Calogero-Moser spaces and KP hierarchy for the cyclic quiver

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Classical and Quantum Integrability

Glasgow

11–12 March 2014

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- [Airault, McKean, Moser (1976)]: The Korteweg–de Vries (KdV) equation has rational solutions with poles moving as particles of a Calogero–Moser (CM) system of type A;
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- [Olshanetsky, Perelomov (1976)]: CM systems for all root systems (in rational case: for the reflection groups, that is for finite Coxeter groups); classical series: types A and B.
- **Our result**: There is a generalization of the KP hierarchy that admits rational solutions whose pole dynamics is governed by the CM system for the generalized symmetric group $G = (\mathbb{Z}/m\mathbb{Z}) \wr S_n = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$.
  $m = 1$: type A, $m = 2$: type B,
  $m \geqslant 3$: complex reflection group case

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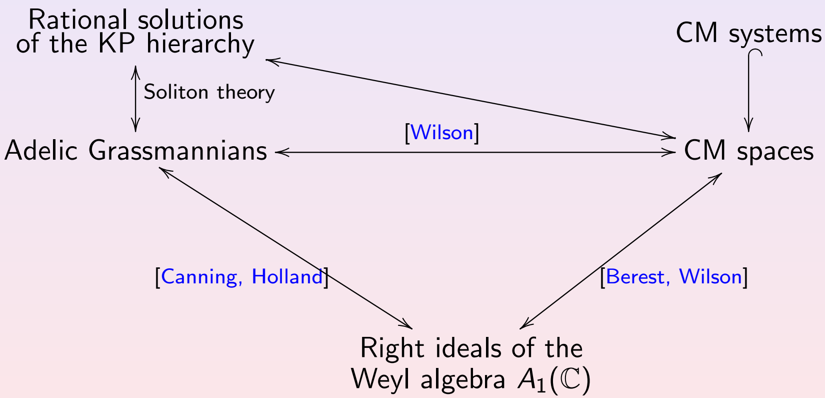
Plan

1. A case
   - KP hierarchy
   - Wilson's solutions
   - Calogero–Moser system
2. Spherical case
   - Generalized KP hierarchy
   - Quiver solutions
   - Calogero–Moser systems for G
3. Non-spherical case
   - Non-spherical framing
   - Non-equivariant quiver solutions
   - Spin CM systems (for G)

Scheme of CM correspondence



KP equation

- *Kadomtsev–Petviashvili* (KP) *equation*:

$$3\partial_y^2 u = \partial_x\left(4\partial_t u + 6u\partial_x u - \partial_x^3 u\right), \qquad u = u(x, y, t).$$

- The rational solutions [Chudnovsky, Chudnovsky], [Krichever]:

$$u = \sum_{i=1}^{n} \frac{2}{(x - x_i)^2},$$

where $x_i(t, y)$ are coordinate solutions of the n-particle classical *Calogero–Moser* (CM) *system*:

$$\partial_y p_i = \{H_2, p_i\}, \qquad \partial_t p_i = \{H_3, p_i\},$$
$$\partial_y x_i = \{H_2, x_i\}, \qquad \partial_t x_i = \{H_3, x_i\}$$

where $H_2 = \sum_{i=1}^{n} p_i^2 - 2\sum_{i<j} \frac{1}{(x_i - x_j)^2}$, $H_3 = \sum_{i=1}^{n} p_i^3 + \ldots$, $\{H_2, H_3\} = 0$.

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KP hierarchy

- Consider the algebra $\mathcal{A}$ generated by $f(x)$, $\partial = \partial_x$, ∂^{-1}:

$$\partial^N f(x) = \sum_{j=0}^{\infty} \binom{N}{j} f^{(j)}(x)\partial^{N-j}, \quad N \in \mathbb{Z};$$

$$\mathcal{A} = \left\{ F = \sum_{j=-\infty}^{N} f_j(x)\partial^j \mid N \in \mathbb{Z} \right\}; \qquad \text{denote } F_+ = \sum_{j\geqslant 0}^{N} f_j(x)\partial^j.$$

- *KP hierarchy*:

$$\frac{\partial}{\partial t_k} L = [(L^k)_+, L], \qquad L = \partial + \sum_{j=1}^{\infty} u_j \partial^{-j},$$

where $u_j = u_j(x, t_2, t_3, t_4, \ldots)$.

- KP equation: $u = -2u_1$, $t_2 = y$, $t_3 = t$.

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Calogero–Moser space

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$$\mathcal{C}_n = \{(X,Y) \in \mathrm{Mat}_{n\times n}(\mathbb{C})^2 \mid \mathrm{rank}\left([X,Y]-1\right) = 1\}/GL_n$$
$$= \{(X,Y,v,w) \mid [X,Y] = 1 - vw\}/GL_n,$$

where $X, Y \in \mathrm{Mat}_{n\times n}(\mathbb{C})$, $v \in \mathbb{C}^n$, $w \in (\mathbb{C}^n)^*$.

- Dynamics on $\mathcal{C}_n$: $\quad X(t) = X - \sum_{k=2}^{\infty} k t_k Y^{k-1}, \quad Y(t) = Y,$

$v(t) = v, \quad w(t) = w, \qquad$ where $t = (t_2, t_3, t_4, \ldots)$.

- It gives a solution of KP hierarchy [Wilson]:

$$L = M\partial M^{-1}, \qquad M = 1 - w(X(t)-x)^{-1}(Y-\partial)^{-1}v,$$

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Calogero–Moser system

- Generic point of $\mathcal{C}_n$:

$$X_{ij} = x_i\delta_{ij}, \quad Y_{ij} = p_i\delta_{ij} - (1-\delta_{ij})(x_i - x_j)^{-1}, \quad v_i = w_i = 1.$$

- Dynamics on $\mathcal{C}_n$ in these coordinates:

$$\left(g(t)X(t)g(t)^{-1}\right)_{ij} = x_i(t)\delta_{ij},$$

$$\left(g(t)Yg(t)^{-1}\right)_{ij} = p_i(t)\delta_{ij} - (1-\delta_{ij})(x_i(t) - x_j(t))^{-1},$$

where $g(t) \in GL_n$.

- These gives the solutions of the rational classical CM system:

$$\partial_{t_k} p_i = \{H_k, p_i\}, \qquad \partial_{t_k} x_i = \{H_k, x_i\}, \qquad k = 2, 3, 4, \dots,$$

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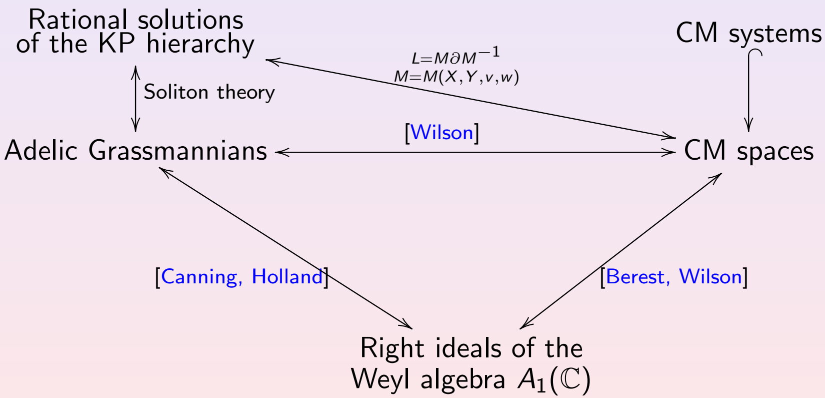

CM correspondence for the cyclic quiver

Solutions of the spherical sub-hierarchy

CM systems for $G = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$

[Etingof, Ginzburg] if $\alpha = (n, \dots, n)$

[Chalykh, A.S.]

CM spaces $\mathcal{C}_{\alpha,\lambda}$ (quiver varieties)

[Baranovsky, Ginzburg, Kuznetsov]

Right ideals of the spherical Cherednik algebra $\epsilon_0 \mathcal{H}_\lambda \epsilon_0$

Cherednik algebra for cyclic group

- Cyclic group: $\Gamma = \mathbb{Z}/m\mathbb{Z} = \{1, \sigma, \dots, \sigma^{m-1}\}$.
- This is a complex reflection group of rank 1: σ acts on $\mathbb{C}^1$ as multiplication by $\mu = e^{2\pi i/m}$.
- The *(rational) Cherednik algebra* for Γ is $\mathcal{H}_\lambda = \langle x, y, \sigma \rangle$ over

$$\sigma x \sigma^{-1} = \mu^{-1} x, \qquad \sigma y \sigma^{-1} = \mu y, \qquad xy - yx = \lambda$$

(and $\sigma^m = 1$), where $\lambda \in \mathbb{C}\Gamma$.

- The algebra $\mathcal{H}_\lambda$ parametrized by $\lambda_0, \dots, \lambda_{m-1}$, where

$$\lambda = \sum_{k=0}^{m-1} \lambda_k \epsilon_k, \qquad \epsilon_k = \frac{1}{m} \sum_{r=0}^{m-1} \mu^{-kr} \sigma^r.$$

We will suppose $\sum_{k=0}^{m-1} \lambda_k = -1$.

- PBW basis: $x^k \sigma^r y^\ell$, where $k, \ell \geqslant 0$, $r = 0, \dots, m-1$.

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Generalized KP hierarchy

- Consider the extension of the Cherednik algebra $\mathcal{H}_\lambda$ by rational functions $f(x) \in \mathbb{C}(x)$ and y^{-1}:

$$\mathcal{P} = \Big\{ F = \sum_{j=-\infty}^{N} \sum_{r=0}^{m-1} f_{r,j}(x)\sigma^r y^j \mid f_{r,j}(x) \in \mathbb{C}(x),\ N \in \mathbb{Z} \Big\}.$$

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Spherical sub-hierarchy

- By imitating $\sigma y \sigma^{-1} = \mu y$, it is natural to require the *equivariance condition*

$$\sigma L \sigma^{-1} = \mu L.$$

- The flow $\frac{\partial}{\partial t_k}$ preserve the equivariance condition if an only if $k = mp$ for some integer p.
- *Spherical sub-hierarchy*:

$$\frac{\partial}{\partial t_{mk}} L = [(L^{mk})_+, L], \qquad L = y + \sum_{j=0}^{\infty} f_j y^{-j},$$

where $f_j = \sum_{r=0}^{m-1} \tilde{f}_{r,j} x^{-j} \sigma^r, \qquad \tilde{f}_{r,j} = \tilde{f}_{r,j}(x^m; t_m, t_{2m}, t_{3m}, \ldots);$

$$\tilde{f}_{0,0} = 0.$$

Spherical sub-hierarchy

- By imitating $\sigma y \sigma^{-1} = \mu y$, it is natural to require the *equivariance condition*

$$\sigma L \sigma^{-1} = \mu L.$$

- The flow $\frac{\partial}{\partial t_k}$ preserve the equivariance condition if an only if $k = mp$ for some integer p.
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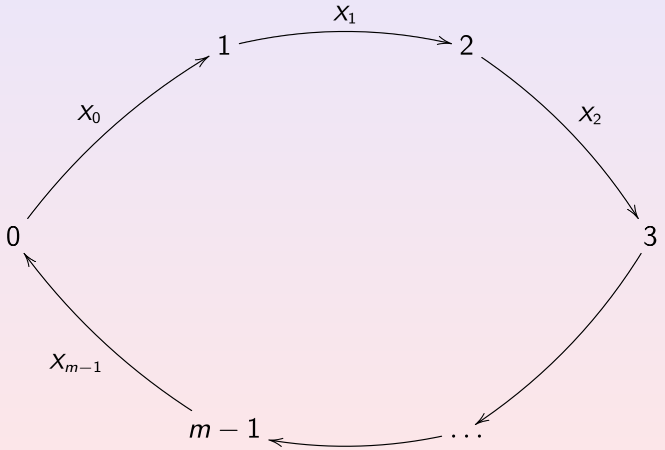
Cyclic quiver

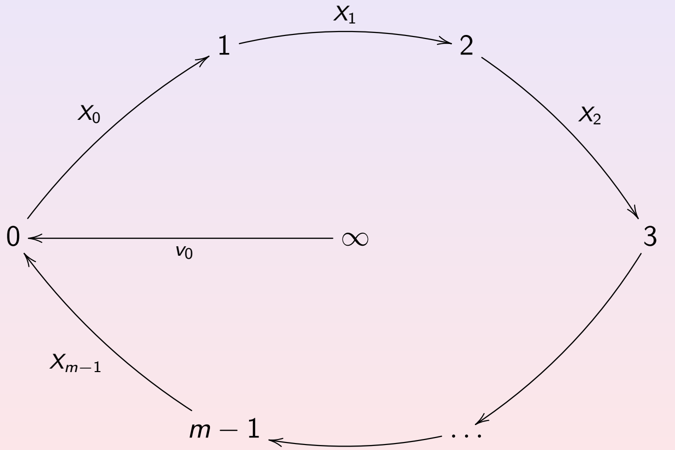
- *Cyclic quiver* Q_0:



The set of vertices: $I_0 = \{0, 1, \ldots, m-1\}$.

Quiver Q

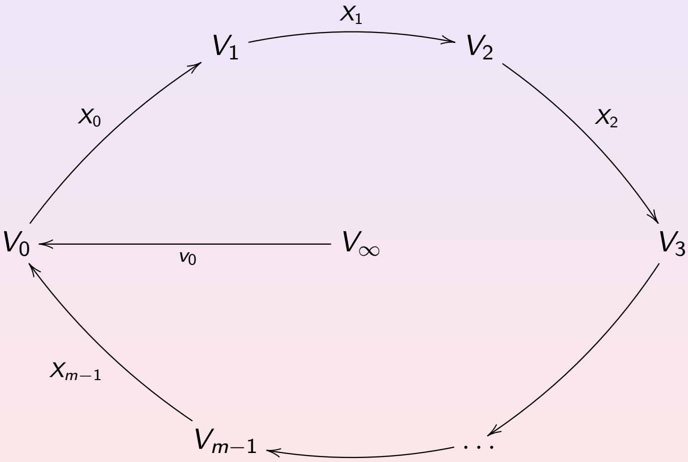
- Cyclic quiver with a (special) *framing* – quiver Q:



The set of vertices: $I = \{\infty, 0, 1, \ldots, m-1\}$.

Representation of quiver

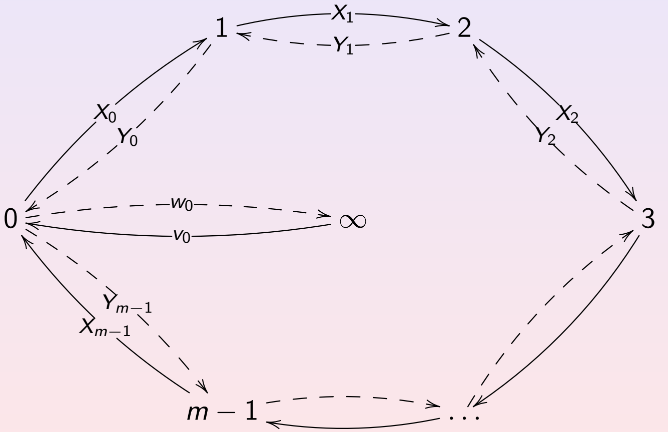
- Representation of quiver on vector spaces V_i, $i \in I$:



Space of representations: $\mathrm{Rep}(Q, \boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in \mathbb{Z}^I$, $\alpha_i = \dim V_i$.

Doubled quiver

- *Doubled quiver* $\overline{Q}$:



$$\mathrm{Rep}(\overline{Q}, \boldsymbol{\alpha}) = T^* \, \mathrm{Rep}(Q, \boldsymbol{\alpha}).$$

Preprojective algebra

- Let $\boldsymbol{\lambda} = (\lambda_\infty, \lambda_0, \ldots, \lambda_{m-1}) \in \mathbb{C}^I = \mathbb{C}^{m+1}$, where $\lambda_0, \ldots, \lambda_{m-1}$ are identified with the parameters of $\mathcal{H}_\lambda$.
  [Crawley-Boevey, Holland]: *Preprojective algebra* $\Pi^\lambda(Q)$ is the algebra of paths of $\overline{Q}$ over

$$Y_0 X_0 - X_{m-1} Y_{m-1} - v_0 w_0 = \lambda_0 e_0,$$
$$Y_r X_r - X_{r-1} Y_{r-1} = \lambda_r e_r, \qquad r = 1, \ldots, m-1,$$
$$w_0 v_0 = \lambda_\infty e_\infty,$$

  $e_k \colon k \to k$ are trivial paths (in a representation $e_k = \mathrm{id}_{V_k}$).
- $\mathrm{Rep}\,(\Pi^\lambda(Q_\infty), \boldsymbol{\alpha}) \neq \varnothing$ only if $\alpha_\infty \lambda_\infty + \sum_{k=0}^{m-1} \alpha_k \lambda_k = 0$.
- $\mathfrak{M}^\lambda_\alpha = \mathrm{Rep}\,(\Pi^\lambda(Q), \alpha)/GL(\alpha)$, where $GL(\alpha) = GL_{\alpha_\infty} \times GL_{\alpha_0} \times \cdots \times GL_{\alpha_{m-1}}$, $\mathfrak{M}^\lambda_\alpha$ is obtained by Hamiltonian reduction from $\mathrm{Rep}(\overline{Q}, \alpha) = T^* \mathrm{Rep}(Q, \alpha)$.

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$$\begin{aligned} Y_0X_0 - X_{m-1}Y_{m-1} - v_0w_0 &= \lambda_0 e_0, \\ Y_rX_r - X_{r-1}Y_{r-1} &= \lambda_r e_r, \qquad r = 1, \ldots, m-1, \\ w_0v_0 &= \lambda_\infty e_\infty, \end{aligned}$$

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- Fix $\alpha_\infty = 1$, then $\lambda_\infty = -\sum_{k=0}^{m-1} \alpha_k \lambda_k$.
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- Let

$$V = \bigoplus_{r=0}^{m-1} V_r, \qquad X = \sum_{r=0}^{m-1} X_r, \qquad Y = \sum_{r=0}^{m-1} Y_r$$

- The Hamiltonians $H_k = -\frac{1}{m} \operatorname{tr} Y^{mk} = -w_0 Y^{mk} v_0 \in \mathbb{C}[\mathcal{C}_{\alpha,\lambda}]$ Poisson-commute:

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Equivariant quiver solution

- The Hamiltonians $H_k = -\frac{1}{m}\operatorname{tr} Y^{mk} = -w_0 Y^{mk} v_0$ induce the flows

$$X(t) = X(0) - \sum_{k\geqslant 1} k t_{mk} Y^{mk-1}, \qquad Y(t) = Y(0),$$
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where the variable t_{mk} is associated with H_k.

- Solution of spherical sub-hierarchy:

$$L = MyM^{-1}, \quad M = 1 - \epsilon_0 w_0 (X(t) - x)^{-1} (Y - y)^{-1} v_0 \epsilon_0,$$

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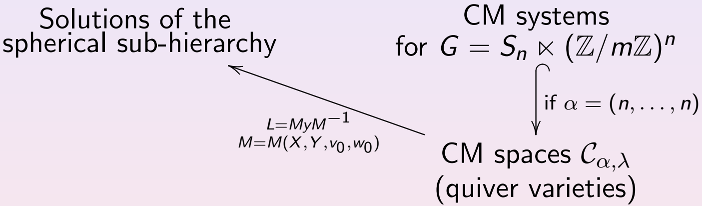
CM correspondence for the cyclic quiver



Calogero–Moser systems for the complex reflection group

- The complex reflection group $G = S_n \ltimes \Gamma^n$, where $\Gamma = \mathbb{Z}/m\mathbb{Z}$. It is generated by transpositions $\sigma_{ij} \in S_n$ and $\sigma_i = (1, \dots, 1, \sigma, 1, \dots, 1) \in \Gamma^n$.
- The *classical Dunkl operators for the group* G are

$$D_i = p_i - c_{00} \sum_{j \neq i} \sum_{r=0}^{m-1} \frac{1}{x_i - \mu^r x_j} \sigma_i^r \sigma_{ij} \sigma_i^{-r} - \sum_{r=1}^{m-1} \frac{c_r}{x_i} \sigma_i^r,$$

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Diagonalization

- Let $\alpha_0 = \alpha_1 = \ldots = \alpha_{m-1} = n$, so that $\boldsymbol{\alpha} = (1, n, \ldots, n)$.
- Generic point $[(X, Y, v_0, w_0)] \in \mathcal{C}_{n,\lambda} = \mathcal{C}_{(n,\ldots,n),\lambda}$:

$$X_k = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}, \quad Y_k = \begin{pmatrix} p_1^{(k)} & & (Y_k)_{ij} \\ & \ddots & \\ (Y_k)_{ji} & & p_n^{(k)} \end{pmatrix}, \quad v_0 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

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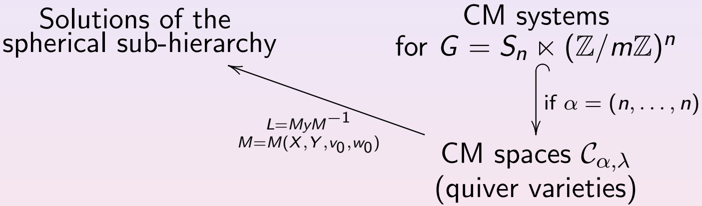
$w_0 = (1, \ldots, 1)$, where $(Y_k)_{ij} = -x_i^k x_j^{m-1-k}/(x_i^m - x_j^m)$,
$p_i^{(k)} = p_i + \frac{1}{x_i}\left(\frac{1}{m}\sum_{\ell=1}^{m-1} \ell\lambda_\ell - \sum_{\ell=k+1}^{m-1} \lambda_\ell\right)$.

- x_i's and p_i's are Darboux coordinates on $\mathcal{C}_{n,\lambda}$: $\{p_i, x_j\} = \delta_{ij}$.
- Quiver dynamics $X(t) = X(0) - \sum_{k=1}^{\infty} k t_k Y^{mk-1}$ in these coordinates coincides with the CM system dynamics.

CM correspondence for the cyclic quiver



CM correspondence for the cyclic quiver

Solutions of the full generalized KP hierarchy

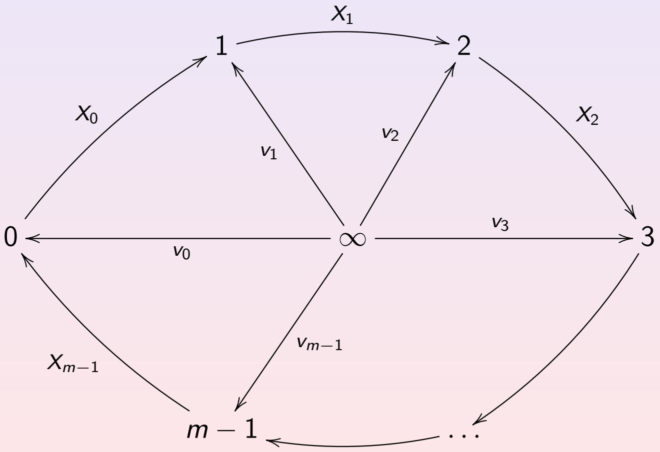
Spin CM systems for $G = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$

$\downarrow$ if $\alpha = (n, \ldots, n)$

CM spaces $\widetilde{\mathcal{C}}_{\alpha,\lambda}$ (quiver varieties)

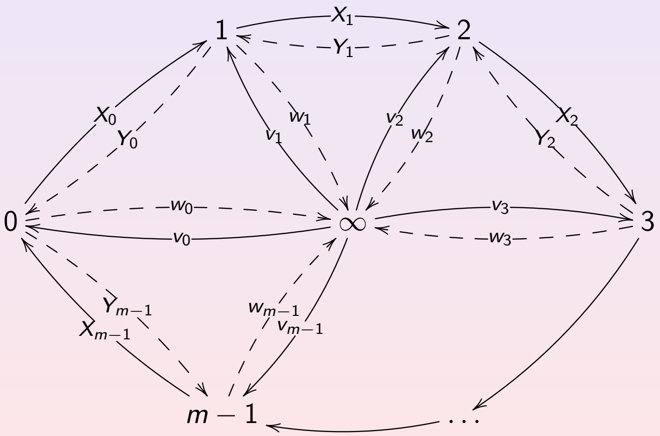
Another framing of the cyclic quiver Q_0

- Quiver $\widetilde{Q}$:



Double quiver

- Quiver $\overline{\widetilde{Q}}$:



Preprojective algebra

- Preprojective algebra $\Pi^\lambda(\widetilde{Q})$:

$$Y_r X_r - X_{r-1} Y_{r-1} - v_r w_r = \lambda_r e_r, \qquad r = 0, 1, \ldots, m-1,$$

$$\sum_{\ell=0}^{m-1} w_\ell v_\ell = \lambda_\infty e_\infty.$$

- Let $\alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{C}^m$, $\lambda_\infty = -\sum_{k=0}^{m-1} \alpha_k \lambda_k$. *CM space*:

$$\widetilde{\mathcal{C}}_{\alpha,\lambda} = \mathrm{Rep}\left(\Pi^\lambda(\widetilde{Q}), \boldsymbol{\alpha}\right)/GL(\alpha),$$

$$\boldsymbol{\lambda} = (\lambda_\infty, \lambda_0, \ldots, \lambda_{m-1}), \qquad \boldsymbol{\alpha} = (1, \alpha_0, \ldots, \alpha_{m-1}).$$

- Commuting Hamiltonians on $\widetilde{\mathcal{C}}_{\alpha,\lambda}$:

$$\widetilde{H}_k = -\sum_{r=0}^{m-1} w_r Y^k v_{r+k}, \qquad \{\widetilde{H}_k, \widetilde{H}_\ell\} = 0.$$

where $Y = \sum_{r=0}^{m-1} Y_r$.

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Non-equivariant solutions

- The Hamiltonian $\widetilde{H}_k$ generates the flow on $\widetilde{\mathcal{C}}_{\alpha,\lambda}$:

$$\frac{\partial}{\partial t_k}X = -\sum_{a=0}^{k-1} Y^a \sum_{\ell=0}^{m-1} v_\ell w_{\ell-k} Y^{k-a-1}, \qquad \frac{\partial}{\partial t_k}Y = 0,$$

$$\frac{\partial}{\partial t_k}v_\ell = -Y^k v_{\ell+k}, \qquad \frac{\partial}{\partial t_k}w_\ell = w_{\ell-k}Y^k.$$

- It gives the quiver solution of the (full) KP hierarchy:

$$L = MyM^{-1}, \quad M = 1 - \sum_{r,\ell=0}^{m-1} \epsilon_r w_r (X-x)^{-1}(Y-y)^{-1} v_\ell \epsilon_\ell,$$

where $X = X(t)$, $Y = Y(t)$, $v_\ell = v_\ell(t)$, $w_\ell = w_\ell(t)$ and $t = (t_1, t_2, t_3, \dots)$.

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Spin CM integrable systems

- Let $\alpha_0 = \alpha_1 = \ldots = \alpha_{m-1} = n$. Then $\widetilde{\mathcal{C}}_{n,\lambda} = \widetilde{\mathcal{C}}_{(n,\ldots,n),\lambda}$ is an $2mn$-dimensional symplectic (affine) variety with Darboux coordinates

$$x_i, \quad p_i, \quad (v_r)_i, \quad (w_r)_i, \qquad i = 1, \ldots, n, \qquad r = 1, \ldots, m-1.$$

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Thank you for your attention