Burchnall-Chaundy polynomials and Dodgson's condensation method

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Plan

- **Burchnall-Chaundy polynomials**
- **Dodgson's condensation method and octahedral equation**
- **Relation with Hirota-Miwa equation**
- **Reductions: difference BCh and Dodgson equations**
- **Cauchy problem and Laurent property**
- **Explicit Casoratian form**
- **Continuum limit: Laurent form of BCh polynomials**
- **Discussion**

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Reference

A.P. Veselov and R. Willox J Phys A. **48** (2015)

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$$P'_{n+1}(z)P_{n-1}(z) - P_{n+1}(z)P'_{n-1}(z) = P_n(z)^2$$

with $P_{-1}(z) = P_0(z) = 1$:

$$P_1 = z, \; P_2 = \frac{1}{3}(z^3 + \tau_2), \; P_3 = \frac{1}{45}(z^6 + 5\tau_2 z^3 + \tau_3 z - 5\tau_2^2),$$

$$P_4 = \frac{1}{4725}(z^{10} + 15\tau_2 z^7 + 7\tau_3 z^5 - 35\tau_2\tau_3 z^2 + 175\tau_2^3 z - \frac{7}{3}\tau_3^2 + \tau_4 z^3 + \tau_4\tau_2), \ldots$$

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Later rediscovered by **Stellmacher ang Lagnese** (1967) and independently by **Adler and Moser** (1978).

The existence is not obvious at all: indeed the above relation is equivalent to

$$\frac{d}{dz}\frac{P_{n+1}}{P_{n-1}} = \frac{P_n^2}{P_{n-1}^2},$$

which means that all the residues of the right-hand side must be zero.

Fomin and Zelevinsky (2002): **cluster algebras and Laurent phenomenon**

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As an example consider the difference equation (related to cluster algebra of type A_1)

$$p_{n+1}p_{n-1} = p_n^2 + 1.$$

If we choose the initial data $p_0 = p_1 = 1$ then the corresponding p_n will be surprisingly integer for all n: $1, 2, 5, 13, 34, 89, \ldots$

Parallel: Laurent phenomenon

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In fact, there is a more general result, which is the simplest example of the Laurent phenomenon:

Theorem (FZ, 2002): *p_n are Laurent polynomials of the initial data p_0, p_1 with integer coefficients.*

Difference Burchnall-Chaundy equation

Consider the following natural difference analogue of the Burchnall-Chaundy relation:

$$Q_{n+1}(z+1)Q_{n-1}(z) - Q_{n+1}(z)Q_{n-1}(z+1) = Q_n(z)Q_n(z+1),$$

with $Q_{-1}(z) = Q_0(z) = 1$.

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Willox-AV (2015): The difference BCh equation has polynomial solutions $Q_n(z)$ with degree $n(n+1)/2$ and coefficients that are Laurent polynomials of the initial data $q_k = Q_k(0)$, such that

$$A_n\, Q_n(z) \in \mathbb{Z}[z; q_1^{\pm 1}, \dots, q_{n-2}^{\pm 1}, q_{n-1}, q_n], \qquad A_n = \prod_{j=1}^{n} (2j-1)!!,$$

where $(2k+1)!! = 1 \times 3 \times 5 \times \cdots \times (2k+1)$.

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The first three polynomials are

$$Q_1 = z + q_1, \quad Q_2 = \frac{z(z^2-1)}{3} + q_1 z^2 + q_1^2 z + q_2,$$

$$Q_3 = \frac{z^2(z^2-1)(z^2-4)}{45} + \frac{2q_1 z^5}{15} + \frac{q_1^2 z^4}{3} + \frac{(q_1^3 - q_1 + q_2)z^3}{3} + \frac{(3q_1q_2 - q_1^2)z^2}{3}$$
$$+\left(\frac{q_3}{q_1} + \frac{q_2^2}{q_1} + \frac{2q_2}{3} - \frac{q_1^3}{3} + \frac{q_1}{5}\right)z + q_3.$$

Dodgson's method of computing determinants

Ch. Dodgson (aka L. Carroll) **(1866)**: **condensation method** for computing determinants

$$A=\begin{pmatrix}1 & & 1 & & 1\\ & 4 & & -4 & \\ 1 & & 5 & & 1\\ & -3 & & 13 & \\ 1 & & 2 & & 3\end{pmatrix}$$

$$\det A = 8$$

$$\det A = \frac{\begin{vmatrix}4 & -4\\ -3 & 13\end{vmatrix}}{5} = \frac{40}{5} = 8$$

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Proof is based on **Desnanot-Jacobi identity**:

$$\Delta\Delta_{1,n}^{1,n} = \Delta_1^1\Delta_n^n - \Delta_1^n\Delta_n^1,$$

where $\Delta = \det A$, $\Delta_i^j = \det A_i^j$ etc.

Dodgson's condensation method and octahedral equation

One can view Dodgson's method as the solution of a very special Cauchy problem for the discrete **Dodgson octahedral equation**

$$u_{l,m+1,n+1}u_{l,m-1,n-1} - u_{l,m+1,n-1}u_{l,m-1,n+1} = u_{l-1,m,n}u_{l+1,m,n},$$

where $m, n, l \in \mathbb{Z}$, $m \equiv n \equiv l \pmod 2$.

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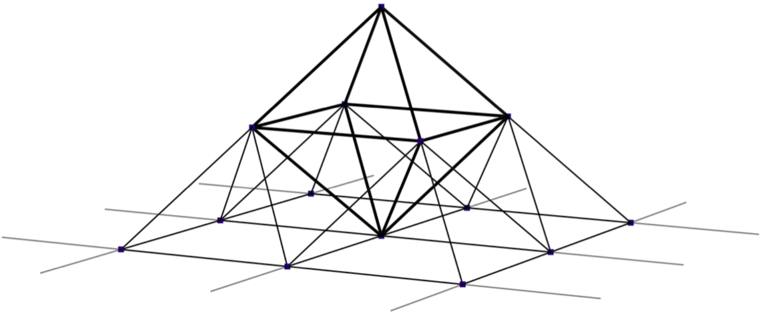

Figure: Dodgson's Cauchy pyramid for computing 3×3 determinants

Relation with Hirota-Miwa equation

Hirota (1981), Miwa (1982): a discrete version of KP equation on a standard cubic lattice

$$a\, v_{l+1,m,n} v_{l,m+1,n+1} + b\, v_{l,m+1,n} v_{l+1,m,n+1} + c\, v_{l,m,n+1} v_{l+1,m+1,n} = 0,$$

where $l, m, n \in \mathbb{Z}$ and a, b, c are arbitrary non-zero parameters.

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where $l, m, n \in \mathbb{Z}$ and a, b, c are arbitrary non-zero parameters.

Formally the Hirota-Miwa equation may be considered as a version of the Dodgson equation if one interprets these six vertices of the cube as the vertices of the octahedron:

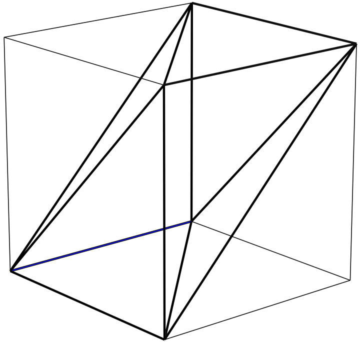

Figure: Support of the Hirota-Miwa equation in the cube

Reductions

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For the octahedral equation a natural reduction would be $u_{l+1,m,n} = u_{l-1,m,n}$ leading to the **discrete Dodgson equation**

$$u_{m+1,n+1}u_{m-1,n-1} - u_{m+1,n-1}u_{m-1,n+1} = u_{m,n}^2,$$

or, in the functional version, the **difference Dodgson equation**:

$$R_{n+1}(z+1)R_{n-1}(z-1) - R_{n+1}(z-1)R_{n-1}(z+1) = R_n^2(z).$$

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For Hirota-Miwa a natural reduction is $v_{l+1,m,n+1} = v_{l,m,n}$ which, for $a = 1$, $b = c = -1$, leads to the **discrete KdV equation**

$$Q_{m+1,n+1}Q_{m,n-1} - Q_{m,n+1}Q_{m+1,n-1} = Q_{m,n}Q_{m+1,n},$$

the functional version of which is the difference Burchnall-Chaundy equation

$$Q_{n+1}(z+1)Q_{n-1}(z) - Q_{n+1}(z)Q_{n-1}(z+1) = Q_n(z)Q_n(z+1).$$

Note that the support of the dKdV equation has a domino shape, while in the Dodgson case we have a 2×2 square, consisting of two dominos:

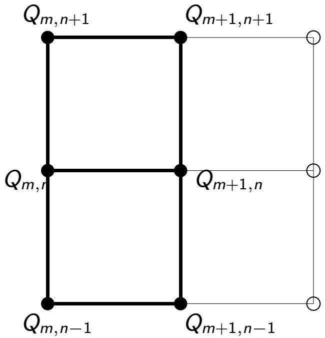

Figure: Domino-type support for the discrete KdV equation.

Relationship

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Willox-AV (2015): *The difference Burchnall-Chaundy and Dodgson equations are equivalent on the set of initial data satisfying* $\Phi_0 = 0$, *where*

$$\Phi_n(z) := Q_n(z+1)Q_{n-1}(z-1) + Q_{n-1}(z+1)Q_n(z-1) - 2Q_{n-1}(z)Q_n(z).$$

More precisely, if the initial data $Q_{-1}(z), Q_0(z)$ *of the Cauchy problem for the dBCh equation satisfy the constraint*

$$Q_0(z+1)Q_{-1}(z-1) + Q_{-1}(z+1)Q_0(z-1) - 2Q_{-1}(z)Q_0(z) = 0,$$

then $R_n(z) = 2^{-\frac{n(n+1}{2}}Q_n(z)$ *satisfy the difference Dodgson equation.*

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Remark. If we modify the Dodgson equation as

$$R_{n+1}(z+1)R_{n-1}(z-1) - R_{n+1}(z-1)R_{n-1}(z+1) = 2R_n^2(z),$$

then modulo constraint $\Phi_0 = 0$ we simply have $Q_n(z) = R_n(z)$.

Cauchy problem and Laurent property

Fomin and Zelevinsky (2002): Laurent property for discrete KdV

$$Q_{m+1,n+1}Q_{m,n-1} - Q_{m,n+1}Q_{m+1,n-1} = Q_{m,n}Q_{m+1,n}.$$

Fomin and Zelevinsky (2002): Laurent property for discrete KdV

$$Q_{m+1,n+1}Q_{m,n-1} - Q_{m,n+1}Q_{m+1,n-1} = Q_{m,n}Q_{m+1,n}.$$

For $Q_{m,-1} = Q_{m,0} = 1, \quad Q_{0,n} = q_n, \quad m,n \in \mathbb{Z}$, this means that $Q_{m,n}$ is a Laurent polynomial in q_i with integer coefficients. In particular, this implies that when all $q_i = 1$, all the $Q_{m,n}$ are integers:

12181	−507	−455	−91	21	5	1	21	397	6469	104145	1332565	15181325
377	13	13	21	9	−3	1	13	149	1629	14001	115245	908245
615	−26	−23	−4	3	2	1	8	59	350	2109	11492	52375
249	51	5	1	1	−1	1	5	21	91	329	977	2477
−39	−19	−7	−1	1	1	1	3	9	21	41	71	113
−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5
113	71	41	21	9	3	1	1	1	−1	−7	−19	−39
2477	977	329	91	21	5	1	−1	1	1	5	51	249
52375	11492	2109	350	59	8	1	2	3	−4	−23	−26	615
908245	115245	14001	1629	149	13	1	−3	9	21	13	13	377
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Explicit determinantal formulae

We follow essentially a difference analogue of Adler-Moser procedure. Let us define the polynomials $x_n(z)$ by the generating function

$$F(z,t,u)F(z,u) := \sum_{k=0}^{\infty} x_k(z)u^k = e^{\sum_{k=1}^{\infty}(-1)^{k+1}(z+t_k)\frac{u^k}{k}} :$$

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$$x_0 = 1, \quad x_1 = z + t_1, \quad x_2 = \frac{1}{2}[(z+t_1)^2 - (z+t_2)], \quad \ldots .$$

They satisfy the relation $x_n(z+1) - x_n(z) = x_{n-1}(z), \quad x_0 = 1$ and can be given as the determinants:

$$x_k(z) = \frac{1}{k!}\begin{vmatrix} z_1 & -1 & 0 & \ldots & \ldots & 0 \\ z_2 & z_1 & -2 & \ldots & \ldots & 0 \\ z_3 & z_2 & z_1 & -3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ z_{k-1} & z_{k-2} & \ldots & \ldots & z_1 & 1-k \\ z_k & z_{k-1} & \ldots & \ldots & z_2 & z_1 \end{vmatrix}, \quad z_k = (-1)^{k+1}(z+t_k).$$

Casoratians

Let now $y_k = x_{2k-1}$ and consider the **Casoratians** $Q_n(z) = C(y_1, \dots y_n)$, where by definition

$$C(f_1, \dots f_n) = \det ||f_i(z + j - 1)||, \;\; i, j = 1, \dots, n.$$

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The Casoratians $Q_k = C(y_1, \dots, y_k)$ can be written as the determinants

$$Q_k = \begin{vmatrix} x_1 & x_3 & x_5 & \dots & \dots & x_{2k-1} \\ 1 & x_2 & x_4 & \dots & \dots & x_{2k-2} \\ 0 & x_1 & x_3 & x_5 & \dots & x_{2k-3} \\ 0 & 1 & x_2 & x_4 & \dots & x_{2k-4} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & x_{k-2} & x_k \end{vmatrix}. \tag{1}$$

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Willox-AV: *The Casoratians $Q_k(z)$ satisfy the difference Burchnall-Chaundy equation.*

Note that the coefficients of the polynomials $Q_k(z)$ are polynomial in the parameters t_j:

$$Q_1 = z + t_1, \qquad Q_2 = \frac{1}{3}\left(z(z^2 - 1) + 3t_1 z^2 + 3t_1^2 z + t_1^3 - t_3\right)$$

$$Q_3 = \frac{1}{45}\left(z^2(z^2-1)(z^2-4) + 6t_1 z^5 + 15t_1^2 z^4 + (20t_1^3 - 5t_3 - 15t_1)z^3\right.$$

$$\left. + 15t_1(t_1^3 - t_1 - t_3)z^2 + (9t_1 - 10t_3 + 9t_5 - 15t_1^2 t_3 - 5t_1^3 + 6t_1^5)z + t_1^6 - 5t_3^2 - 5t_1^3 t_3 + 9t_1 t_5\right).$$

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Parameters and initial data

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We need to express now the KdV parameters t_j via Cauchy data $q_k = Q_k(0)$.

Substituting $z = 0$ to Casoratian we have the relations

$$q_1 = t_1, \quad q_2 = -\frac{1}{3}t_3 + \frac{1}{3}t_1^3, \quad q_3 = \frac{1}{5}t_1 t_5 - \frac{1}{9}t_3^2 - \frac{1}{9}t_1^3 t_3 + \frac{1}{45}t_1^6,$$

$$q_4 = \frac{1}{21}(t_3 - t_1^3)t_7 - \frac{1}{25}t_5^2 + \frac{1}{15}t_1^2 t_3 t_5 + \frac{1}{75}t_1^5 t_5 - \frac{1}{27}t_1 t_3^3 - \frac{1}{315}t_1^7 t_3 + \frac{1}{4725}t_1^{10}, \ldots$$

Further analysis gives the following result:

Willox-AV: *The polynomial q_k depends only on odd parameters t_{2i-1} with $i = 1, \dots, k$ and has the form*

$$q_k = \frac{(-1)^{k+1}}{2k-1} q_{k-2} t_{2k-1} + \psi_k(t_1, t_3, \dots, t_{2k-3}),$$

for some polynomials ψ_k with rational coefficients.
The parameter t_{2k-1} can be expressed in terms of q_j as a Laurent polynomial with integer coefficients

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Further analysis gives the following result:

Willox-AV: *The polynomial q_k depends only on odd parameters t_{2i-1} with $i = 1, \ldots, k$ and has the form*

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As a corollary we have Laurent phenomenon for the difference Burchnall-Chaundy equation:

$$A_n\, Q_n(z) \in \mathbb{Z}[z; q_1^{\pm}, \ldots, q_{n-2}^{\pm}, q_{n-1}, q_n], \quad q_k = Q_k(0).$$

Continuum limit: usual Burchnall-Chaundy polynomials

Burchnall-Chaundy polynomials P_n are known to be the τ-functions

$$u(x, T_1, \dots, T_n) = -2D^2 \log P_n(x, T_1, \dots, T_n)$$

of the rational solutions of the KdVries equation $u_{T_1} = D^3 u - 6uDu, \quad D = \frac{d}{dx}$ and its higher analogues $u_{T_k} = D^{2k+1}u + \dots$. Our parameters t_{2k+1} are simply related to the KdV times by the scaling $t_{2k+1} = 4^k (2k+1) T_k$.

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Willox-AV: *The continuum limit*

$$P_n(x, t_1, t_3, \dots, t_{2n-1}) = \lim_{\varepsilon \to 0} \varepsilon^{\frac{n(n+1)}{2}} Q_n\left(\frac{x}{\varepsilon}, \frac{t_3}{\varepsilon^3}, \dots, \frac{t_{2n-1}}{\varepsilon^{2n-1}}\right),$$

yields the usual Burchnall-Chaundy polynomials parametrized by the scaled KdV times $t_3, \dots, t_{2n-1}$.

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As a corollary we have Laurent phenomenon for the usual Burchnall-Chaundy equation:

$$A_n P_n(z) \in \mathbb{Z}[z; c_1^{\pm}, \dots, c_{n-2}^{\pm}, c_{n-1}, c_n], \quad c_k = P_k(0).$$

New form of the original Burchnall-Chaundy polynomials

In terms of the initial values $c_k = P_k(0)$ we have the new Laurent formulae

$$P_1 = z + c_1, \;\; P_2 = \frac{1}{3}\big(z^3 + 3c_1 z^2 + 3c_1^2 z + 3c_2\big),$$

$$P_3 = \frac{1}{45}\big(z^6 + 6c_1 z^5 + 15c_1^2 z^4 + 15(c_1^3 + c_2)z^3 + 45c_1 c_2 z^2 + 45(\frac{c_2^2}{c_1} + \frac{c_3}{c_1})z + 45c_3\big),$$

$$P_4 = \frac{1}{4725}\big(z^{10} + 10c_1 z^9 + 45c_1^2 z^8 + 15(7c_1^3 + 3c_2)z^7 + 105(c_1^4 + 3c_2 c_1)z^6$$

$$+315(\frac{c_2^2}{c_1} + \frac{c_3}{c_1} + 2c_1^2 c_2)z^5 + 1575(c_2^2 + c_3)z^4$$

$$+1575(\frac{c_2^3}{c_1^2} + c_1 c_3 + \frac{c_4}{c_2} + \frac{c_3^2}{c_1^2 c_2} + 2\frac{c_3 c_2}{c_1^2})z^3$$

$$+4725(\frac{c_3^2}{c_1 c_2} + \frac{c_2 c_3}{c_1} + \frac{c_1 c_4}{c_2})z^2 + 4725(\frac{c_3^2}{c_2} + \frac{c_1^2 c_4}{c_2})z + 4725c_4\big), \dots$$

Discussion

One of the main open questions in the theory of integrable systems is the role of the Cauchy problem, especially at the discrete level.

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As an example, consider discrete KdV in the m direction, and its functional version by replacing n by x:

$$F_{m+1}(x+1)F_m(x-1) - F_{m+1}(x)F_m(x) - F_{m+1}(x-1)F_m(x+1) = 0$$

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In particular, what is the "right" interpolation of the numbers on the vertical lines of the next table? Note that on the second line we have the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$, on the third line the sequence satisfying

$$F_{n-1}G_{n+1} = F_{n+1}G_{n-1} + F_n G_n, \quad G_{-1} = G_0 = 1.$$

Table

12181	−507	−455	−91	21	5	1	21	397	6469	104145	1332565	15181325
377	13	13	21	9	−3	1	13	149	1629	14001	115245	908245
615	−26	−23	−4	3	2	1	8	59	350	2109	11492	52375
249	51	5	1	1	−1	1	5	21	91	329	977	2477
−39	−19	−7	−1	1	1	1	3	9	21	41	71	113
−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1
7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5
113	71	41	21	9	3	1	1	1	−1	−7	−19	−39
2477	977	329	91	21	5	1	−1	1	1	5	51	249
52375	11492	2109	350	59	8	1	2	3	−4	−23	−26	615
908245	115245	14001	1629	149	13	1	−3	9	21	13	13	377
15181325	1332565	104145	6469	397	21	1	5	21	−91	−455	−507	12181

Figure: Solution to the dKdV equation with Cauchy data $Q_{0,n} = Q_{m,-1} = Q_{m,0} = 1$. The axes are given by the column of 1's (n axis) and the top row of 1's (m axis).