Cylindric RPP and 2D TQFT

David Palazzo

Joint work with Christian Korff

C. Korff, D. Palazzo. *Cylindric Reverse Plane Partitions and 2D TQFT*, proceedings article for FPSAC2018.

Frobenius Structures and Relations

University of Glasgow, 23rd of March 2017.

Structure of the talk

- Symmetric functions and reverse plane partitions (RPP)
- Affine symmetric group and cylindric RPP
- Frobenius algebras (2D TQFT) and generalised symmetric group

Symmetric functions and RPP

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Coproduct $\Delta : \Lambda \to \Lambda \otimes \Lambda$ such that $\langle \Delta f, g \otimes h\rangle = \langle f, gh\rangle$.

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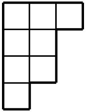
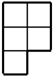
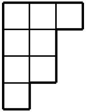
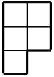
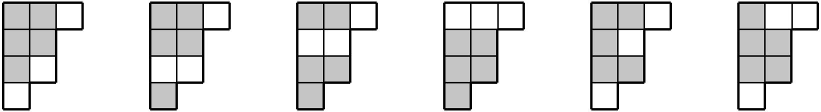
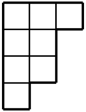
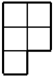
We will give a combinatorial description of $h_{\lambda/\mu}$ in terms of RPP.

Example. Let $\lambda = (3, 2, 2, 1) =$  , $\mu = (2, 2, 1) =$  .

Example. Let $\lambda = (3,2,2,1) =$  , $\mu = (2,2,1) =$  .

All the permutations α of μ ($\alpha \sim \mu$) such that $\alpha \subset \lambda$ are (in grey):



Example. Let $\lambda = (3,2,2,1) =$  , $\mu = (2,2,1) =$  .

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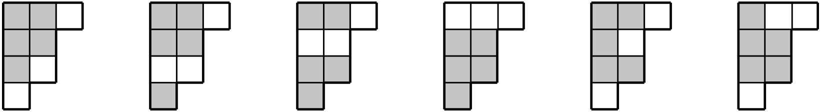

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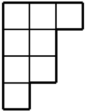
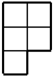
Example. Let $\lambda = (3,2,2,1) =$  , $\mu = (2,2,1) =$  .

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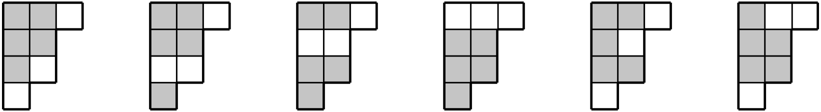

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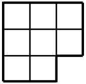
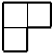
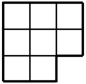
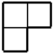
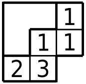
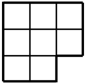
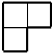
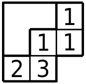
Here $\chi_{\lambda/\mu} = 6$.

Example. Let $\lambda =$  , $\mu =$  .

Example. Let $\lambda =$  , $\mu =$  .

RPP of shape λ/μ: $\pi =$ 

Example. Let $\lambda =$  , $\mu =$  .

RPP of shape λ/μ: $\pi =$ 

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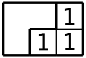

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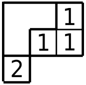

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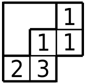
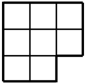
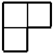
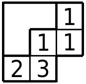
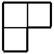
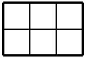
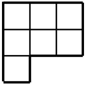
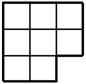
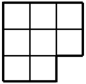
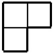
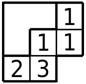


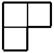
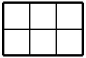
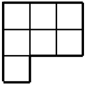
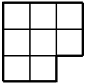
Example. Let $\lambda =$  , $\mu =$  .

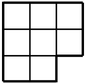
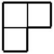
RPP of shape λ/μ: $\pi =$ 

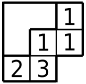
 ,  ,  , 

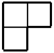
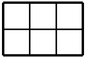
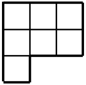
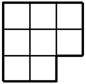
Example. Let $\lambda =$  , $\mu =$  .

RPP of shape λ/μ: $\pi =$ 

$\lambda^{(0)} =$  , $\lambda^{(1)} =$  , $\lambda^{(2)} =$  , $\lambda^{(3)} =$ 

Example. Let $\lambda =$  , $\mu =$  .

RPP of shape λ/μ: $\pi =$ 

$\lambda^{(0)} =$  , $\lambda^{(1)} =$  , $\lambda^{(2)} =$  , $\lambda^{(3)} =$ 

$$x^{\pi} = x_1^{\text{multiplicity of 1 in } \pi} \, x_2^{\text{multiplicity of 2 in } \pi} \cdots = x_1^3 \, x_2 \, x_3$$

Example. Let $\lambda = \ydiagram{3,3,2}$, $\mu = \ydiagram{2,1}$.

RPP of shape λ/μ: $\pi = \begin{ytableau} \ & \ & 1 \\ \ & 1 & 1 \\ 2 & 3 \end{ytableau}$

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Lemma. $\boxed{h_{\lambda/\mu} = \sum_\pi \chi_\pi x^\pi}$, $\qquad \chi_\pi = \chi_{\lambda^{(1)}/\lambda^{(0)}} \cdot \chi_{\lambda^{(2)}/\lambda^{(1)}} \cdots$

The sum is over all RPP of shape λ/μ.

Denote with S_k the symmetric group in k letters, with generators $\langle \sigma_1, \ldots, \sigma_{k-1} \rangle$ and relations

$$\sigma_i^2 = 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1$$

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Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_k$. Right action of S_k on $\mathcal{P}_k$:

$$(\lambda_1, \ldots, \lambda_k).\sigma_i = (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_k)$$

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$S_\lambda = \{w \in S_k \mid \lambda.w = \lambda\}$: stabilizer subgroup of λ.

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$$(\lambda_1, \ldots, \lambda_k).\sigma_i = (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_k)$$

$S_\lambda = \{w \in S_k \mid \lambda.w = \lambda\}$: stabilizer subgroup of λ.

S^λ: coset representatives in $S_\lambda \setminus S_k$

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How can we generalise this to the affine symmetric group?

Affine Symmetric Group and Cylindric RPP

The affine symmetric group $\tilde{S}_k$ is generated by $\langle \sigma_0, \sigma_1, \ldots, \sigma_{k-1} \rangle$ where

$$\sigma_i^2 = 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1$$

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$$\tau \sigma_{i+1} = \sigma_i \tau$$

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_k$. The level n action of $\hat{S}_k$ on $\mathcal{P}_k$ is

$$\begin{aligned} \lambda.\sigma_i &= (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_k) \\ \lambda.\sigma_0 &= (\lambda_k + n, \lambda_2, \ldots, \lambda_{k-1}, \lambda_1 - n) \\ \lambda.\tau &= (\lambda_k + n, \lambda_1, \ldots, \lambda_{k-1}) \end{aligned}$$

For each $\lambda \in \mathcal{P}_k$ we construct a doubly infinite sequence

$$\hat{\lambda} = (\dots, \hat{\lambda}_{-1}, \hat{\lambda}_0, \hat{\lambda}_1, \dots)$$

such that $\hat{\lambda}_i = \lambda_i$ for $i = 1, \dots, k$ and $\hat{\lambda}_{i+k} = \hat{\lambda}_i - n$.

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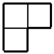
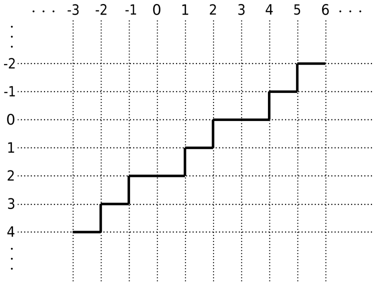
Example. $\lambda =$  , $\hat{\lambda} =$ 

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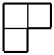
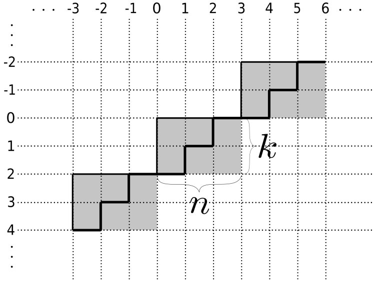
Example. $\lambda =$  , $\hat{\lambda} =$ 

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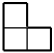
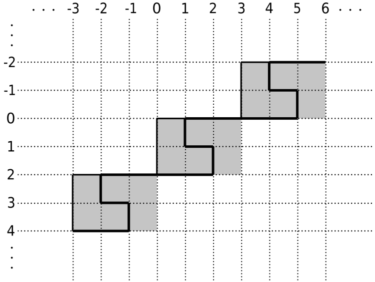
Example. $\lambda.\sigma_1 =$  , $\hat{\lambda}.\sigma_1 =$ 

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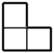
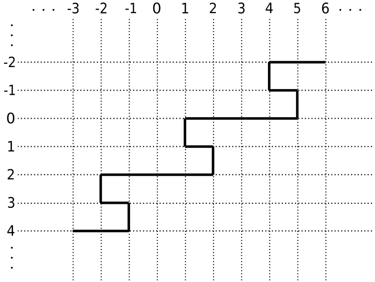
Example. $\lambda.\sigma_1 =$  , $\hat{\lambda}.\sigma_1 =$



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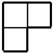
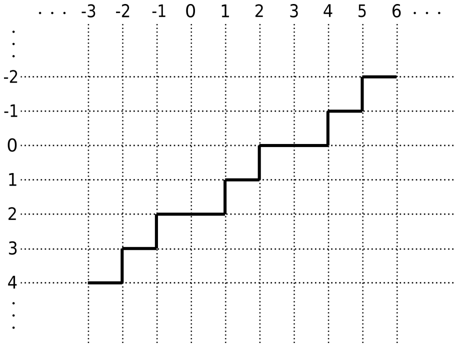
Example. $k = 2$, $n = 3$, $\lambda =$ .

$\hat{\lambda} =$ 

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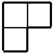
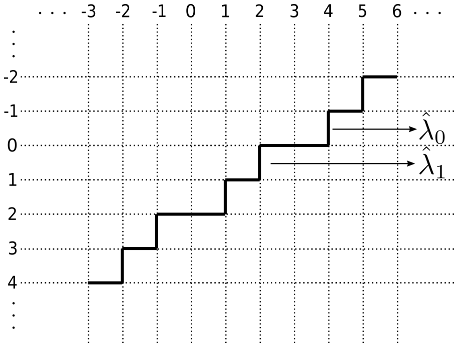
Example. $k = 2$, $n = 3$, $\lambda =$ .

$\hat{\lambda} \quad = $ 

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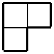
Example. $k = 2$, $n = 3$, $\lambda =$ .

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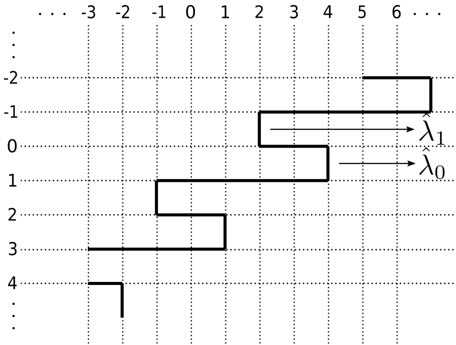

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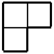
Example. $k = 2$, $n = 3$, $\lambda =$  .

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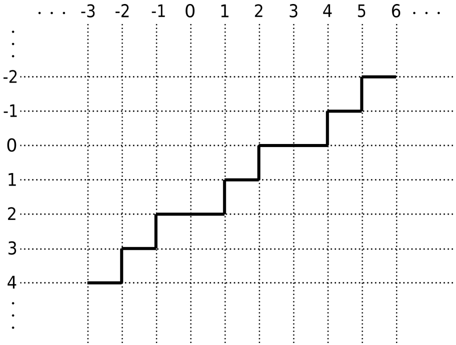

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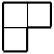
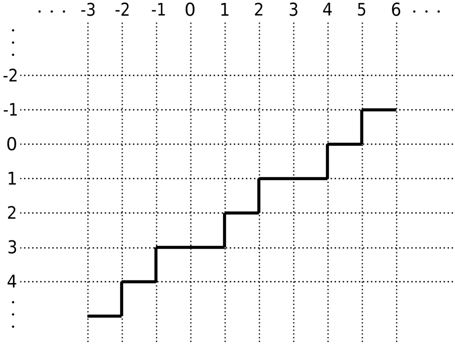
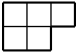
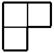
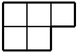
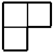
Example. $k = 2$, $n = 3$, $\lambda =$ .

$\hat{\lambda}.\tau \quad =$ 

Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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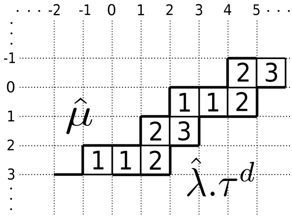
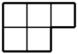
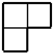
$\hat{\pi} =$ 

Example. $k = 2,\ n = 3,\ \lambda =$  , $\mu =$  , $d = 1$.

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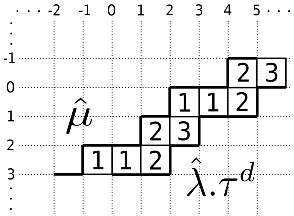
$\hat{\pi} =$ 

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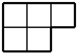
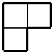
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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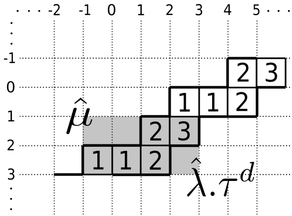
$\hat{\pi} =$ 

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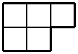
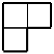
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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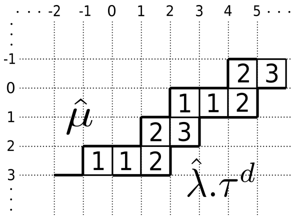
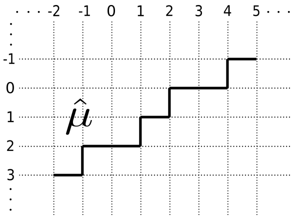
$\hat{\pi} =$ 

$\hat{\lambda}^{(0)} =$ 

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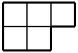
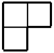
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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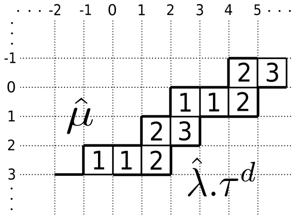
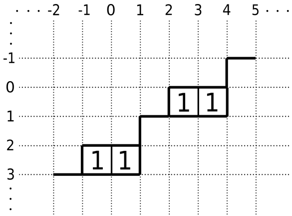
$\hat{\pi} =$ 



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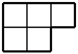
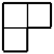
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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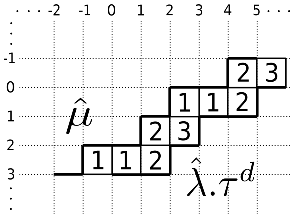
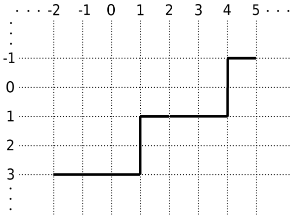
$\hat{\pi} =$ 

$\hat{\lambda}^{(1)} =$ 

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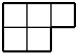
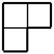
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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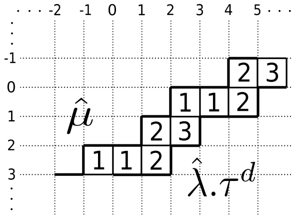
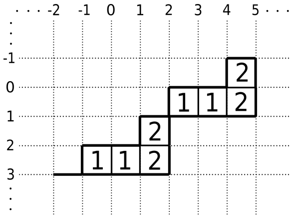
$\hat{\pi} =$ 



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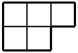
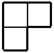
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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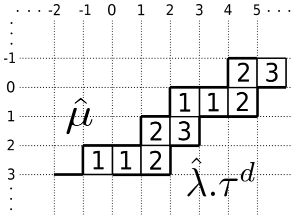
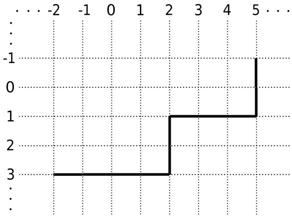
$\hat{\pi} =$ 

$\hat{\lambda}^{(2)} =$ 

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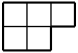
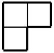
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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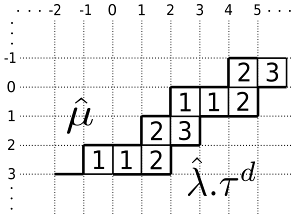
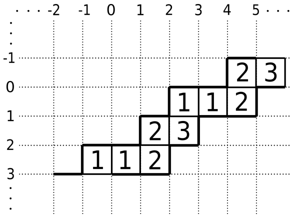
$\hat{\pi} =$ 



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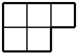
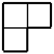
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  , $d = 1$.

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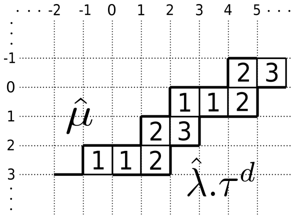
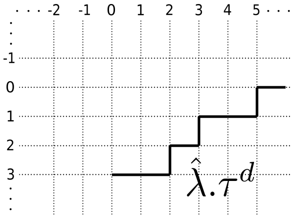
$\hat{\pi} =$ 

$\hat{\lambda}^{(3)} =$ 

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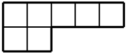
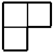
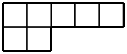
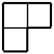
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  .

Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  .

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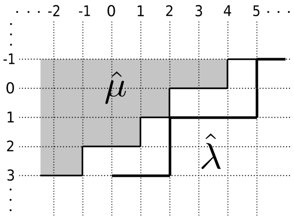
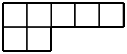
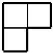


Example. $k = 2,\ n = 3,\ \lambda =$  $,\ \mu =$  .

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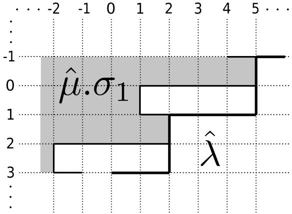
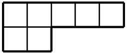
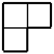


Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  .

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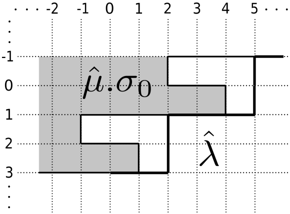
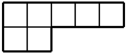
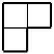


Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  .

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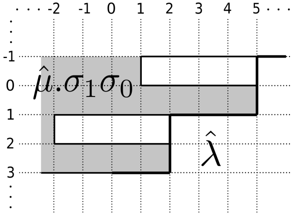
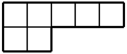
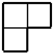


Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  .

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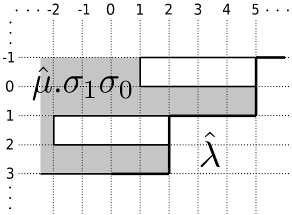

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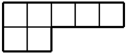
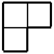
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  .

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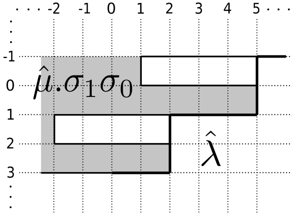

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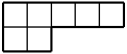
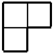
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  .

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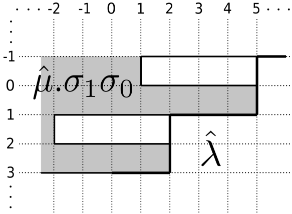

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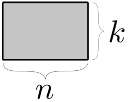
Remark. If $\lambda \in$ 

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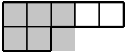
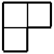
Example. $k = 2$, $n = 3$, $\lambda =$  , $\mu =$  .

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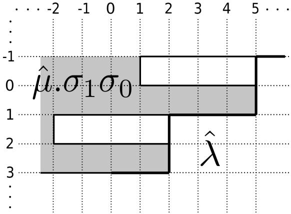

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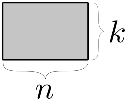
Remark. If $\lambda \in$ 

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Remark. If $d = 0$ then $h_{\lambda/0/\mu} = h_{\lambda/\mu}$ and $N^{\lambda}_{\mu\nu} = f^{\lambda}_{\mu\nu}$.

Relation with Frobenius algebras (2D TQFT)

Fix k and n. Consider the quotient

$$\boxed{\mathcal{V}_k(n) = \underbrace{\mathbb{C}[x_1, \ldots, x_k]^{S_k}}_{\Lambda_k} \backslash \langle x_i^n = 1 \rangle}$$

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Theorem [Korff, DP, '17]. $\{m_\lambda\}_{\lambda \in \mathcal{A}_k(n)}$ *is a basis of* $\mathcal{V}_k(n)$ *and*

$$m_\mu m_\nu = \sum_{\lambda \in \mathcal{A}_k(n)} N_{\mu\nu}^\lambda m_\lambda$$

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$\mathcal{V}_k(n)$ *is a Frobenius algebra (i.e. a 2D TQFT) with bilinear form*

$$\langle m_\mu, m_\nu \rangle = \frac{\delta_{\lambda\mu^*}}{|S_\lambda|}, \qquad \mu^* = (n-\mu_k, \ldots, n-\mu_1)$$

Theorem [Korff, DP, '17]. *Let* $\zeta^n = 1$ *and* $\zeta^\lambda = (\zeta^{\lambda_1}, \ldots, \zeta^{\lambda_k})$.

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- *Modular group relations* :

$$\boxed{(ST)^3 = S^2 = C}$$

$$T_{\lambda\mu} = \delta_{\lambda\mu} \zeta^{\frac{-kn(n-1)}{24} + \frac{1}{2} \sum_{i=1}^{k} \lambda_i (n - \lambda_i)}$$

$$C_{\lambda\mu} = \delta_{\lambda\mu^*}$$

Theorem [Korff, DP, '17]. *Let* $\zeta^n = 1$ *and* $\zeta^\lambda = (\zeta^{\lambda_1}, \ldots, \zeta^{\lambda_k})$.

- *Verlinde formula* :

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Remark. For $k = 1$, $\mathcal{V}_1(n)$ is the $\hat{\mathfrak{sl}}_n$-Verlinde algebra (Grothendieck ring of a modular tensor category) at level 1.

Relation with the generalised symmetric group

We give a representation theoretic interpretation of $N^{\lambda}_{\mu\nu}$ in terms of the generalised symmetric group

$$S(n,k) = \mathbb{Z}_n^{\times k} \rtimes S_k$$

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The simple modules $\mathcal{L}_{\underline{\lambda}}$ of $S(n,k)$ are labelled in terms of multipartitions $\underline{\lambda} = \left(\lambda^{(1)}, \ldots, \lambda^{(n)}\right)$, $\lambda^{(i)}$ partitions [Osima, '54].

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Representation ring Rep$S(n,k)$: $\mathcal{L}_{\underline{\lambda}} \otimes \mathcal{L}_{\underline{\mu}} = \bigoplus_{\underline{\nu}} c^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}} \mathcal{L}_{\underline{\nu}}$.

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Proposition [Korff, DP, '17]. *The structure constants $N^{\lambda}_{\mu\nu}$ of $\mathcal{V}_k(n)$ have the alternative expression*

$$N^{\lambda}_{\mu\nu} = \sum_{\underline{\lambda}} c^{\underline{\lambda}}_{\underline{\mu}\underline{\nu}} \frac{f_{\underline{\lambda}}}{f_{\underline{\mu}} f_{\underline{\nu}}}, \qquad f_{\underline{\lambda}} = \prod_{i=1}^{n} f_{\lambda^{(i)}}$$

- $f_{\lambda^{(i)}}$: number of standard tableaux of shape $\lambda^{(i)}$;
- The sum runs over $\underline{\lambda}$ such that $|\lambda^{(i)}| = n_i(\lambda)$, where $|\lambda^{(i)}|$ is the number of boxes in $\lambda^{(i)}$;
- Similarly for $\underline{\mu}$ and $\underline{\nu}$.

Example. Let $n = 3$, $k = 2$. For all the multipartitions below we have $f_{\underline{\mu}} = f_{\underline{\nu}} = f_{\underline{\lambda}} = 1$. Let

$$\underline{\mu} = \left(\emptyset\,, \square, \square\right) \quad \to \quad \mu = (3, 2)$$
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We have

$$\mathcal{L}_{\underline{\mu}} \otimes \mathcal{L}_{\underline{\nu}} = \mathcal{L}_{\underbrace{(\emptyset\, , \emptyset\, , \square\square)}_{(3,3)}} \oplus \mathcal{L}_{\underbrace{(\emptyset\, , \emptyset\, , \begin{smallmatrix}\square\\ \square\end{smallmatrix})}_{(3,3)}} \oplus \mathcal{L}_{\underbrace{(\square\, , \square\, , \emptyset)}_{(2,1)}}$$

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Thus, the only non-zero $N_{\mu\nu}^{\lambda}$ are

$$N_{\mu\nu}^{(3,3)} = 2, \qquad N_{\mu\nu}^{(2,1)} = 1$$

Conclusions and outlook

We defined cylindric complete symmetric functions $h_{\lambda/d/\mu}$ by means of cylindric RPP and the affine symmetric group.

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Plan for the future:

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Plan for the future:

- Have a geometrical interpretation of $N^\nu_{\lambda\mu}$.

  Cylindric Schur functions: $s_{\lambda/d/\mu} = \sum_\nu C^\lambda_{\mu\nu} s_\nu$ [Postnikov, 05']. Here $C^\lambda_{\mu\nu}$ are 3-point, genus 0 Gromov-Witten invariants (counting of curves intersecting 3 Schubert varieties).

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- Generalise to other (affine) Coxeter groups?

Thank you for your attention!