 + \vec{\alpha})$$

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Remark: When $N = 1$ we obtain the known Bäcklund transformation of the SG equation

Related Yang-Baxter map

The re-factorisation problem of the product of two Darboux matrices can lead to a Yang-Baxter map.

1. Yu.B. Suris and A.P. Veselov. *Lax matrices for Yang-Baxter maps*. J. Nonlinear Math. Phys., 10:223–230, 2003.
2. A. P. Veselov, *Yang-Baxter maps and integrable dynamics*, Phys. Lett. A 314, 214–221 2003.
3. T. Kouloukas and V. Papageorgiou. *Poisson Yang-Baxter maps with binomial Lax matrices*. J. Math. Phys., 52 12:404012, 2011.
4. S. Konstantinou-Rizos and A. V. Mikhailov. *Darboux transformations, finite reduction groups and related Yang-Baxter maps*. J. Phys. A: Math. Theor., 46, 2013.

For the Darboux matrix $M_\nu(x;\lambda)$

$$M_\nu(x;\lambda) = \mathbb{1} + \frac{\mathrm{i}\nu}{\lambda - \mathrm{i}\nu} Q \left|a\right\rangle \left\langle a\right| - \frac{\mathrm{i}\nu}{\lambda + \mathrm{i}\nu} \left|a\right\rangle \left\langle a\right| Q$$

with $\left|a\right\rangle = (\mathrm{i}, x)^T$ and $x \in \mathbb{S}^N$ we consider the re-factorisation problem

$$M_\nu(x;\lambda) M_\mu(y;\lambda) = M_\mu(Y;\lambda) M_\nu(X;\lambda).$$

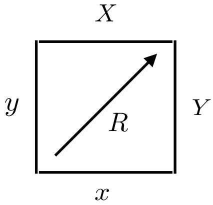
If for a given x and y the re-factorisation problem has a unique solution for X and Y then we can define a map

$$\begin{array}{rlcl} R(\nu,\mu): & \mathbb{S}^N \times \mathbb{S}^N & \rightarrow & \mathbb{S}^N \times \mathbb{S}^N \\ & (x,y) & \rightarrow & (X,Y) = (X(x,y;\nu,\mu), Y(x,y;\nu,\mu)) \end{array}$$

In this case

$$I(\lambda) = \mathrm{Tr}\left(M_\nu(x;\lambda) M_\mu(y;\lambda)\right)$$

is a generating function for invariants of the maps.



X
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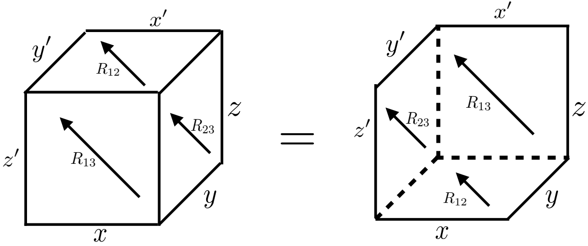


x'
y'
R_{12}
z
R_{23}
z'
R_{13}
y
x
$=$
x'
y'
R_{13}
z
z'
R_{23}
R_{12}
y
x

We also define the extended maps in $\mathbb{S}^N \times \mathbb{S}^N \times \mathbb{S}^N$ as follows:

$$\begin{array}{rccl} R_{1,2}(\nu,\mu): & (x,y,z) & \mapsto & (X(x,y;\nu,\mu), Y(x,y;\nu,\mu), z) \\ R_{1,3}(\nu,\kappa): & (x,y,z) & \mapsto & (X(x,z;\nu,\kappa), y, Y(x,z;\nu,\kappa)) \\ R_{2,3}(\mu,\kappa): & (x,y,z) & \mapsto & (x, X(y,z;\mu,\kappa), Y(y,z;\mu,\kappa)) \end{array}$$

and we say that $R(\nu,\mu)$ is a parametric Yang-Baxter map if it satisfies the (set theoretical) parametric Yang-Baxter equation

$$R_{1,2}(\nu,\mu)\circ R_{1,3}(\nu,\kappa)\circ R_{2,3}(\mu,\kappa) = R_{2,3}(\mu,\kappa)\circ R_{1,3}(\nu,\kappa)\circ R_{1,2}(\nu,\mu).$$

In our case we obtain the vector Yang-Baxter map $(X,Y) = R(\nu,\mu)$ defined on the unit sphere, where

$$\begin{aligned} X &= \frac{(\nu^2-\mu^2)x + 2\mu(\nu+\mu\langle x,y\rangle)y}{\nu^2+\mu^2+2\mu\nu\langle x,y\rangle}, \\ Y &= \frac{(\mu^2-\nu^2)y + 2\nu(\mu+\nu\langle x,y\rangle)x}{\nu^2+\mu^2+2\mu\nu\langle x,y\rangle} \end{aligned}$$

- $R(\nu,\mu)\circ R(\nu,\mu) = id$ (involution)
- $(X,Y) = R(\nu,\mu)(x,y) = (f_{\nu,\mu}(x,y), f_{\mu,\nu}(y,x))$

$$f_{\nu,\mu}(x,y) = \frac{(\nu^2-\mu^2)x + 2\mu(\nu+\mu\langle x,y\rangle)y}{\nu^2+\mu^2+2\mu\nu\langle x,y\rangle}$$

- If $P(x,y) = (y,x)$ then
  $P\circ R(\mu,\nu)\circ P = R(\nu,\mu) = R(\nu,\mu)^{-1}$ (reversible)
- $I(x,y) = \langle x,y\rangle$ (scalar invariant)
- $H(x,y) = \nu x + \mu y$ (vector invariant)

1. V.G. Papageorgiou, A.G. Tongas, and A.P. Veselov, *Yang-Baxter maps and symmetries of integrable equations on quad-graphs*, Journal of mathematical physics, 47 (2006), no. 8, 083502
2. V. E. Adler, *Integrable deformations of a polygon*, Physica D 87, 52–57 (1995).

$$(U, V) = \widetilde{R}(a, b)(u, v) = (\phi_\nu^{-1} \times \phi_\mu^{-1}) \circ R(\nu, \mu) \circ (\phi_\nu \times \phi_\mu)$$

$$x = \phi_\nu(u) = \nu^{-1} u, \quad y = \phi_\mu(v) = \mu^{-1} v$$

$$\begin{tikzcd}
\mathbb{S}^N \times \mathbb{S}^N \arrow[r, "\widetilde{R}(\nu^2-2,\mu^2-2)"] \arrow[d, "\phi_\nu \times \phi_\mu"'] & \mathbb{S}^N \times \mathbb{S}^N \\
\mathbb{S}^N \times \mathbb{S}^N \arrow[r, "R(\nu,\mu)"'] & \mathbb{S}^N \times \mathbb{S}^N \arrow[u, "\phi_\nu^{-1} \times \phi_\mu^{-1}"']
\end{tikzcd}$$

$$\begin{aligned}
U &= v + \frac{a-b}{||u+v||^2}(u+v), \\
V &= u - \frac{a-b}{||u+v||^2}(u+v),
\end{aligned}$$

$$||u||^2 = 2 + a, \quad ||v||^2 = 2 + b, \quad a = \nu^2 - 2, \quad b = \mu^2 - 2$$

Discrete vector sine-Gordon

Starting from the refactorisation problem of the YB map

$$M_\nu(x;\lambda)M_\mu(y;\lambda) = M_\mu(Y;\lambda)M_\nu(X;\lambda)$$

we set

$$y = \frac{\mathbf{v}_{0,1}+\mathbf{v}}{||\mathbf{v}_{0,1}+\mathbf{v}||}, \;\; X = \frac{\mathbf{v}_{1,0}+\mathbf{v}}{||\mathbf{v}_{1,0}+\mathbf{v}||}$$

and

$$x = \mathcal{S}_\mu(X) = \frac{\mathbf{v}_{1,1}+\mathbf{v}_{0,1}}{||\mathbf{v}_{1,1}+\mathbf{v}_{0,1}||}, \;\; Y = \mathcal{S}_\nu(y) = \frac{\mathbf{v}_{1,1}+\mathbf{v}_{1,0}}{||\mathbf{v}_{1,1}+\mathbf{v}_{1,0}||}$$

where

$$\mathbf{v} = \begin{pmatrix} \beta \\ \vec{\alpha} \end{pmatrix}$$

where we use the notation $\mathbf{v}_{n,m} = \mathcal{S}_\nu^n \mathcal{S}_\mu^m \mathbf{v}$, $\mathbf{v}_{0,0} = \mathbf{v}$.

From the first component of the YB map

$$(X, Y) = (f_{\nu,\mu}(x, y), f_{\mu,\nu}(y, x))$$

we obtain that

$$\frac{\mathbf{v}_{1,0} + \mathbf{v}}{||\mathbf{v}_{1,0} + \mathbf{v}||} = f_{\nu,\mu}\left(\frac{\mathbf{v}_{1,1} + \mathbf{v}_{0,1}}{||\mathbf{v}_{1,1} + \mathbf{v}_{0,1}||}, \frac{\mathbf{v}_{0,1} + \mathbf{v}}{||\mathbf{v}_{0,1} + \mathbf{v}||}\right)$$

which is equivalent to

$$\mathbf{v}_{1,0} = -\mathbf{v} + 2\left\langle f_{\nu,\mu}(x, y), \mathbf{v}\right\rangle f_{\nu,\mu}(x, y) = F(\mathbf{v}, \mathbf{v}_{0,1}, \mathbf{v}_{1,1}; \nu, \mu)$$

Similarly, from the second component of the inverse YB map

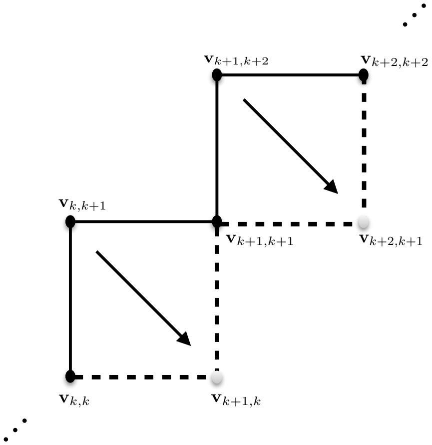
$$(x, y) = (f_{\nu,\mu}(X, Y), f_{\mu,\nu}(Y, X))$$

we obtain

$$\mathbf{v}_{0,1} = -\mathbf{v} + 2\left\langle f_{\mu,\nu}(Y, X), \mathbf{v}\right\rangle f_{\mu,\nu}(Y, X) = G(\mathbf{v}, \mathbf{v}_{1,0}, \mathbf{v}_{1,1}; \nu, \mu)$$

For the dvSG equation there is a well defined initial value problem with the initial data given by the staircase

$$\{\mathbf{v}_{k,k} \in \mathbb{S}^N, \mathbf{v}_{k,k+1} \in \mathbb{S}^N;\ k \in \mathbb{Z},\ \mathbf{v}_{k,k} + \mathbf{v}_{k,k+1} \neq 0,\ \mathbf{v}_{k-1,k} + \mathbf{v}_{k,k} \neq 0\}$$



1. The vector invariant $H(x, y) = \nu x + \mu y$ of the YB map i.e. the relation
$$\nu X + \mu Y = \nu x + \mu y$$
implies the conservation law for the dvSG
$$(\mathcal{S}_\mu - 1)\nu \frac{\mathbf{v}_{1,0} + \mathbf{v}}{||\mathbf{v}_{1,0} + \mathbf{v}||} = (\mathcal{S}_\nu - 1)\mu \frac{\mathbf{v}_{0,1} + \mathbf{v}}{||\mathbf{v}_{0,1} + \mathbf{v}||}$$

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2. When $N = 1$ we obtain the classical discrete sine-Gordon equation

A local flow on the sphere

Inverse Problem: Given a Darboux transformation find all the Lax operators that are associated with it.

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We consider a Lax operator

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First we average over the group $G_r \times G_s \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by the automorphisms r and s

$$\mathcal{W}(\lambda) = \left\langle \frac{\mathrm{i}\nu b}{\lambda - \mathrm{i}\nu} \right\rangle_{G_r \times G_s} = \frac{\mathrm{i}\nu B}{\lambda - \mathrm{i}\nu} - \frac{\mathrm{i}\nu Q B Q^{-1}}{\lambda + \mathrm{i}\nu}, \quad B^T = -B$$

where $B = 4^{-1}(b - b^T)$

The compatibility condition of the linear system

$$\Psi_\tau = \mathcal{W}\Psi, \quad \mathcal{S}_\nu(\Psi) = M_\nu \Psi$$

is

$$D_\tau M_\nu = \mathcal{W}_1 M_\nu - M_\nu \mathcal{W}$$

and implies the following conditions:

$$B_1 A = AB,$$

$$\begin{aligned} D_\tau A &= \mathrm{i}\nu(B_1 - B) - (2)^{-1}(B_1 QAQ - QAQB \\ &\quad + QB_1 QA - AQBQ). \end{aligned}$$

Remember:

$$M_\nu(\lambda) = \mathbb{1} + \frac{A}{\lambda - \mathrm{i}\nu} - \frac{QAQ}{\lambda + \mathrm{i}\nu}, \quad A = \mathrm{i}\nu Q \left|a\right\rangle \left\langle a\right|, \;\; \left|a\right\rangle = \begin{pmatrix} \mathrm{i} \\ \mathbf{u} \end{pmatrix}$$

and $\mathbf{u} \in \mathbb{S}^N$.

It can be verified that

$$B = \frac{Q\left|a_{-1}\right\rangle\left\langle a\right| - \left|a\right\rangle\left\langle a_{-1}\right| Q}{\left\langle a\right| Q \left|a_{-1}\right\rangle}$$

is skew symmetric and satisfies $B_1 A = AB$. Moreover with this choice of B the reality condition

$$\overline{\mathcal{W}(\bar{\lambda})} = \mathcal{W}(\lambda)$$

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The other equation is equivalent to

$$\mathbf{u}_\tau = 2\left(\mathcal{S}_\nu - 1\right)\frac{\mathbf{u} + \mathbf{u}_{-1}}{||\mathbf{u} + \mathbf{u}_{-1}||^2}, \quad \mathbf{u} \in \mathbb{S}^N$$

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1. V.E. Adler. *Classification of integrable Volterra-type lattices on the sphere: isotropic case*. Journal of Physics A: Mathematical and Theoretical, 41(14):145201, 2008.
2. O. Ragnisco and P.M. Santini. *A unified algebraic approach to integral and discrete evolution equations*. Inverse Problems, 6(3):441, 1990.

But

$$\mathbf{u} = \frac{\mathbf{v}_1 + \mathbf{v}}{||\mathbf{v}_1 + \mathbf{v}||}, \quad \mathbf{v} = \begin{pmatrix} \beta \\ \vec{\alpha} \end{pmatrix}$$

and can be proven that $\mathbf{v}$ satisfies

$$D_\tau \mathbf{v} = \frac{||\mathbf{v} + \mathbf{v}_{-1}||^2(\mathbf{v}_1 + \mathbf{v}) - ||\mathbf{v}_1 + \mathbf{v}||^2(\mathbf{v} + \mathbf{v}_{-1})}{\langle \mathbf{v}_1 + \mathbf{v}, \mathbf{v} + \mathbf{v}_{-1} \rangle + ||\mathbf{v}_1 + \mathbf{v}|| \cdot ||\mathbf{v} + \mathbf{v}_{-1}||}$$

$$\mathbf{v} \xrightarrow{\mathit{Miura}} \mathbf{u}$$

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$$\mathbf{v} \xrightarrow{Miura} \mathbf{u}$$

When $N = 1$ we obtain

$$\phi_\tau = \tan\left(\frac{\phi_1 - \phi_{-1}}{4}\right)$$

F. Nijhoff and H. Capel. *The discrete Korteweg-de Vries equation*. Acta Applicandae Mathematica, 39(1–3):133–158, 1995.

Conclusions

1. Construction of a Darboux transformation for the vSG using the reduction group
2. Derived the corresponding Bäcklund transformation
3. Constructed a related YB map
4. Constructed the dvSG
5. Constructed a new Lax operator and derived a differential-difference equation and established its Miura equivalence with another differential-difference equation.

A.V. Mikhailov, and G. Papamikos, and J.P. Wang, *Darboux transformation for the vector sine-Gordon equation and integrable equations on a sphere*, arxiv:1507.07248