Monodromy local moduli of semisimple coalescent Frobenius structures

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Frobenius Structures and Relations

Glasgow, March 22^{nd}, 2018

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Definition

A *Frobenius manifold* (FM, for short) *M of charge d* is a complex manifold endowed with

1. an $\mathcal{O}_M$-bilinear metric tensor η with flat Levi-Civita connection ∇;
2. a $(1,2)$-tensor $c \in \Gamma\left(TM \otimes \bigodot^2 T^*M\right)$ s.t. $c^\flat \in \Gamma\left(\bigodot^3 T^*M\right)$, $\nabla c^\flat \in \Gamma\left(\bigodot^4 T^*M\right)$;
3. a *parallel* vector field e, called *unit*, s.t. $c(-,e,-) \in \Gamma(\mathrm{End}(TM))$ is the identity;
4. a vector field E, called *Euler v.f.*, s.t. $\mathfrak{L}_E c = c, \quad \mathfrak{L}_E \eta = (2-d)\eta$.

The rich geometry of a Frobenius manifold is encoded in the *flatness* condition of an *extended deformed connection* $\widehat{\nabla}$ defined on $\pi^* TM$.

$$\begin{tikzcd}
\pi^* TM \arrow[r] \arrow[d] & TM \arrow[d] \\
\mathbb{C}^* \times M \arrow[r, "\pi"] & M
\end{tikzcd}$$

For $X, Y \in \Gamma(\pi^* T_M)$ define

$$\widehat{\nabla}_X Y := \nabla_X Y + z \cdot X \circ Y,$$

$$\widehat{\nabla}_{\partial_z} Y := \nabla_{\partial_z} Y + \mathcal{U}(Y) - \frac{1}{z}\mu(Y),$$

where we have introduced the $(1,1)$-tensors

$$\mathcal{U}(Y) := E \circ Y, \quad \mu(Y) := \frac{2-d}{2} Y - \nabla_Y E.$$

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There is a *local identification*

$$\left\{\begin{array}{l}\text{semisimple points } t \text{ of}\\ \text{Frobenius } n\text{-manifold } M\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{deformation parameters of isomonodromic}\\ \text{families of differential systems } n\times n\\ \dfrac{dY(z,t)}{dz} = \left(U(t) + \dfrac{1}{z}V(t)\right)Y(z,t)\\ \quad U \text{ diagonal, } V \text{ anti-symmetric}\end{array}\right\}$$

Here, U, V are the components of the tensors $\mathcal{U}, \mu$ w.r.t. an orthonormalized idempotent vielbein $(f_1,\dots,f_n)$

$$f_j|_t := \frac{1}{\eta(\pi_j|_t, \pi_j|_t)^{\frac{1}{2}}}\pi_j|_t \qquad \pi_1|_t,\dots,\pi_n|_t \text{ idempotents at } t.$$

$$\left.\frac{\partial}{\partial t^\alpha}\right|_t = \sum_i \Psi_{i\alpha}(t) f_i|_t, \qquad U := \Psi\mathcal{U}\Psi^{-1}, \qquad V := \Psi\mu\Psi^{-1}.$$

Definition (ℓ-chamber)

Let $\phi \in \mathbb{R}$ and $\ell(\phi) := \{z\colon \arg z = \phi\}$ be an oriented ray in the universal cover $\widehat{\mathbb{C}\setminus\{0\}}$. We call ℓ-chamber of a given FM M any connected component of the open set of points $t \in M$ s.t.

1. the eigenvalues $u_i(t)$ are pairwise distinct,
2. no Stokes ray $R_{ij}(t) := \left\{-i\rho(\bar{u}_i(t) - \bar{u}_j(t))\colon \rho \in \mathbb{R}_+\right\}$, with $i \neq j$, is covered by $\ell(\phi)$.

Notice that being an element of an ℓ-chamber is a sufficient but *not necessary* condition for being a semisimple point of M.

$$\frac{dY(z,t)}{dz} = \left(U(t) + \frac{1}{z}V(t)\right)Y(z,t), \quad U(t) = \mathrm{diag}(u_1(t),\dots,u_n(t)).$$

Fuchsian singularity at $z = 0$

Fundamental solutions in Levelt form:

$$Y_0(z,t) = \Psi(t)\Phi(z,t)z^{\mu}z^{R},$$

$$\Phi(z,t) = 1 + \sum_{n=1}^{\infty}\Phi_k(t)z^k,$$

$$\Phi(-z,t)^T\eta\Phi(z,t) = \eta.$$

Irregular singularity at $z = \infty$

Genuine solutions $Y_{R/L}$

$$Y_{R/L}(z,t) \sim \Theta(z,t)e^{zU(t)}, \quad z \in \Pi_{R/L}, \quad z \to \infty$$

$$\Theta(z,t) = 1 + \sum_{n=1}^{\infty}\Theta_k(t)\frac{1}{z^k}, \quad \Theta(-z,t)^T\Theta(z,t) = 1,$$

$$Y_L(z,t) = Y_R(z,t)S, \quad Y_R(z,t) = Y_0(z,t)C.$$

Theorem (B. Dubrovin, Isomonodromy Theorem)

The data (μ, R, S, C) are constants in any ℓ-chamber. Thus they are local invariants of M.

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If we cross a *wall* of an ℓ-chamber, the monodromy data S and C manifest a jump discontinuity. The data (S, C) mutate according to the action of the mapping class group of a disk with n ordered punctures (representing the u_i's), that is the braid group $\mathcal{B}_n$: f.g. group with $n-1$ generators, $\beta_{12}, \beta_{23}, \dots, \beta_{n-1,n}$ satisfying

$$\beta_{i,i+1}\beta_{j,j+1} = \beta_{j,j+1}\beta_{i,i+1}, \quad |i-j| \geq 2$$

$$\beta_{i,i+1}\beta_{i+1,i+2}\beta_{i,i+1} = \beta_{i+1,i+2}\beta_{i,i+1}\beta_{i+1,i+2}.$$

If we relabel the canonical coordinates u_i's in the *ℓ-lexicographical order*, the Stokes matrix is put in triangular form. The elementary braid $\beta_{i,i+1}$ acts on the moduli S and C as follows:

$$S^{\beta_{i,i+1}} = A^{\beta_{i,i+1}}(S) \cdot S \cdot A^{\beta_{i,i+1}}(S),$$
$$C^{\beta_{i,i+1}} = C \cdot (A^{\beta_{i,i+1}}(S))^{-1},$$

$$A^{\beta_{i,i+1}}(S) := \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & & \\ & & 1 & -s_{i,i+1} & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

The monodromy data (S, C), at a point $p \in M$ in a ℓ-chamber, are defined up to some natural ambiguities, due to several choices:

1. action of $\mathfrak{S}_n$ (choice of ordering of canonical coordinates):
$$S \mapsto PSP^{-1}, \quad C \mapsto CP^{-1};$$
2. action of $(\mathbb{Z}/2\mathbb{Z})^n$ (choice of sings of normalized idempotents):
$$S \mapsto JSJ^{-1}, \quad C \mapsto CJ^{-1};$$
3. action of $\mathcal{C}_0(\eta, \mu, R)$ (choice of solution in Levelt normal form at $z = 0$):
$$C \mapsto GC;$$
4. action of $\mathbb{Z}$ (choice of a determination of the slope of the line ℓ):
$$C \mapsto M_0^k C, \quad M_0 := \exp(2\pi i \mu) \exp(2\pi i R).$$

Remarkably, from the knowledge of the data (μ, R, S, C) the whole Frobenius structure of an ℓ-chamber can be reconstructed trough a Riemann-Hilbert Problem[1].

[1] Caveat: Do not think that the RH problem is solvable at any point of the chamber!

Let X be a smooth projective variety, s.t. $H^{\mathrm{odd}}(X;\mathbb{C})=0$.

Let $(T_1, T_2, \dots, T_r, T_{r+1}, \dots, T_N)$ be a homogeneous basis of $H^\bullet(X) := \bigoplus_k H^{2k}(X;\mathbb{C})$, and denote by $(t^1, \dots, t^N)$ the relative coordinates.
Assume that the Gromov-Witten potential of genus 0

$$F_0^X(t) := \sum_{n=0}^{\infty} \sum_{\beta \in \mathrm{Eff}(X)} \sum_{\alpha_1, \dots, \alpha_n = 1}^{N} \frac{t^{\alpha_1} \dots t^{\alpha_n}}{n!} \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}} \bigwedge_{i=1}^{n} \mathrm{ev}_i^* \, T_{\alpha_i},$$

$$\mathrm{ev}_i \colon \overline{\mathcal{M}}_{0,n}(X,\beta) \to X \colon ((C, \mathbf{x}); f) \mapsto f(x_i)$$

is convergent on a non-empty domain $\Omega \subseteq H^\bullet(X)$.
Remarkably, the domain Ω admits a Frobenius manifold structure where

$$\eta(T_\alpha, T_\beta) := \int_X T_\alpha \wedge T_\beta, \qquad (c^\flat)_{\alpha\beta\gamma} := \frac{\partial^3 F_0^X}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \qquad e := T_1 \equiv 1,$$

$$E|_t := c_1(X) + \sum_{\alpha=1}^{N} \left(1 - \frac{1}{2} \deg T_\alpha\right) t^\alpha T_\alpha.$$

The locus $\Omega \cap H^2(X;\mathbb{C})$ is called small quantum cohomology of X. The whole Frobenius structure can be analytically continued to an unramified covering of the domain Ω: in general almost nothing is explicitly known about this Big Quantum Cohomology.

Conjecture (G.C. - B.A. Dubrovin - D. Guzzetti, to appear)

Let X be a Fano manifold of Hodge-Tate type:

$$h^{p,q}(X)=0, \quad \text{if } p\neq q.$$

1. *The Frobenius manifold $QH^\bullet(X)$ is semisimple iff there exists a full exceptional collection $(E_1,\dots,E_n)$ in $\mathcal{D}^b(X)$ with $n=\sum\beta_{2i}(X)$. Moreover*
2. *the Stokes matrix S is equal to the inverse of the Gram matrix of the Euler-Poincaré-Grothendieck product χ on $K_0(X)$ w.r.t. the basis $([E_1],\dots,[E_n])$;*
3. *the central connection matrix C is the one associated to the morphism*

$$Д^-_X\colon K_0(X)\otimes_{\mathbb{Z}}\mathbb{C}\to H^\bullet(X,\mathbb{C})\colon E\mapsto \frac{i^{\overline{d}}}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}^-_X\cup e^{-\pi i c_1(X)}\cup \mathrm{Ch}(E),$$

$$\widehat{\Gamma}^-_X:=\prod_{j=1}^d\Gamma(1-\delta_j),\quad \mathrm{Ch}(E):=\sum_j(2\pi i)^j\,\mathrm{ch}_j(E)$$

where d is the dimension of X, and $\overline{d}$ its residue class (mod 2).

Some of the contributions and partial confirmations of the Conjecture:

- B. Dubrovin and D. Guzzetti: proof of the first and second part of the Conjecture for $\mathbb{P}^n(\mathbb{C})$.
- G. Ciolli proved the first part of the Conjecture for 36 (out of 59) classes of Fano threefolds with odd vanishing cohomology.
- A. Bayer and Yu. I. Manin: semisimplicity is stable under blow-ups along points. Probably, the Fano assumption is not needed.
- C. Hertling, Yu. I. Manin and C. Teleman proved that a necessary condition for semisimplicity is the Hodge-Tate condition.
- K. Ueda proved the validity of the first and second part of the conjecture for cubic surfaces and suggested its validity also for Grassmannians.
- Results of H. Iritani and Y. Kawamata confirm the validity of the first part of the Conjecture for projective toric varieties.
- J.A. Cruz Morales, A. Mellit, N. Perrin and M. Smirnov proved the validity of the first part of the Conjecture for the isotropic Grasmmannians $IG(n, 2n)$.

Theorem (G.C., B. Dubrovin, D. Guzzetti, to appear)

The Conjecture holds true for all complex Grassmannians.

The central connection matrix C is intended to be computed wrt the topological-enumerative solution:

$$\begin{array}{c}\mathcal{L}_1\\ \Big\downarrow T^*_{x_1}C\\ \overline{\mathcal{M}}_{0,n}(X,\beta)\end{array}$$

$$Y_0(z,t)=\Psi(t)\cdot\Theta_{\rm top}(z,t)z^{\mu}z^{c_1(X)\cup},$$

$$(\Theta_{\rm top})^{\alpha}_{\beta}:=\delta^{\alpha}_{\beta}+\sum_{\lambda}\left\langle\left\langle\frac{zT_\beta}{1-z\psi_1},T_\lambda\right\rangle\right\rangle_0\eta^{\lambda\alpha},$$

$\psi_1:=c_1(\mathcal{L}_1)$, 1-st ψ class on the moduli spaces $\overline{\mathcal{M}}_{0,n}(X,\beta)$.

In 2013, B. Dubrovin suggested a first formulation for the third part of the Conjecture:

$$\sum_{\alpha}C^{\alpha}_{k}T_{\alpha}=\frac{1}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}^{-}_{X}\cup\mathrm{Ch}(E_k). \quad (1)$$

In 2014, S. Galkin, V. Golyshev and H. Iritani claimed a refinement of the third part of original Dubrovin's Conjecture (Γ-conjecture II):

$$\sum_{\alpha}C^{\alpha}_{k}T_{\alpha}=\frac{1}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}^{+}_{X}\cup\mathrm{Ch}(E_k). \quad (2)$$

Both (1) and (2) actually correspond to different choices of a solution in Level normal form! Such an ambiguity is described by the group $\mathcal{C}_0(\eta,\mu,R)$.

Achtung!

Depending on (k, n), the small quantum cohomology of the Grassmannian $\mathbb{G}(k, n)$ may present some coalescence

$$u_i = u_j \text{ for some } i \neq j.$$

Points of the bifurcation set, the *skeleton* of the walls of ℓ-chambers, for any chosen ℓ.

We have at least two foundational problems:

1. are the monodromy data S, C defined at these points?
2. is there any hope for isomonodromicity near these points?

Before addressing this problem, let us focus on some side questions:

1. for which (k, n) the Grassmannian $\mathbb{G}(k, n)$ is coalescing?
2. how much frequent is the coalescence phenomenon among all Grassmannians?

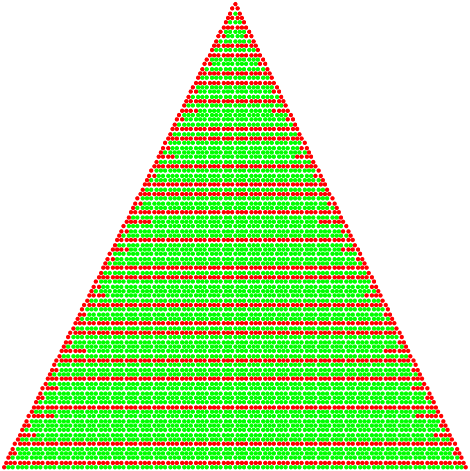


Figure: In this figure we represent complex Grassmannians as disposed in a Tartaglia-Pascal triangle: the k-th element (from the left) in the n-th row (from the top of the triangle) represents the Grassmannian $\mathbb{G}(k, n+1)$, where $n \leq 97$. The dots colored in red represent *non-coalescing* Grassmannians, while the dots colored in green the *coalescing* ones.

The study of $QH^\bullet(\mathbb{G}(r,k))$ can be reduced to the one of $QH^\bullet(\mathbb{P}^{k-1})$, using the Quantum Satake Identification, and the notion of Alternate product of Frobenius Manifolds: let

$$\mathbb{P} := \mathbb{P}^{k-1}_{\mathbb{C}}, \quad \mathbb{G} := \mathbb{G}(r,k), \quad \Pi := \underbrace{\mathbb{P}\times\cdots\times\mathbb{P}}_{r \text{ times}}, \quad \mathbb{F} := \mathrm{Fl}(1,2,\dots,r,k).$$

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "p"'] \arrow[dr, hook, "\iota"] & \\ \mathbb{G} & & \Pi \end{tikzcd}$$

- Cohomology class $\alpha \in H^\bullet(\mathbb{G})$ can be lifted to a cohomology class $\tilde{\alpha} \in H^\bullet(\Pi)$ s.t. $\iota^*\tilde{\alpha} = p^*\alpha$.
- The map

$$\vartheta\colon H^\bullet(\mathbb{G}) \to H^\bullet(\Pi)\colon \alpha \mapsto \tilde{\alpha} \cup_\Pi \Delta,$$

with $\Delta := \prod_{1\le i<j\le r}(x_i - x_j)$ defines a $\mathbb{C}$-isomorphism between $H^\bullet(\mathbb{G})$ and $[H^\bullet(\Pi)]^{\mathrm{ant}} \cong \bigwedge^r H^\bullet(\mathbb{P})$.

Such an identification extends also at the quantum level:

$$T_p QH^\bullet(\mathbb{G}) \cong \bigwedge^r T_{\bar{p}} QH^\bullet(\mathbb{P}), \quad p = t^1\sigma_1, \quad \bar{p} = t^1\sigma_1 + (r-1)\pi i \sigma_1.$$

The coalescence phenomenon on $\mathbb{G}$ can be rephrased in terms of vanishing sums of roots of unity

Theorem (G.C., 2016)

The Grassmannian $\mathbb{G}(r,k)$ is coalescing if and only if $\pi_1(k) \le r \le k - \pi_1(k)$.

Let us focus on

$$\frac{dY}{dz} = A(z,t)Y, \quad A(z,t) = \sum_{k=0}^{\infty} A_k(t) z^{-k}, \quad r_\infty = 1,$$

$$A_0(t) = \mathrm{diag}(u_1(t), \dots, u_n(t)), \quad \text{w.l.o.g. } u_i(t) = t_i + c_i, \quad A_k \in \mathcal{O}(\Omega)$$

where the eigenvalues u_i's can coalesce along a locus $\Delta \subseteq \Omega$. For $t \notin \Delta$, we have a unique formal solution of the form

$$Y_F(z,t) = \left(1 + \sum_{k=1}^{\infty} F_k(t) z^{-k}\right) z^{B_1(t)} e^{A_0(t)z}, \quad B_1(t) := \mathrm{diag}(A_1(t)), \tag{3}$$

there exist and are unique fundamental solutions $Y_r(z,t)$, $r = 1, 2, 3$, s.t.

$$Y_r(z,t) \sim Y_F(z,t), \quad |z| \to \infty \text{ in suitable } t\text{-independent sectors, uniformly in } t \in \mathcal{B} \Subset \Omega \setminus \Delta,$$

$$Y_2(z,t) = Y_1(z,t) S_1(t), \quad Y_3(z,t) = Y_2(z,t) S_2(t).$$

There are several difficulties concerning fundamental solutions at $z = \infty$ when $t \to t_0 \in \Delta$:

1. In general, for $t \in \Delta$, formal solutions of the form (3) do not exist: their form is much more complicated;
2. Even if formal solutions of type (3) exist, they are not unique;
3. In general, the coefficients $F_k(t)$ diverge for $t \to t_0 \in \Delta$;
4. Even if $F_k(t)$ converge, in general $\lim_{t \to t_0 \in \Delta} F_k(t) \neq F_k(t_0)$.

Theorem (G.C., B. Dubrovin, D. Guzzetti, 2016 - Part I)

- *If $\hat{t} \in \Delta$, the differential system*

$$\frac{dY}{dz} = A(z, \hat{t})Y$$

admits a fundamental solution of the form $\widehat{Y}(z) = \widehat{G}(z) z^{B_1(\hat{t})} e^{A_0(\hat{t})z}$, $B_1(\hat{t}) = \mathrm{diag}(A_1(\hat{t}))$, *with*

$$\widehat{G}(z) \sim 1 + \sum_{k=1}^{\infty} \widehat{F}_k z^{-k}, \quad |z| \to \infty \text{ in a suitable sector,}$$

if and only if the following vanishing conditions hold: whenever $u_a(\hat{t}) = u_b(\hat{t})$ *then*

1. $\left(A_1(\hat{t})\right)_{ab} = 0$;
2. *if moreover* $\left(A_1(\hat{t})\right)_{aa} - \left(A_1(\hat{t})\right)_{bb} = 1 - \ell$ *for some* $\ell \in \mathbb{N}_{\geq 2}$, *then*

$$\sum_{\gamma:\ u_\gamma(\hat{t}) \neq u_a(\hat{t})} \left(A_1(\hat{t})\right)_{a\gamma} \left(\widehat{F}_k\right)_{\gamma b} + \sum_{j=1}^{\ell-2} \left(A_{\ell-j}(\hat{t})\widehat{F}_j\right)_{ab} + \left(A_1(\hat{t})\right)_{ab} = 0.$$

If the second resonance never manifests, then the $\widehat{F}_k$*'s are unique.*

- *If $\mathcal{B} \subseteq \Omega \setminus \Delta$, the coefficients F_k's computed in $\mathcal{B}$ can be holomorphically continued to $F_k \in \mathcal{O}(\Omega)$ if and only if the functions $(A_1(t))_{ab}$, and for any $\ell \in \mathbb{N}_{\geq 2}$*

$$\left[(A_1(t))_{aa} - (A_1(t))_{bb} + \ell - 1\right] (F_{\ell-1}(t))_{ab} + \sum_{\gamma \neq a} (A_1(t))_{a\gamma} (F_k(t))_{\gamma b} + \sum_{j=1}^{\ell-2} (A_{\ell-j}(t)F_j(t))_{ab} + (A_1(t))_{ab}$$

are vanishing as fast as $\mathcal{O}(u_a(t) - u_b(t))$ along Δ, whenever u_a and u_b coalesce.

Theorem (G.C., B. Dubrovin, D. Guzzetti, 2016 - Part II)

Consider the system

$$\frac{dY}{dz} = A(z,t)Y, \quad A(z,t) = A_0(t) + \frac{1}{z}A_1(t),$$

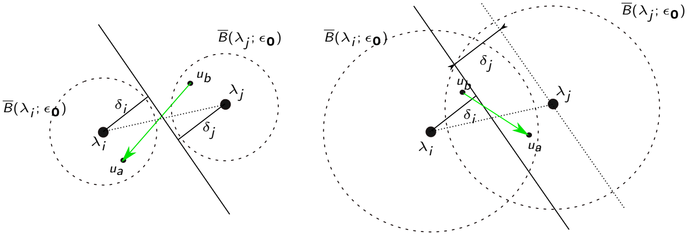
with A_0, A_1 holomorphic in a sufficiently small closed polydisc $\Omega \subseteq \mathbb{C}^n$. Let A_0 admit coalescence of eigenvalues along a locus $\Delta \subseteq \Omega$. Suppose that

1. *A_0 is holomorphically similar to its diagonal Jordan form;*
2. *the matrix entries of A_1 satisfy the vanishing condition*
$$(A_1(t))_{ab} = \mathcal{O}(u_a(t) - u_b(t)) \quad \text{if } u_a(t), \text{ and } u_b(t) \text{ coalesce as } t \to \bar{t} \in \Delta;$$
3. *the matrix A_1 is non resonant along Δ, i.e. for any a, b and $t \in \Delta$*
$$(A_1(t))_{aa} - (A_1(t))_{bb} \notin \mathbb{Z} \setminus \{0\}$$
4. *let the dependence on $t \in \Omega$ be isomonodromic on a sufficiently small simply connected open subset $\mathcal{B} \subseteq \Omega$ where A_0 has distinct eigenvalues, and where the results of Jimbo, Miwa and Ueno apply.*

Then

- *formal solutions $Y_F(z,t)$ holomorphically depend on $t \in \Omega$;*
- *the genuine solutions $Y(z,t)$, with $t \in B$, determined by the condition*
$$Y(z,t) \sim Y_F(z,t) \text{ as } z \to \infty \text{ in suitable sectors,}$$
can be holomorphically continued for all $t \in \Omega$. The asymptotic expansion still holds in suitable sectors, uniformly w.r.t. $t \in \Omega' \subseteq \Omega$, where Ω' is a slight smaller polydisc.

Consequently, the Stokes matrices S_1, S_2 (describing the Stokes phenomenon of the solutions near $z = \infty$) and central connection matrix C are well-defined and constant in the whole polydisc Ω'.



For a Frobenius manifold M we have that:

1. the matrix Ψ and its inverse Ψ^{-1} are holomorphic at any semisimple point (even coalescing ones);
2. from the compatibility conditions of the system
$$\frac{\partial Y}{\partial z} = A(z,u)Y, \quad A(z,u) := U + \frac{1}{z}V(u),$$
$$\frac{\partial Y}{\partial u_k} = M_k(z,u)Y, \quad M_k(z,u) := zE_k + V_k(u), \quad (E_k)_{ab} := \delta_{ak}\delta_{bk}, \quad V_k(u) := \frac{\partial \Psi}{\partial u_k}\Psi^{-1},$$
we deduce that $[U, V_k] = [E_k, V]$, so that $V_{ij}(u) \to 0$ if u_i and u_j coalesce;
3. from the η-skew-symmetry of μ we deduce that V is antisymmetric, and so non-resonant;
4. outside the coalescence locus, we already know the validity of Isomonodromy Theorems.

Corollary

In the case related to Frobenius manifolds, the assumptions (1)-(2)-(3)-(4) of the previous Theorem are satisfied in any simply connected open neighborhood of semisimple points, even in presence of coalescence. The monodromy data are thus well defined and locally constant near any semisimple point.

Constraints on the exceptional collections arising at a semisimple coalescent point: If the eigenvalues u_i's coalesce, at some semisimple point t_0, to $s \leq n$ values $\lambda_1, \dots, \lambda_s$ with multiplicities $p_1, \dots, p_s$ (with $p_1 + \dots + p_s = n$), then the corresponding monodromy data can be expressed in terms of Gram matrices and characteristic classes of objects of a *full s-block exceptional collection*, i.e. a collection of the type

$$\mathcal{E} := (\underbrace{E_1, \dots, E_{p_1}}_{\mathcal{B}_1}, \underbrace{E_{p_1+1}, \dots, E_{p_1+p_2}}_{\mathcal{B}_2}, \dots, \underbrace{E_{p_1+\dots+p_{s-1}+1}, \dots, E_{p_1+\dots+p_s}}_{\mathcal{B}_s}), \quad E_j \in \mathrm{Obj}\left(\mathcal{D}^b(X)\right),$$

where for each pair (E_i, E_j) in the same block $\mathcal{B}_k$ the orthogonality conditions hold

$$\mathrm{Ext}^{\ell}(E_i, E_j) = 0, \quad \text{for any } \ell.$$

In particular, any reordering of the objects inside a single block $\mathcal{B}_j$ preserves the exceptionality of $\mathcal{E}$.

Consider the complex manifold $X = \mathbb{G}(2,4)$:

$$\dim_{\mathbb{C}} X = 4, \quad \beta_0(X) = \beta_2(X) = \beta_6(X) = \beta_8(X) = 1, \quad \beta_4(X) = 2,$$

$$\text{Schubert basis} \quad \left(\sigma_0, \sigma_{\square}, \sigma_{\square\square}, \sigma_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, \sigma_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}, \sigma_{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}}\right).$$

The Frobenius manifold $QH^\bullet(X)$ has dimension 6: its structure is explicitly known only along the small quantum 1-dimensional locus $H^2(X,\mathbb{C})$.
If x_1, x_2 represent the Chern roots of the dual-tautological bundle $\mathcal{S}^\vee$, then

$$p = t^2\sigma_{\square} \in H^2(X,\mathbb{C}), \quad q := \exp(t^2), \quad QH^\bullet_p(X) \cong \frac{\mathbb{C}[x_1,x_2]^{\mathfrak{S}_2}[q]}{\langle h_3, h_4+q\rangle}, \quad \sigma_\lambda = \frac{\begin{vmatrix} x_1^{\lambda_1+1} & x_1^{\lambda_2} \\ x_2^{\lambda_1+1} & x_2^{\lambda_2}\end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1\end{vmatrix}}.$$

Since $c_1(X) = 4\sigma_{\square}$, we deduce that at the point $p = t^2\sigma_{\square}$ we have

$$u_1(p) = u_2(p) = 0, \quad u_3(p) = -4i\sqrt{2}q^{\frac{1}{4}}, \quad u_4(p) = 4i\sqrt{2}q^{\frac{1}{4}},$$

$$u_5(p) = -4\sqrt{2}q^{\frac{1}{4}}, \quad u_6(p) = 4\sqrt{2}q^{\frac{1}{4}}.$$

The small quantum cohomology is completely contained in the coalescence locus: the computation of the monodromy data is justified by our previous Theorems.

Consider the complex manifold $X = \mathbb{G}(2,4)$:

$$\dim_{\mathbb{C}} X = 4, \quad \beta_0(X) = \beta_2(X) = \beta_6(X) = \beta_8(X) = 1, \quad \beta_4(X) = 2,$$

$$\text{Schubert basis} \quad \left(\sigma_0, \sigma_{\square}, \sigma_{\square\square}, \sigma_{\substack{\square\\ \square}}, \sigma_{\substack{\square\square\\ \square}}, \sigma_{\substack{\square\square\\ \square\square}}\right).$$

The Frobenius manifold $QH^\bullet(X)$ has dimension 6: its structure is explicitly known only along the small quantum 1-dimensional locus $H^2(X, \mathbb{C})$.
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The differential system defining *deformed flat 1-form* $\widehat{\nabla}\xi = 0$, with $\xi := \xi_j(z,t)dt^j$ is

$$\partial_z\xi_1 = 4\xi_2 + \frac{2}{z}\xi_1, \qquad \partial_z\xi_4 = 4\xi_5,$$

$$\partial_z\xi_2 = 4(\xi_3 + \xi_4) + \frac{1}{z}\xi_2, \qquad \partial_z\xi_5 = 4q\xi_1 + 4\xi_6 - \frac{1}{z}\xi_5,$$

$$\partial_z\xi_3 = 4\xi_5, \qquad \partial_z\xi_6 = 4q\xi_2 - \frac{2}{z}\xi_6.$$

The whole system can be reduced to the study of the quantum differential equation

$$\vartheta^5\Phi(w) - 1024w^4\vartheta\Phi(w) - 2048w^4\Phi(w) = 0, \quad \vartheta := w\frac{d}{dw},$$

and the solution can be reconstructed through the formulae

$$\xi_1 = z^2\Phi\left(zq^{\frac{1}{4}}\right), \quad \xi_2 = \frac{1}{4}z^2\partial_z\left[\Phi\left(zq^{\frac{1}{4}}\right)\right], \quad \xi_3 = \frac{1}{32}\left(z\partial_z\left[\Phi\left(zq^{\frac{1}{4}}\right)\right] + z^2\partial_z^2\left[\Phi\left(zq^{\frac{1}{4}}\right)\right]\right)+h,$$

$$\xi_4 = \frac{1}{32}\left(z\partial_z\left[\Phi\left(zq^{\frac{1}{4}}\right)\right] + z^2\partial_z^2\left[\Phi\left(zq^{\frac{1}{4}}\right)\right]\right) - h,$$

$$\xi_5 = \frac{1}{128}\left(\partial_z\left[\Phi\left(zq^{\frac{1}{4}}\right)\right] + 3z\partial_z^2\left[\Phi\left(zq^{\frac{1}{4}}\right)\right] + z^2\partial_z^3\left[\Phi\left(zq^{\frac{1}{4}}\right)\right]\right),$$

$$\begin{aligned}\xi_6 = \frac{1}{512}\Bigg(&-512qz^2\Phi\left(zq^{\frac{1}{4}}\right) + \frac{1}{z}\partial_z\left[\Phi\left(zq^{\frac{1}{4}}\right)\right]\\ &+ 7\partial_z^2\left[\Phi\left(zq^{\frac{1}{4}}\right)\right] + 6z\partial_z^3\left[\Phi\left(zq^{\frac{1}{4}}\right)\right] + z^2\partial_z^4\left[\Phi\left(zq^{\frac{1}{4}}\right)\right]\Bigg),\end{aligned}$$

where $h \in \mathbb{C}$. By taking the Mellin transform, we found two solutions

$$\Phi_1(w) := \frac{1}{2\pi i}\int_{\Lambda_1}\frac{\Gamma(s)^5}{\Gamma(s+\frac{1}{2})}4^{-s}w^{-4s}ds, \quad \Phi_2(w) = \frac{1}{2\pi i}\int_{\Lambda_1}\Gamma(s)^5\Gamma\left(\frac{1}{2}-s\right)e^{i\pi s}4^{-s}w^{-4s}ds$$

and reconstruct both $\Xi_{\text{left/right}}$, w.r.t. a line ℓ of slope $0 < \phi < \frac{\pi}{6}$.

We consider the topological solution defined near $z=0$ by the expansion

$$\begin{array}{c} \mathcal{L}_1 \\ \Big\downarrow{\scriptstyle T^*_{x_1}C} \\ \overline{\mathcal{M}}_{0,k+2}(X,\beta) \end{array} \qquad \begin{aligned} \Xi_0(z,t) &:= \eta\cdot\Theta_{top}(z,t)\cdot z^{\mu}z^{c_1(X)\wedge}, \\ \Theta_{top}(z,t)^{\gamma}_{\lambda} &:= \delta^{\gamma}_{\lambda} + \sum_{k,n=0}^{\infty}\sum_{\beta\in\mathrm{Eff}(X)}\sum_{\alpha_1,\dots,\alpha_k}\frac{h^{\gamma}_{\lambda,k,n,\beta,\underline{\alpha}}}{k!}\cdot t^{\alpha_1}\dots t^{\alpha_k}\cdot z^{n+1}, \end{aligned}$$

$$h^{\gamma}_{\lambda,k,n,\beta,\underline{\alpha}} := \sum_{\nu}\eta^{\nu\gamma}\int_{[\overline{\mathcal{M}}_{0,k+2}(X,\beta)]^{\mathrm{virt}}} c_1(\mathcal{L}_1)^n\wedge \mathrm{ev}_1^*\sigma_{\lambda}\wedge\mathrm{ev}_2^*\sigma_{\nu}\wedge\bigwedge_{j=1}^{k}\mathrm{ev}^*_{j+2}\sigma_{\alpha_j}.$$

The Stokes matrix is given by

$$\Xi_{\mathrm{left}} = \Xi_{\mathrm{rigth}}S, \quad S = \begin{pmatrix} 1 & 6 & -20 & 20 & -70 & 20 \\ 0 & 1 & -4 & 4 & -16 & 6 \\ 0 & 0 & 1 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Analogously, the central connection matrix can be explicitly computed.

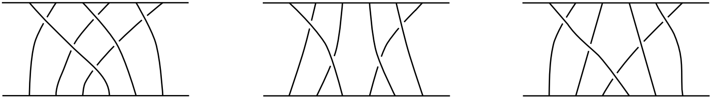
The central connection matrix

$$\Xi_{right} = \Xi_0 C$$

is the matrix associated to the morphism

$$Д_X^-\colon K_0(X)\otimes_{\mathbb{Z}}\mathbb{C}\to H^\bullet(X,\mathbb{C})\colon E\mapsto \frac{1}{(2\pi)^2}\widehat{\Gamma}_X^-\wedge e^{-c_1(X)\pi i}\wedge \mathrm{Ch}(E),$$

w.r.t. the Schubert basis and the exceptional basis $([E_1],\dots,[E_6])$ obtained from the Kapranov exceptional collection, twisted by $\wedge^2\mathcal{S}^\vee$, by mutation along the braids



$\beta_{34}\beta_{12}\beta_{56}\beta_{23}\beta_{45}\beta_{34}$ $\beta_{12}\beta_{56}\beta_{23}\beta_{45}$ $\beta_{12}\beta_{56}\beta_{23}\beta_{45}\beta_{34}$

Moreover, we have that

$$(S^{-1})_{ij} = \sum_h (-1)^h \dim_{\mathbb{C}} \mathrm{Hom}^h(E_i,E_j).$$

Theorem (G.C., B. Dubrovin, D. Guzzetti, to appear)

The monodromy data for $\mathbb{G}(r,k)$ are the ones related to an explicit mutation of the twisted Kapranov collection

$$\left(\mathbb{S}^\lambda\mathcal{S}^\vee\otimes\mathcal{L}\right)_\lambda,\quad \mathcal{L}:=\det\left(\bigwedge^2\mathcal{S}^\vee\right).$$

Future directions:

- generalization to higher Poincaré ranks;
- applications to the study of (holomorphic branches of) Painlevé transcendents;
- up to which extent is it possible to extend the isomonodromic theory of Frobenius manifolds at points of the caustic?
- explicit computations of the monodromy data for $IG(n, 2n)$ (work in progress, joint with M. Smirnov);
- beyond the Fano assumption: the case of Hirzebruch surfaces (work in progress);
- study of the freedom in choice of the calibration at $z = 0$ in the Dubrovin-Zhang Theory of Normal Forms (work in progress, actually still at the beginning, joint with D. Yang);
- many other directions...

Thank You!

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