Aperiodic Order

To repeat, or not to repeat, that is the question

Uwe Grimm
Department of Mathematics & Statistics
The Open University, Milton Keynes

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What is a crystal?

macroscopic features ⟷ **microscopic atomic order**

symmetry

facetting

crystal lattice

unit cell

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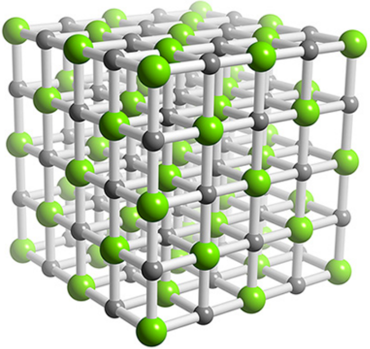

Symmetry

- translational symmetry (periodicity)
- rotational symmetry
  order n $\Longleftrightarrow$ rotation by $360^\circ/n$
- reflection symmetry (mirror symmetry)
- permutation ('colour') symmetry
  (symmetry under exchange)
- ...

Crystallographic restriction

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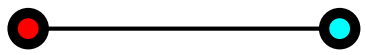

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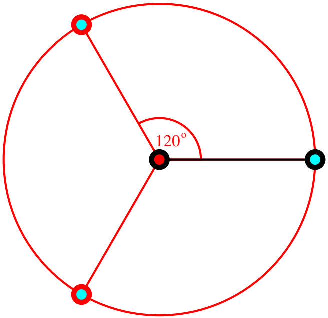

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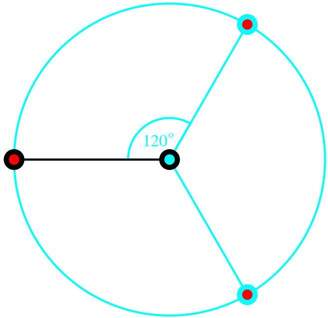

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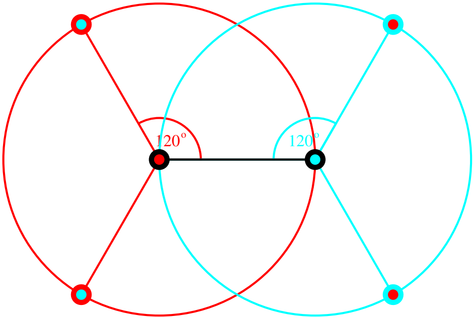

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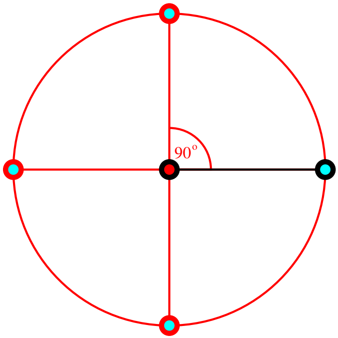

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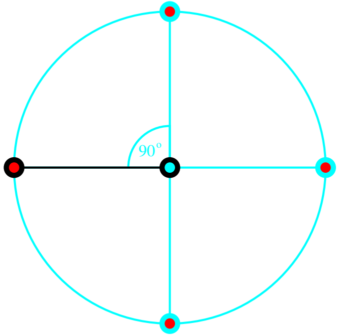

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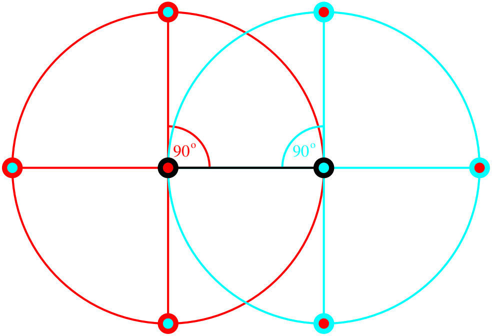

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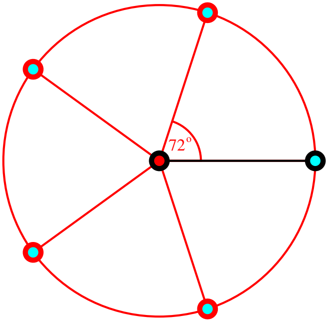

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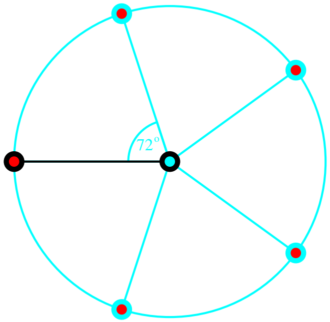

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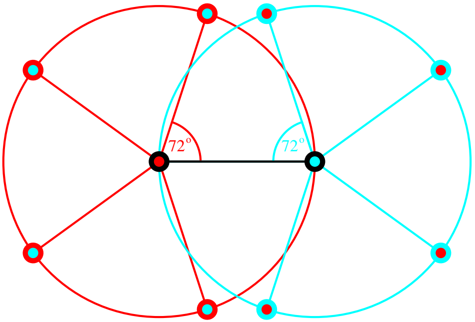

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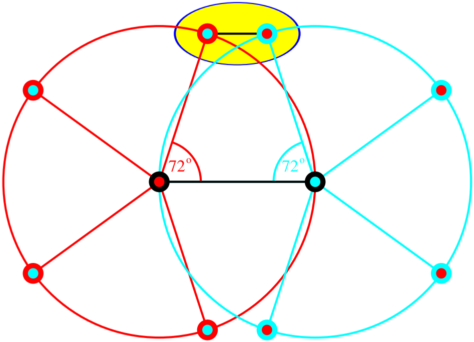

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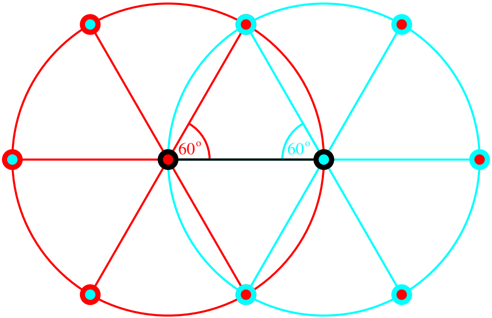

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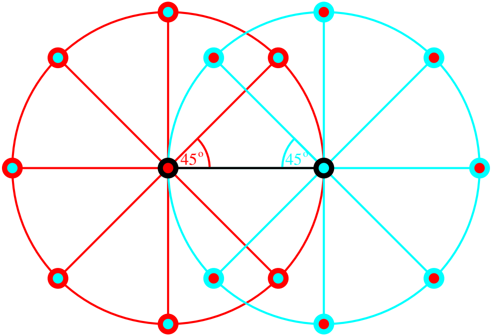

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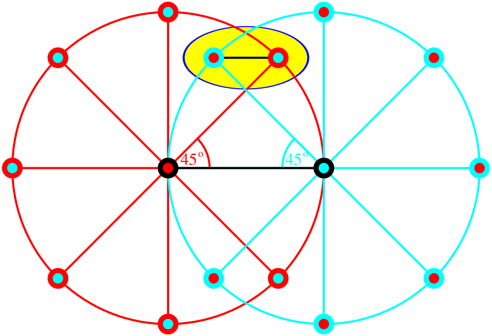
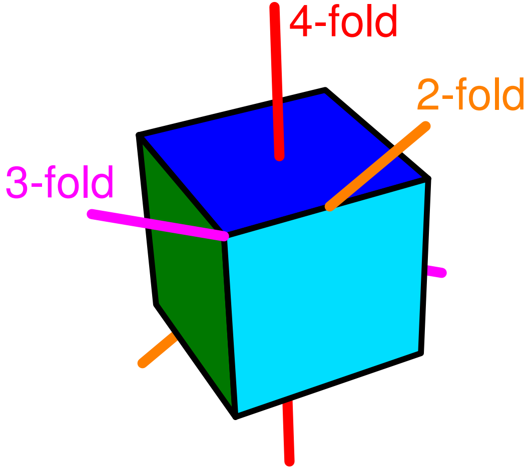
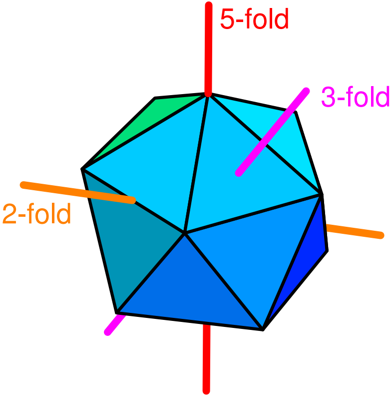
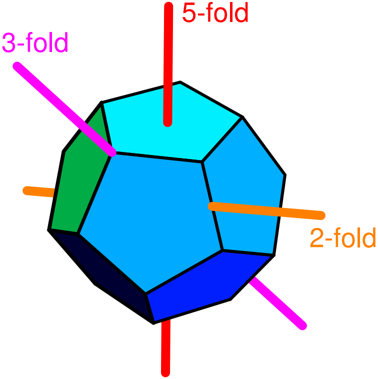


Cubic symmetry



Icosahedral symmetry



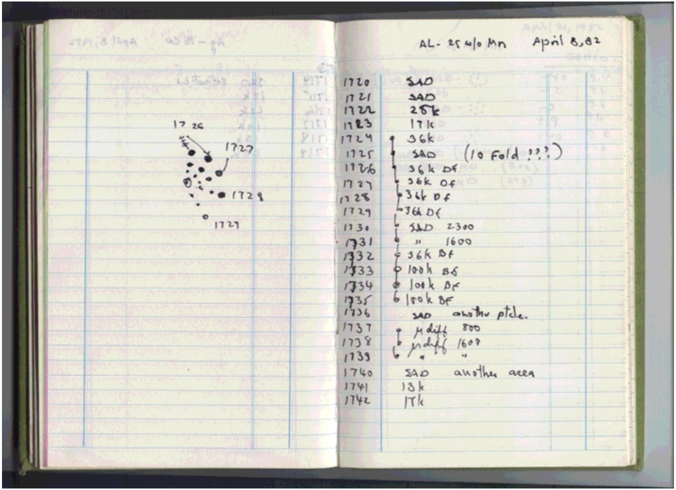


An unexpected discovery...

(Photo courtesy of the Ames Laboratory)

Dan Shechtman, Nobel prize for Chemistry 2011

Forbidden crystals



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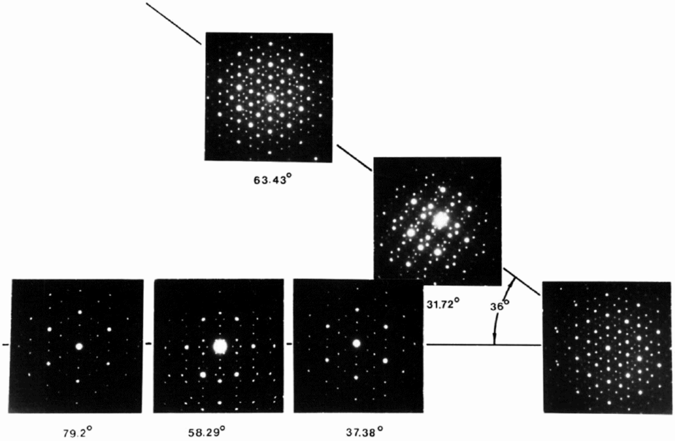

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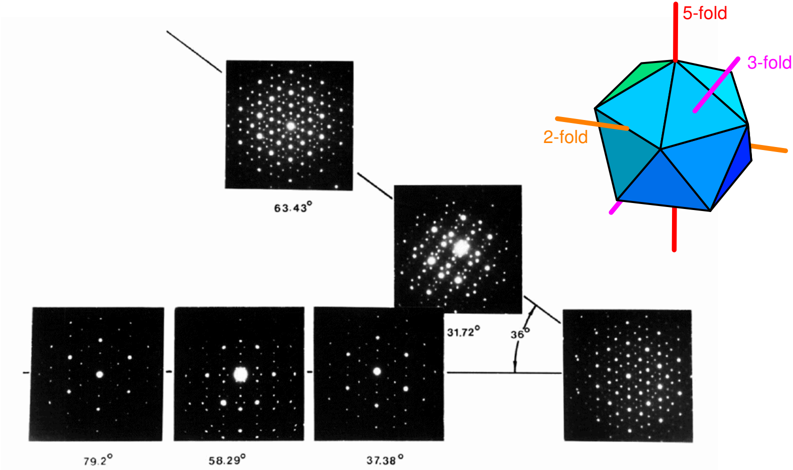

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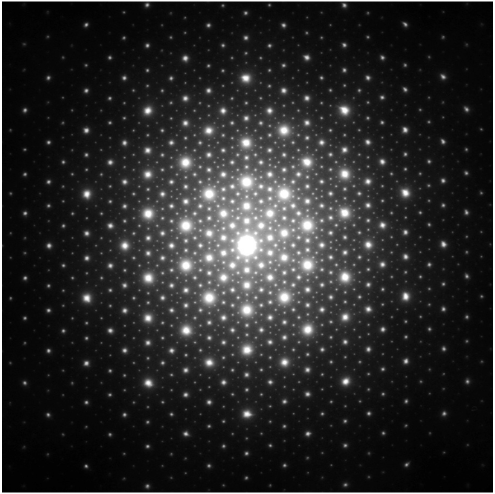
Quasicrystals

(Photo courtesy of Paul Canfield, Ames Laboratory)

Quasicrystals

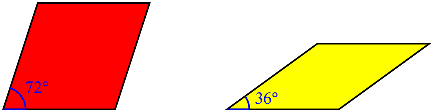


(Picture courtesy of Conradin Beeli)

Aperiodic Tilings

- ‘forbidden’ symmetries
- pointlike diffraction
- atomic structure of quasicrystals
- generating tilings:
  - local (matching) rules
  - inflation
  - projection

Penrose Tilings



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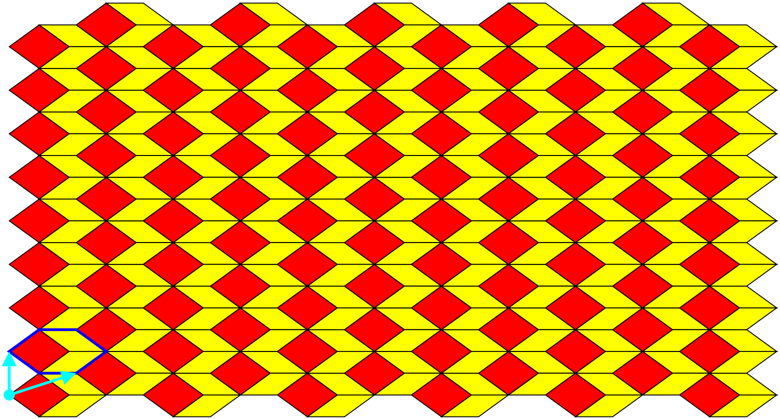

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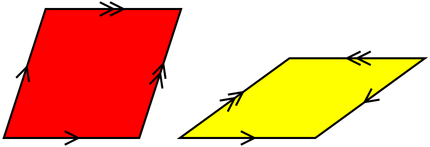

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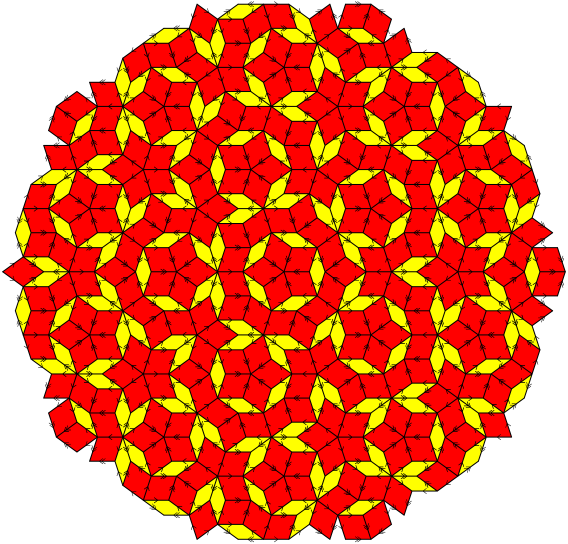

Penrose Tilings

Penrose Tilings

Sir Roger Penrose on his pattern...

Penrose Tilings

... and a less desired application

“So often we read of very large companies riding rough-shod over small businesses or individuals, but when it comes to the population of Great Britain being invited by a multi-national to wipe their bottoms on what appears to be the work of a Knight of the Realm without his permission, then a last stand must be made.” (David Bradley, Director of Pentaplex)

Fibonacci Series

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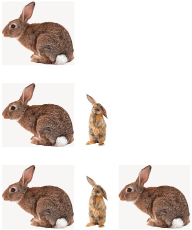

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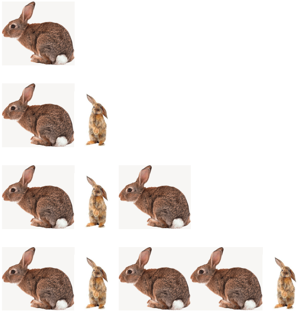

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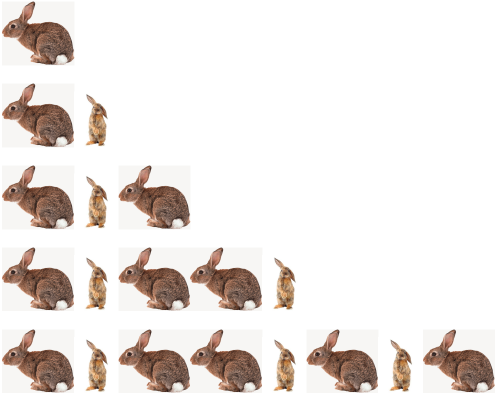

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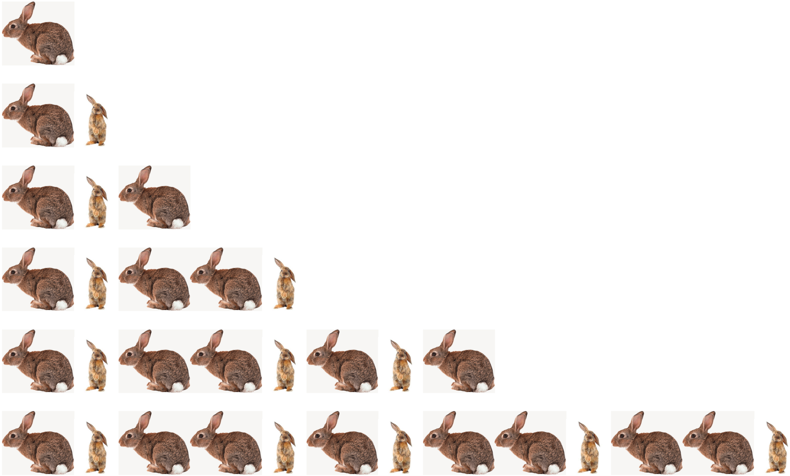


Leonardo da Pisa (Fibonacci), *Liber Abaci* (1202)

Fibonacci Series

Substitution rule for L ('large') and S ('small')

$$L \mapsto LS \quad \text{and} \quad S \mapsto L \quad \text{gives}$$

$$L \mapsto LS \mapsto LSL \mapsto LSLLS \mapsto LSLLSLSL \mapsto \ldots$$

The Fibonacci numbers $f_{n+1} = f_n + f_{n-1}$ ($f_0 = 0$, $f_1 = 1$) are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

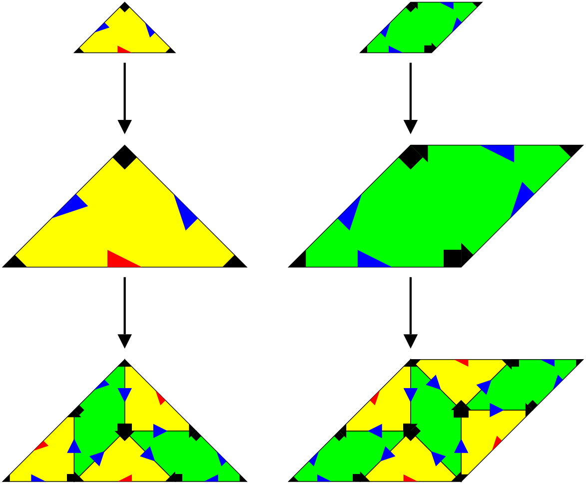
with $\quad \frac{f_n}{f_{n-1}} \to \frac{1+\sqrt{5}}{2} = 1,6180339\ldots \quad$ (golden ratio)

Fibonacci Series

Geometric Realisation (Inflation)

One-dimensional aperiodic pattern

Ammann-Beenker Tilings



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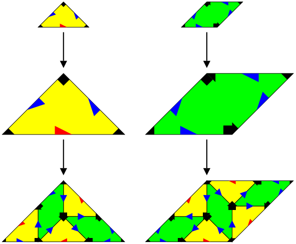

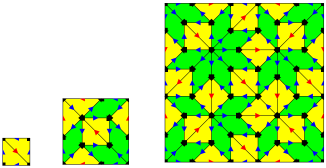

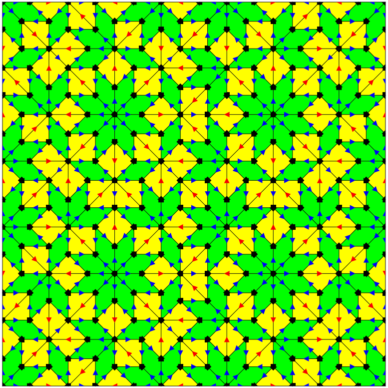

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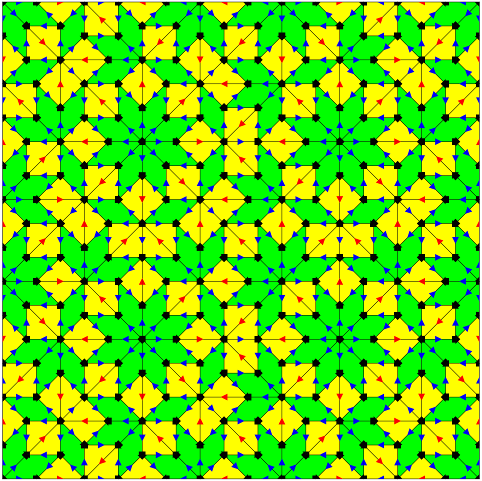

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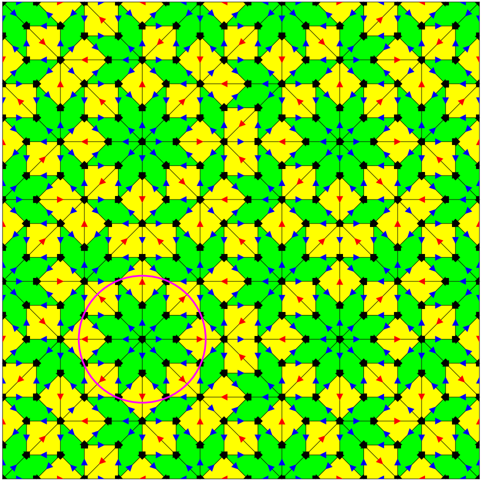

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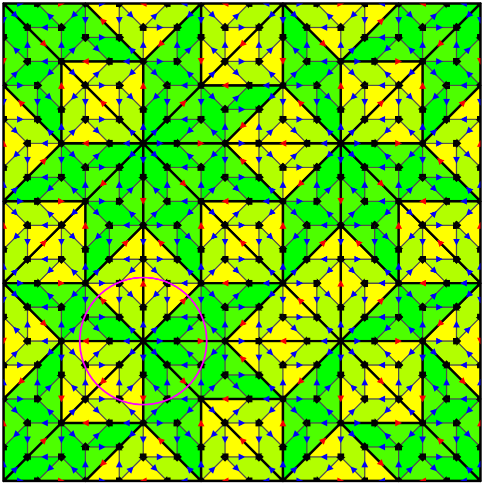

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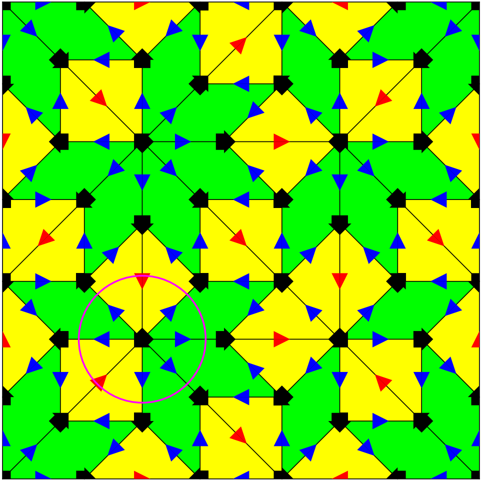

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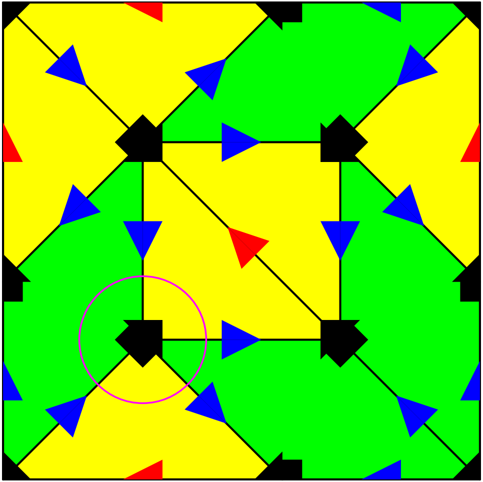

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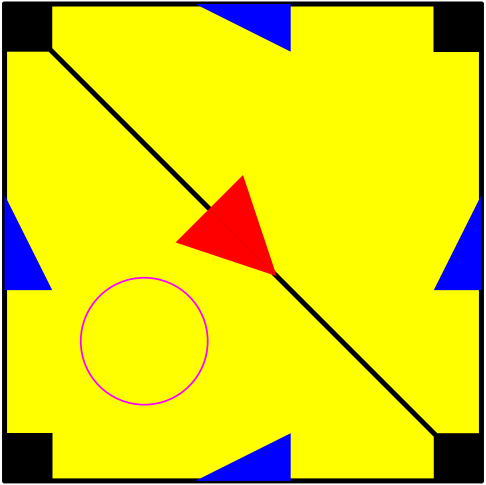

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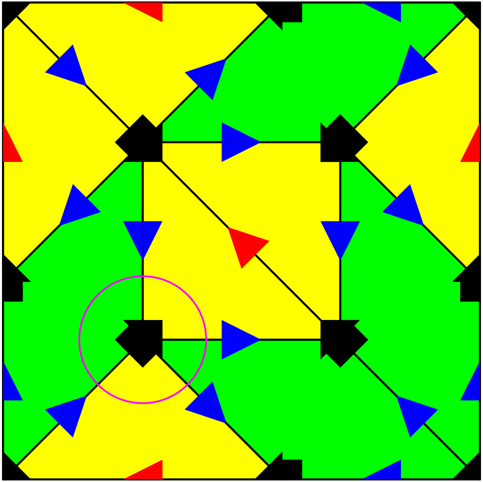

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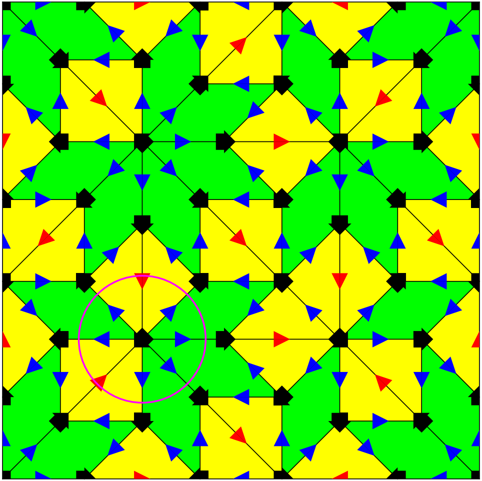

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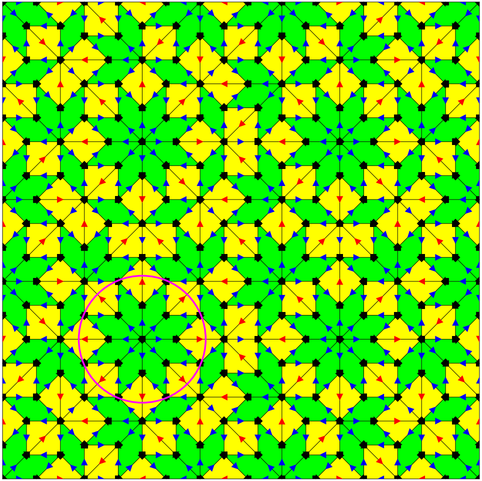
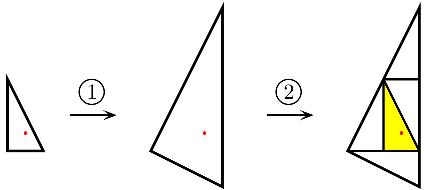


Ammann-Beenker Tilings

The ‘Pinwheel’ Tiling



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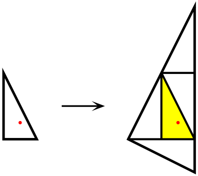

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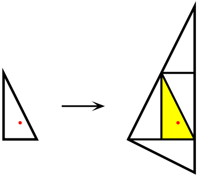

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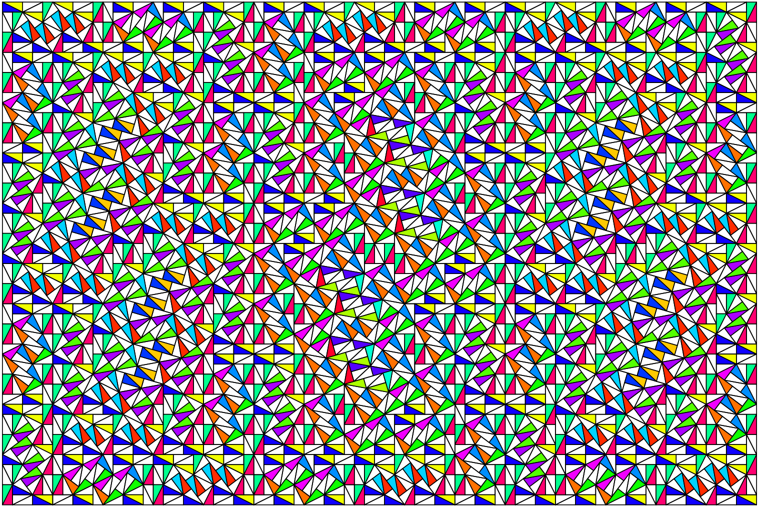
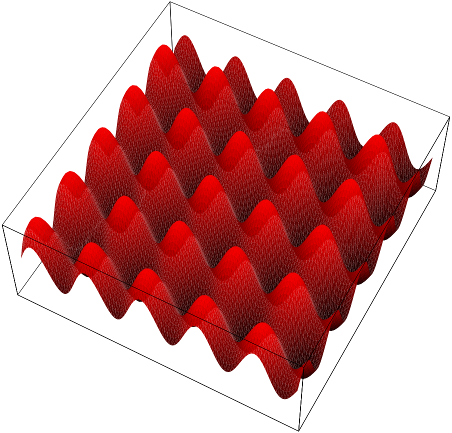


Projection

- crystallographic restriction:
  more symmetries in higher dimensions
- 5-, 8- and 12-fold symmetry realised in 4D, icosahedral symmetry in 6D lattices
- aperiodic tilings can be obtained as ‘slices’ of higher-dimensional periodic lattices
- ‘cut & project’ or ‘model sets’
- inherit almost periodicity and pure point diffractivity

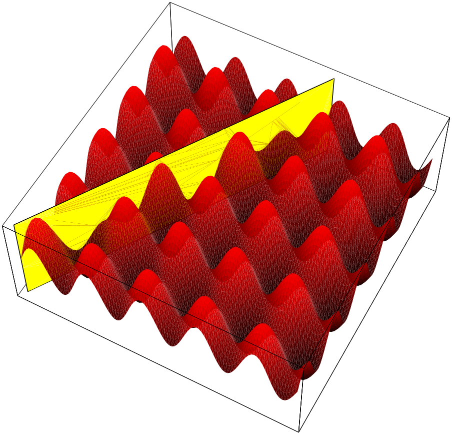
Almost periodicity

$$F(x) = \cos(2\pi x) + \cos(2\pi y)$$



Almost periodicity

$$f(x) = \cos(2\pi x) + \cos(2\pi\sqrt{2}x) = \cos(2\pi x) + \cos(2\pi y)_{|y=\sqrt{2}x}$$



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$$|f(x+n) - f(x)| = |2\sin(\pi\sqrt{2}n)\sin(\pi\sqrt{2}(n+2x))| \leq 2\,|\sin(\pi\sqrt{2}n)|$$

Almost periodicity

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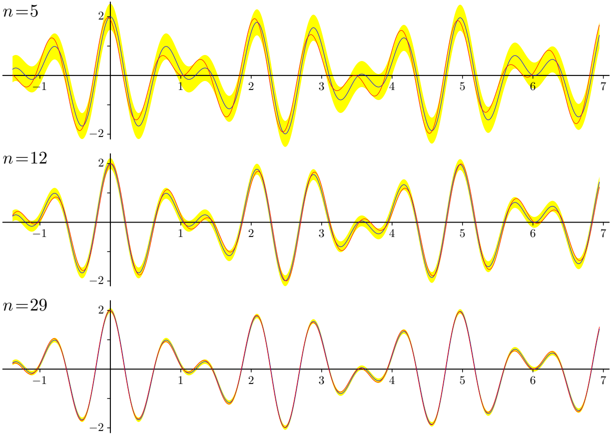


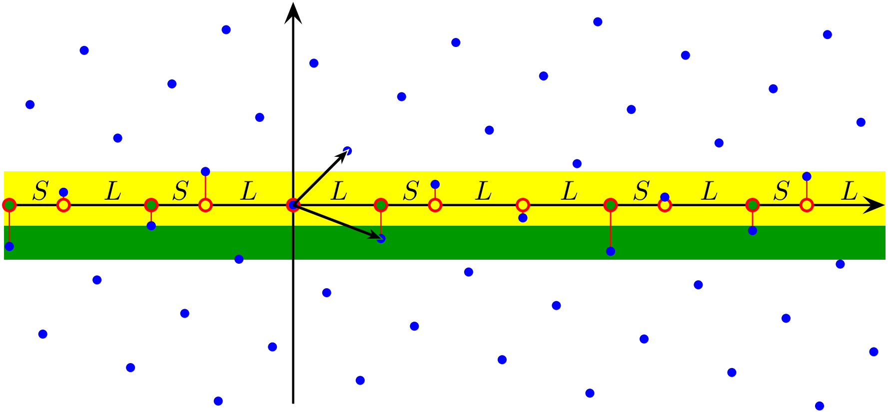
$$\sqrt{2} \simeq 1.41$$

$$5\sqrt{2} \simeq 7.07$$

$$12\sqrt{2} \simeq 16.98$$

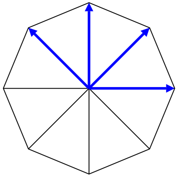
$$29\sqrt{2} \simeq 41.01$$

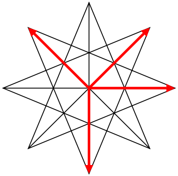
Fibonacci Projection



Ammann-Beenker Projection

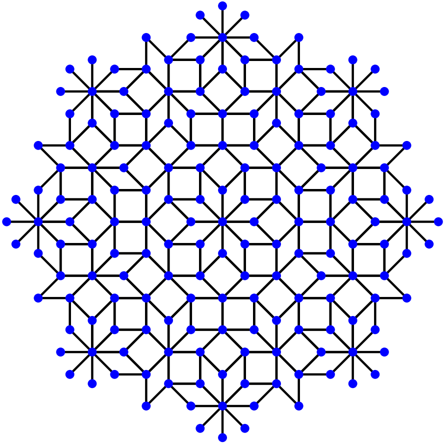
Projection from $\mathbb{Z}^4 \subset \mathbb{R}^4$:

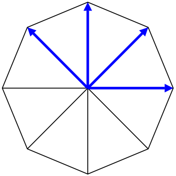


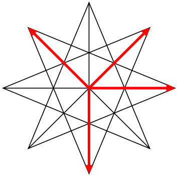


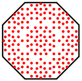
Ammann-Beenker Projection

Projection from $\mathbb{Z}^4 \subset \mathbb{R}^4$:









Euclidean Model Sets

CPS:

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\;\pi\;} & \mathbb{R}^d \times \mathbb{R}^m & \xrightarrow{\;\pi_{\mathrm{int}}\;} & \mathbb{R}^m \\
\cup & & \cup & & \cup \ \text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\mathrm{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\qquad\star\qquad\qquad} & & L^{\star}
\end{array}$$

Model set: $\boxed{\varLambda = \{ x \in L \mid x^{\star} \in W \}}$

with $W \subset \mathbb{R}^m$ compact, $\lambda(\partial W) = 0$

Diffraction: $\boxed{\widehat{\gamma} = \sum_{k \in L^{\circledast}} |A(k)|^2 \, \delta_k}$

with $L^{\circledast} = \pi(\mathcal{L}^{*})$ and amplitude $A(k) = \frac{\mathrm{dens}(\varLambda)}{\mathrm{vol}(W)} \, \widehat{1_W}(-k^{\star})$

The quest for a monotile

Is there a single shape that tiles space without gaps or overlaps, but does not admit any periodic tiling?

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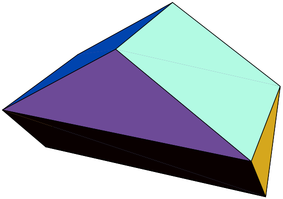

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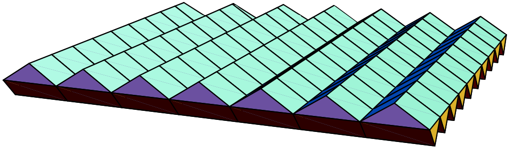

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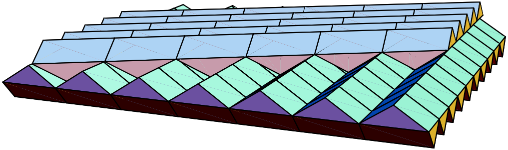

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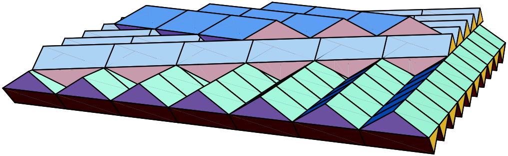

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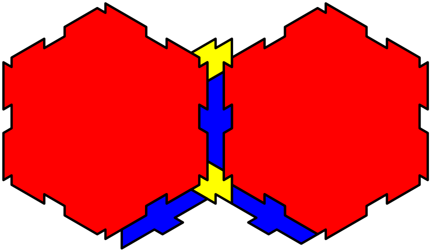

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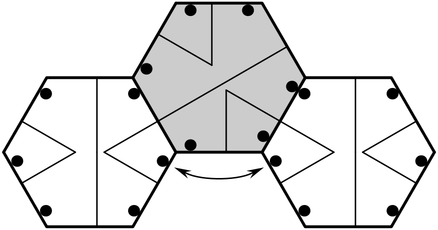
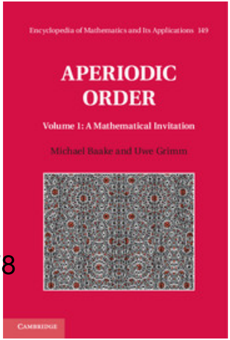
Socolar-Taylor monotile (2011)



What about Integrable Systems?

- Ising-type spin systems on aperiodic structures
  (Korepin 1986/7, Tracy 1988, Au-Yang & Perk 2006)
- aperiodic quantum spin chains
  (Benza 1989, Luck 1993, Hermisson, Grimm & Baake 1997)
- entropy of random tiling ensembles
  (Widom 1993, Kalugin 1994, Nienhuis & de Gier 1996/7)
- aperiodic Schrödinger operators
  (lots of literature, good reviews by Damanik)
- diffraction measure and spectral measure

For more on this...



Michael Baake & UG
Aperiodic Order. Vol 1. A Mathematical Invitation
Cambridge University Press (2013)

Michael Baake, David Damanik & UG
What is Aperiodic Order? arXiv:1512.05104

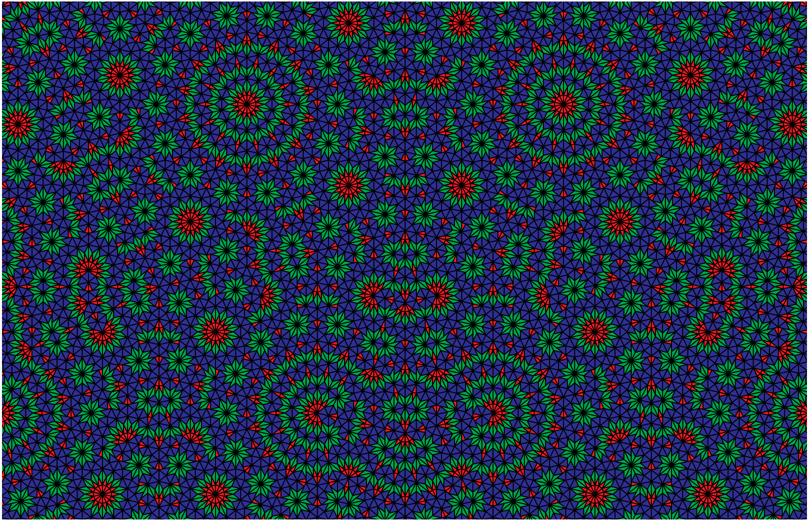
Michael Baake, David Damanik & UG
Aperiodic order and spectral properties. arXiv:1506.04978

UG
Aperiodic crystals and beyond
Acta Crystallographica B 71 (2015) 258-274. arXiv:1506.05276

Michael Baake, UG & Robert V. Moody
What is Aperiodic Order? arXiv:math/0203252

UG & Michael Schreiber
Aperiodic Tilings on the Computer. arXiv:cond-mat/9903010

‘Buffalo’ Tiling



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