Ultradiscrete inverse scattering and an elementary linearization of the Takahashi-Satsuma box-ball system

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Ralph WILLOX (the University of Tokyo)
Saburo KAKEI (Rikkyo University)
Satoshi TSUJIMOTO (Kyoto University)
Jonathan NIMMO (University of Glasgow)

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Overview of the talk

- We give a (brief) review of a solution method for a cellular automaton version of the KdV equation, reminiscent of the well-known IST scheme for the continuous KdV equation. [Joint work with A. Ramani and B. Grammaticos]

- We relate these results, in the case of the so-called Takahashi-Satsuma soliton cellular automaton (or 'Box & Ball' system), to certain simple combinatorial objects (rigged configurations) that offer a linearization of the BBS time evolution in terms of action-angle variables.

- These same techniques can be used, almost without modification, to provide action-angle variables for the Takahashi-Matsukidaira 'BBS with carrier', which is a CA version of the mKdV equation.

Overview of the talk

- (Brief) review of a solution method for a CA version of the KdV equation, reminiscent of the IST scheme for the continuous KdV equation.

The 'ultradiscrete' KdV equation

$$\boxed{U_\ell^{t+1} \;=\; \min\Big[\,1 - U_\ell^t\,,\ \sum_{k=-\infty}^{\ell-1} (U_k^t - U_k^{t+1})\,\Big]} \qquad (\,U : \mathbb{Z}^2 \to \mathbb{R}\,)$$

with boundary conditions $U_\ell^t = 0$ for $\ell \ll -1 \left({}^\forall t > 0\right)$ and initial conditions $U_\ell^0 \in \mathbb{R}$ with finite support, i.e.: $|\ell| \gg 1 : \; U_\ell^0 = 0\,.$

- A solitonic system, with all information evolving from left to right.

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- A solitonic system, with all information evolving from left to right.
- Can be obtained from a suitable discretization of KdV by a special limiting procedure: the ultradiscrete limit. [**Tokihiro et al. PRL 76 (1996) 3247**]

(this is most easily seen on its 'Yang-Baxter' form: $\begin{cases} U_\ell^{t+1} + U_\ell^t = \min[1, V_\ell^t + U_\ell^t] \\ V_{\ell+1}^t + U_\ell^{t+1} = V_\ell^t + U_\ell^t \end{cases}$)

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- A solitonic system, with all information evolving from left to right.
- Can be obtained from a suitable discretization of KdV by a special limiting procedure: the ultradiscrete limit. [**Tokihiro et al. PRL 76 (1996) 3247**]
- If $U_\ell^0 \in \{0, 1\}$, the ud-KdV evolution is closed on this set and the system contains only solitons: the Takahashi-Satsuma Box&Ball system.

The Box&Ball system

[Takahashi & Satsuma J. Phys. Soc. Jpn. 59 (1990) 3514]

$$U_\ell^{t+1} = \min\Big[1 - U_\ell^t \,,\ \sum_{k=-\infty}^{\ell-1} (U_k^t - U_k^{t+1}) \Big] \ , \quad \text{with} \quad \underline{U_\ell^0 \in \{0,1\}}$$

$$\begin{array}{cccccccccccccccccccccccc}
 & \ell & \rightarrow & & & & & & & & & & & & & & & & & & & & & \\
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\downarrow & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}$$

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$$\begin{array}{cccccccccccccccccccccccc}
 & \ell & \rightarrow & & & & & & & & & & & & & & & & & & & & & \\
t & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}$$

- All initial conditions decompose into solitons (i.e. into sequences of 1s that move with a speed equal to their length) and in fact, its Cauchy problem can be solved exactly.
- Soliton interactions give rise to pair-wise additive phase-shifts:
  when two solitons interact, the slower one is retarded by an amount twice its own speed, while the faster one is advanced by that amount.

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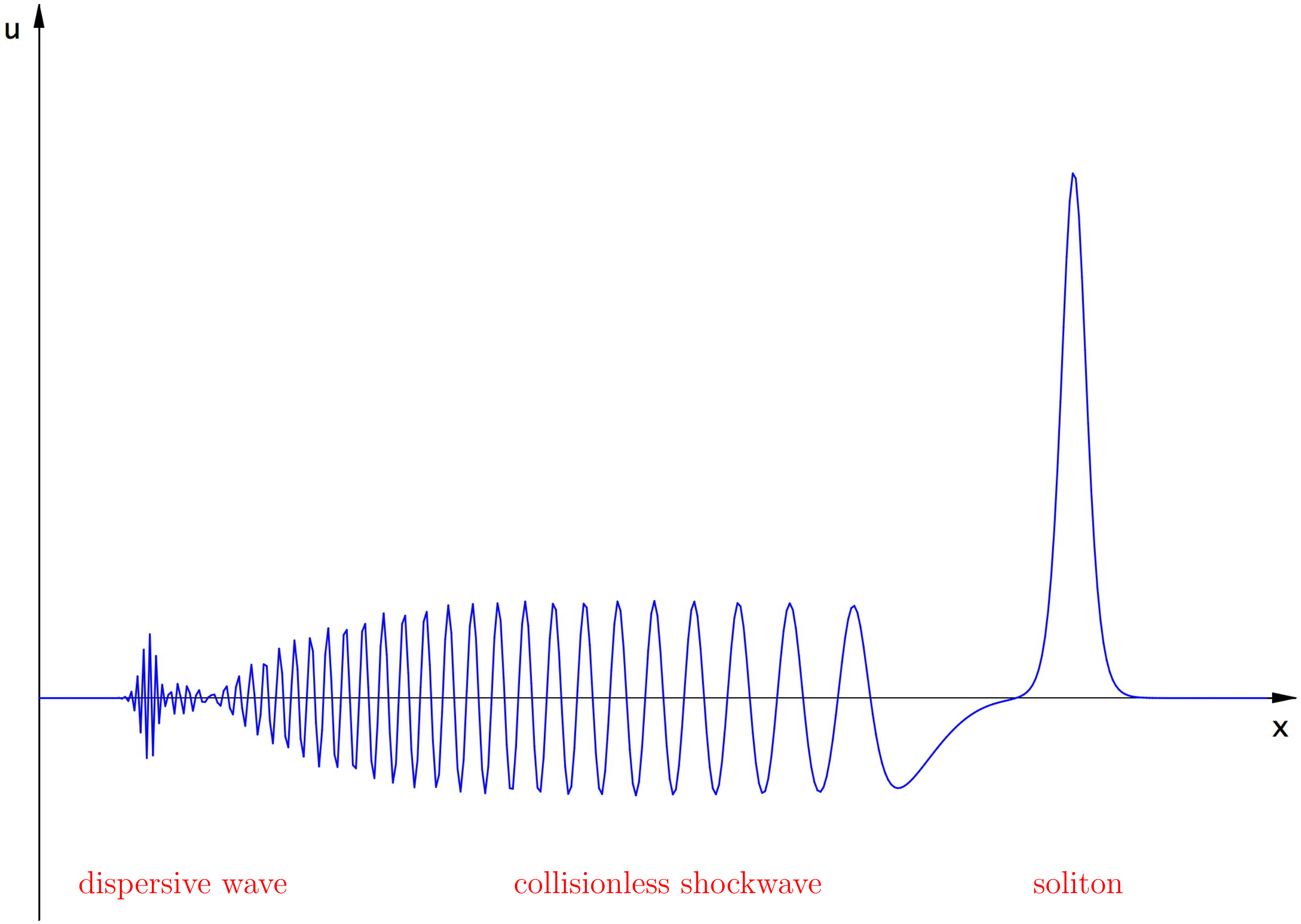
- (Brief) review of a solution method for a CA version of the KdV equation, reminiscent of IST for the continuous KdV equation.

- Can one solve the Cauchy problem for the ud-KdV equation over the reals ?

IST : continuous vs. ultradiscrete



dispersive wave

collisionless shockwave

soliton

IST : continuous vs. ultradiscrete

KdV: $\boxed{\mathrm{u}_{\mathrm{t}} + \mathrm{u}_{3\mathrm{x}} + 6\mathrm{u}\mathrm{u}_{\mathrm{x}} = 0}$

Lax pair for KdV: $$\begin{cases} \frac{\partial^2}{\partial x^2}\phi + \mathrm{u}\,\phi = \lambda^2\phi & (\star) \\ \frac{\partial}{\partial t}\phi = -\big(4\frac{\partial^3}{\partial x^3}\phi + 6\mathrm{u}\,\frac{\partial}{\partial x}\phi + 3\mathrm{u}_{\mathrm{x}}\phi\big) \end{cases}$$

- Suitable initial conditions $\mathrm{u}(\mathrm{x},0) \in L_1^1 := \left\{ p(\xi) \text{ measurable} \,\middle|\, \int_{-\infty}^{+\infty} |p(\xi)|(1+|\xi|)\mathrm{d}\xi < \infty \right\}$ have a finite and simple discrete spectrum, when taken as potentials in $(\star)$.

- Asymptotically, a generic initial condition separates into a right-moving *solitonic* part (≈ discrete spectrum) and a *non-solitonic* remainder consisting of modulated dispersive wave trains and collisionless shock waves (≈ continuous spectrum).

IST : continuous vs. ultradiscrete

KdV: $\boxed{\mathrm{u_t + u_{3x} + 6uu_x = 0}}$

Lax pair for KdV: $$\begin{cases} \frac{\partial^2}{\partial x^2}\phi + \mathrm{u}\,\phi = \lambda^2\phi & (\star) \\ \frac{\partial}{\partial t}\phi = -\big(4\frac{\partial^3}{\partial x^3}\phi + 6\mathrm{u}\,\frac{\partial}{\partial x}\phi + 3\mathrm{u_x}\phi\big) \end{cases}$$

- The contribution from the discrete spectrum is characterized by the eigenvalues λ_j and the right and left normalization coefficients, c_j^r and c_j^ℓ, for the eigenfunctions for the corresponding bound states.

- The phase-shift s_j the j^{th} soliton undergoes as t runs from $-\infty$ to $+\infty$ is given by

$$s_j = \frac{1}{2\lambda_j}\log\Big[\Big(\frac{c_j^r c_j^\ell}{2\lambda_j}\Big)^2 \prod_{k=1}^{j-1}\Big(\frac{\lambda_j-\lambda_k}{\lambda_j+\lambda_k}\Big)^4\Big]$$

for $\lambda_1 > \lambda_2 > \cdots > \lambda_N > 0\,.$ **[Ablowitz & Kodama 1982 ; Ablowitz & Segur 1977]**

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KdV: $\boxed{\mathrm{u_t + u_{3x} + 6uu_x = 0}}$

Lax pair for KdV:
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- The contribution from the continuous spectrum to the phase-shifts of the solitons can be represented in terms of the (right) reflection coefficient, $b_r(\kappa)$, which is also part of the scattering data required in the IST scheme for KdV.

- The phase-shift s_j the j^{th} soliton undergoes as t runs from $-\infty$ to $+\infty$ is given by

$$s_j = \frac{1}{\lambda_j}\sum_{k=1}^{j-1}\log\frac{\lambda_k-\lambda_j}{\lambda_k+\lambda_j} - \frac{1}{\lambda_j}\sum_{k=j+1}^{N}\log\frac{\lambda_j-\lambda_k}{\lambda_j+\lambda_k} + \frac{1}{\pi}\int_0^\infty \frac{\log(1-|b_r(\kappa)|^2)}{\kappa^2+\lambda_j^2}\,d\kappa$$

for $\lambda_1 > \lambda_2 > \cdots > \lambda_N > 0\,.$ **[P.C. Schuur, Lect. Notes Math. 1232, Springer-Verlag (1986)]**

IST : continuous vs. ultradiscrete

. . π . . . .

IST : continuous vs. ultradiscrete

$$\begin{array}{cccccccccccccccccccc}
\cdot & \cdot & \pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1-\pi & 1 & 1 & 1 & 1 & 1 & 2\pi-6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1-\pi & \cdot & \cdot & \cdot & \cdot & 7-2\pi & 1 & 1 & 1 & 1 & 4\pi-12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & {\color{blue}1-\pi} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & {\color{red}13-4\pi} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}6\pi-18} & \cdot
\end{array}$$

background: speed 1

soliton with speed $2\pi - 1$

IST : continuous vs. ultradiscrete

$$\begin{array}{cccccccccccccccccccc}
\cdot & \cdot & \pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1-\pi & 1 & 1 & 1 & 1 & 1 & 2\pi-6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1-\pi & \cdot & \cdot & \cdot & \cdot & 7-2\pi & 1 & 1 & 1 & 1 & 4\pi-12 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1-\pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 13-4\pi & 1 & 1 & 1 & 1 & 6\pi-18 & \cdot
\end{array}$$

$$\begin{array}{cccccccccccccccccccc}
\cdot & 1 & 1 & 1 & \cdot & \cdot & -\pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\pi & 2\pi-3 & 6-2\pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\pi & 2\pi-5 & 1 & 7-2\pi & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\pi & \cdot & 2\pi-6 & 1 & 1 & 7-2\pi & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\pi & \cdot & \cdot & \cdot & 2\pi-6 & 1 & 1 & 7-2\pi & \cdot
\end{array}$$

background

soliton with speed 3

Phase-shifts for ultradiscrete solitons & background

- Phase-shifts for the fast solitons (Δ_ω) and background (Δ_{bg}) are given by:

$$\Delta_{\omega>1} = 2\Big(\sum_{\omega'<\omega} \omega' - \sum_{\omega'>\omega} \omega + \sum_{\ell\in\{bg\}} U_\ell^{(bg)} \Big) \, , \qquad \Delta_{bg} = \; - 2 \, (\sum_{\omega'>1} 1)$$

(slow (= speed 1) solitons can be taken to be part of the background)

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·	1	1	1	·	·	$-\pi$	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	1	1	1	$-\pi$	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	3	$-\pi$	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	-2	$\pi+1$	$4-2\pi$	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	$-\pi$	$2\pi-3$	$6-2\pi$	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	$-\pi$	$2\pi-5$	1	$7-2\pi$	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	$-\pi$	·	$2\pi-6$	1	1	$7-2\pi$	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	$-\pi$	·	·	·	$2\pi-6$	1	1	$7-2\pi$	·

phase shift: -2

phase shift: -2π

Overview of the talk

- (Brief) review of a solution method for a CA version of the KdV equation, reminiscent of IST for the continuous KdV equation.
- Can one solve the Cauchy problem for the ud-KdV equation over the reals ?
- Step 1: relate the ud-KdV equation to a linear system and define the discrete spectrum, bound states etc. of a given initial condition.

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- Step 3: solve the Cauchy problem for the original initial condition by *dressing* the explicit solution constructed for the background.

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A linear system for ud-KdV

[Willox et al. J. Phys. A 43 (2010) 482003]

$$\boxed{U_\ell^{t+1} = \min\Big[1-U_\ell^t\,,\ \sum_{k=-\infty}^{\ell-1}(U_k^t-U_k^{t+1})\Big]}$$

$$\left\{\begin{aligned}
\max\big[\Phi_{\ell+1}^t-\kappa\,,\ \Phi_{\ell-1}^t\big] &= \Phi_\ell^t+\max\big[-U_\ell^t\,,\ U_{\ell-1}^t-1\big]\\
\max\big[\Phi_{\ell+1}^{t+1}-\kappa\,,\ \Phi_{\ell-1}^{t+1}\big] &= \Phi_\ell^{t+1}+\max\big[U_\ell^t-1\,,\ \ -U_{\ell-1}^t\big]\\
\max\big[\Phi_\ell^t+\kappa-\omega\,,\ \Phi_\ell^{t+1}+U_\ell^t+\kappa-1\big] &= \Phi_{\ell+1}^{t+1}\\
\max\big[\Phi_{\ell+1}^{t+1}\,,\ \Phi_{\ell+1}^t+U_\ell^t-1\big] &= \Phi_\ell^t
\end{aligned}\right.$$

for some constants $\kappa,\omega\ \geq 0$

- This system is ‘linear’ in the semi-field $\mathbb{R}\cup\{\infty\}_{\max,+}$, in the sense that its solution Φ_ℓ^t is only defined up to an additive constant : $\Phi_\ell^t\to\Phi_\ell^t+c^t$.

Solving ud-KdV through 'IST' [Willox et al. Contemporary Mathematics 580 (2012)]

$$\boxed{\begin{cases} \max\left[\Phi_{\ell+1}^{t}-\kappa\,,\ \Phi_{\ell-1}^{t}\right] \ = \ \Phi_{\ell}^{t}+\max\left[-U_{\ell}^{t}\,,\ U_{\ell-1}^{t}-1\right] \\ \max\left[\Phi_{\ell+1}^{t+1}-\kappa\,,\ \Phi_{\ell-1}^{t+1}\right] \ = \ \Phi_{\ell}^{t+1}+\max\left[U_{\ell}^{t}-1\,,\ \ -U_{\ell-1}^{t}\right] \\ \max\left[\Phi_{\ell}^{t}+\kappa-\omega\,,\ \Phi_{\ell}^{t+1}+U_{\ell}^{t}+\kappa-1\right] \ = \ \Phi_{\ell+1}^{t+1} \\ \max\left[\Phi_{\ell+1}^{t+1}\,,\ \Phi_{\ell+1}^{t}+U_{\ell}^{t}-1\right] \ = \ \Phi_{\ell}^{t} \end{cases}}$$

for some constants $\kappa,\omega \geq 0$

- Consider the above system at $t=0$, with a 'potential' U_{ℓ}^{0} given by some initial condition for ud-KdV (over the reals, with finite support).

Solving ud-KdV through 'IST' [Willox et al. Contemporary Mathematics 580 (2012)]

$$\boxed{\begin{cases} \max\left[\Phi^0_{\ell+1}-\kappa\,,\ \Phi^0_{\ell-1}\right] \ = \ \Phi^0_\ell+\max\left[-U^0_\ell\,,\ U^0_{\ell-1}-1\right] \\ \max\left[\Phi^1_{\ell+1}-\kappa\,,\ \Phi^1_{\ell-1}\right] \ = \ \Phi^1_\ell+\max\left[U^0_\ell-1\,,\ \ -U^0_{\ell-1}\right] \\ \max[\Phi^0_\ell+\kappa-\omega\,,\ \Phi^1_\ell+U^0_\ell+\kappa-1] \ = \ \Phi^1_{\ell+1} \\ \max\left[\Phi^1_{\ell+1}\,,\ \Phi^0_{\ell+1}+U^0_\ell-1\right] \ = \ \Phi^0_\ell \end{cases}}$$

for some constants $\kappa,\omega \geq 0$

- Consider the above system at $t=0$, with a 'potential' U^0_ℓ given by some initial condition for ud-KdV (over the reals, with finite support).

- The system has solutions with special asymptotics: $\begin{cases} \ell\sim-\infty: & \alpha_t+\kappa\,\ell \\ \ell\sim+\infty: & 0 \end{cases}$

  for $\alpha_t=\alpha_0-\omega\,t$ (linear!), and which obey the dispersion relation : $\kappa=\min\left[1,\ \omega\right]$.

Solving ud-KdV through 'IST'

Definition of a bound state :

If, for U_ℓ^t with finite support, the ultradiscrete linear system has a solution Φ_ℓ^t for some *positive* κ and ω, such that

$$\mathcal{N}_\Phi := \max_{\ell\in\mathbb{Z}}\left[\Phi_\ell^t + \Phi_{\ell-1}^t - \kappa\,\ell + \omega\,t\right] < +\infty\,,$$

we say that the potential U_ℓ^t has a bound state.

- $\mathcal{N}_\Phi$ is invariant under the ud-KdV time evolution.
- It can be shown that, if there is a bound state for some U_ℓ^0, the associated ω is the speed of the fastest soliton contained in that initial condition.

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we say that the potential U_ℓ^t has a bound state.

- The quantity $\Omega_\ell^t := \Phi_\ell^t + \Phi_{\ell-1}^t - \kappa\,\ell + \omega\,t$ is an analogue of the squared eigenfunction for KdV.
- It has the asymptotics : $\begin{cases} \ell \sim -\infty : & 2\alpha_0 + \kappa(\ell-1) - \omega t \\ \ell \sim +\infty : & -\kappa\ell + \omega t \end{cases}$

Solving ud-KdV through 'IST'

Theorem 1 : [Willox et al. Contemporary Mathematics 580 (2012) 135–155]

$$^{\forall}\ell : \quad U_\ell + U_{\ell+1} \leq 0 \qquad \rightarrow \qquad \text{pure background : no bound state exists}$$

$$^{\exists}\ell : \quad 0 < U_\ell + U_{\ell+1} \leq \mu \leq 1 \qquad \rightarrow \qquad \text{bound state(s) exist with } \omega = \kappa = \mu \leq 1$$

(μ: maximal)

$$^{\exists}\ell : \quad U_\ell + U_{\ell+1} > 1 \qquad \rightarrow \qquad \text{bound state(s) exist with } \kappa = 1,\ \omega > 1$$

- ω is obtained uniquely by solving the system for Φ_ℓ^0, Φ_ℓ^1, as $\omega = \alpha_0 - \alpha_1$, and at most one bound state can be found for any initial condition.
- Generically, there are no unique functions Φ_ℓ^0, Φ_ℓ^1 for this bound state.
- However, Φ_ℓ^0, Φ_ℓ^1 can be found algorithmically.

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- Step 2: use these bound states to *undress* the initial condition, leaving only a soliton-less 'background', the Cauchy problem for which is trivial.

Solving ud-KdV through 'IST'

Theorem 2 :

Define the following transformation of U_ℓ , in terms of the functions Φ_ℓ^0, Φ_ℓ^1 that correspond to a bound state ω for that U_ℓ :

$$U_\ell \mapsto \widetilde{U}_\ell \quad : \qquad \boxed{\widetilde{U}_\ell = U_\ell + \Phi_{\ell+1}^0 + \Phi_\ell^1 - \Phi_\ell^0 - \Phi_{\ell+1}^1}$$

This transformation corresponds to an *undressing* of the potential U_ℓ :

- The 'mass' of the potential U_ℓ decreases by ω : $\displaystyle\sum_{\ell=-\infty}^{+\infty} \widetilde{U}_\ell^t = \sum_{\ell=-\infty}^{+\infty} U_\ell^t - \omega$
- the region where large values of the sum $U_\ell + U_{\ell+1}$ occur shrinks under the undressing and $\widetilde{U}_\ell$ corresponds to an initial value for ud-KdV in which a speed ω soliton was eliminated.

Solving ud-KdV through 'IST'

Theorem 2 :

Define the following transformation of U_ℓ , in terms of the functions Φ^0_ℓ, Φ^1_ℓ that correspond to a bound state ω for that U_ℓ :

$$U_\ell \mapsto \widetilde{U}_\ell \quad : \qquad \boxed{\widetilde{U}_\ell = U_\ell + \Phi^0_{\ell+1} + \Phi^1_\ell - \Phi^0_\ell - \Phi^1_{\ell+1}}$$

This transformation corresponds to an *undressing* of the potential U_ℓ :

- The 'mass' of the potential U_ℓ decreases by ω : $\displaystyle\sum_{\ell=-\infty}^{+\infty} \widetilde{U}^t_\ell = \sum_{\ell=-\infty}^{+\infty} U^t_\ell - \omega$
- the region where large values of the sum $U_\ell + U_{\ell+1}$ occur shrinks under the undressing and $\widetilde{U}_\ell$ corresponds to an initial value for ud-KdV in which a speed ω soliton was eliminated.
- Hence, a finite iteration leads to a potential without bound states, i.e. a background without solitons (when considered as an initial condition for ud-KdV).

Example of an undressing

$$\boxed{U_\ell \mapsto \widetilde{U}_\ell}$$

$U_\ell :$	$\cdot$	-1	$\pi/5$	-2	$\cdot$	$\cdot$	$\pi/3$	1	1	$\pi/7$	$\cdot$
$\Phi^0_\ell :$	α_0-10	α_0-9	α_0-7	$\pi/3-6$	$\pi/3-3$	$\pi/3-2$	$\pi/3-1$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\Phi^1_\ell :$	α_1-10	α_1-9	α_1-8	α_1-6	$-10\pi/21-6$	$-10\pi/21-3$	$-10\pi/21-2$	$-\pi/7-2$	$-\pi/7-1$	$-\pi/7$	$\cdot$
$\widetilde{U}_\ell :$	$\cdot$	$\cdot$	-1	$\pi/5$	-2	$\cdot$	$1-\pi/3$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

where $\alpha_0 = 8\pi/15$ and $\alpha_1 = -29\pi/105 - 1$, and thus $\omega := \alpha_0 - \alpha_1 = 1 + 17\pi/21$.

Example of an undressing

$$\boxed{U_\ell \mapsto \widetilde{U}_\ell}$$

$$\begin{array}{llllllllllll}
U_\ell : & \cdot & -1 & \pi/5 & -2 & \cdot & \cdot & \pi/3 & 1 & 1 & \pi/7 & \cdot \\
\Phi^0_\ell : & \alpha_0-10 & \alpha_0-9 & \alpha_0-7 & \pi/3-6 & \pi/3-3 & \pi/3-2 & \pi/3-1 & \cdot & \cdot & \cdot & \cdot \\
\Phi^1_\ell : & \alpha_1-10 & \alpha_1-9 & \alpha_1-8 & \alpha_1-6 & -10\pi/21-6 & -10\pi/21-3 & -10\pi/21-2 & -\pi/7-2 & -\pi/7-1 & -\pi/7 & \cdot \\
 & & & & & & & & & & & \\
\widetilde{U}_\ell : & \cdot & \cdot & -1 & \pi/5 & -2 & \cdot & 1-\pi/3 & \cdot & \cdot & \cdot & \cdot
\end{array}$$

where $\alpha_0 = 8\pi/15$ and $\alpha_1 = -29\pi/105 - 1$, and thus $\omega := \alpha_0 - \alpha_1 = 1 + 17\pi/21$.

Compare this with the time evolution of the initial value U_ℓ for udKdV :

$$\begin{array}{cccccccccccccccccc}
\cdot & -1 & \pi/5 & -2 & \cdot & \cdot & \pi/3 & 1 & 1 & \pi/7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \pi/5 & -2 & \cdot & 1-\pi/3 & \cdot & \cdot & 1-\pi/7 & 1 & 1 & 20\pi/21-2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & \pi/5 & -2 & \cdot & 1-\pi/3 & \cdot & \cdot & \cdot & \cdot & 3-20\pi/21 & 1 & 1 & 1 & 37\pi/21-5 & \cdot
\end{array}$$

result of undressing : $\widetilde{U}_\ell$ soliton with speed $1 + 17\pi/21$

Solving ud-KdV through 'IST'

- Iteration of the undressing $U_\ell \mapsto \widetilde{U}_\ell$ yields an (ordered) list of data :

$$\Big[\big(\omega^{(1)}, \alpha_0^{(1)}\big), \big(\omega^{(2)}, \alpha_0^{(2)}\big), \ldots, \big(\omega^{(N)}, \alpha_0^{(N)}\big)\Big], \qquad \omega^{(1)} \geq \omega^{(2)} \geq \cdots \geq \omega^{(N)}$$

where N = the # of eliminated bound states.

- Ultimately, one obtains a background $\widehat{U}_\ell$ which is free of bound states and which, as an initial value for ud-KdV, evolves undeformed at speed 1.

- The set $\Big\{\Big[\big(\omega^{(1)}, \alpha_0^{(1)}\big), \big(\omega^{(2)}, \alpha_0^{(2)}\big), \ldots, \big(\omega^{(N)}, \alpha_0^{(N)}\big)\Big], \quad \widehat{U}_\ell \quad \Big\}$ constitutes the ultradiscrete scattering data.

Solving ud-KdV through 'IST'

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This is in fact the ultradiscrete analogue of a famous theorem due to P. Deift & E. Trubowitz [**Comm. Pure. Appl. Math. 12 (1979) 121-151**] stating that the fastest soliton in any given initial state can be removed by Darboux transformation, without perturbing the rest of the spectrum.

Solving ud-KdV through ‘IST’

- Iteration of the undressing $U_\ell \mapsto \widetilde{U}_\ell$ yields an (ordered) list of data :

$$\Big[(\omega^{(1)},\alpha_0^{(1)}),(\omega^{(2)},\alpha_0^{(2)}),\ldots,(\omega^{(N)},\alpha_0^{(N)})\Big], \qquad \omega^{(1)} \geq \omega^{(2)} \geq \cdots \geq \omega^{(N)}$$

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- The set $\Big\{\Big[(\omega^{(1)},\alpha_0^{(1)}),(\omega^{(2)},\alpha_0^{(2)}),\ldots,(\omega^{(N)},\alpha_0^{(N)})\Big],\ \ \widehat{U}_\ell\ \ \Big\}$ constitutes the ultradiscrete scattering data.

- In the absence of faster solitons, the evolution of a background $\widehat{U}_\ell$ can be described explicitly, for all times t, as: **[Hirota Stud. Appl. Math. 122 (2009) 361]**

$$\boxed{U_\ell^t = T_{\ell+1}^t + T_\ell^{t+1} - T_\ell^t - T_{\ell+1}^{t+1}, \qquad T_\ell^t = \frac{1}{2}\sum_{k=-\infty}^{+\infty} \widehat{U}_k\,|\ell-k-t|}$$

Overview of the talk

- (Brief) review of a solution method for a CA version of the KdV equation, reminiscent of IST for the continuous KdV equation.

- Can one solve the Cauchy problem for the ud-KdV equation over the reals ?

- Step 1: relate the ud-KdV equation to a linear system and define the discrete spectrum, bound states etc. of a given initial condition.

- Step 2: use these bound states to *undress* the initial condition, leaving only a soliton-less ‘background’, the Cauchy problem for which is trivial.

- Step 3: solve the Cauchy problem for the original initial condition by *dressing* the explicit solution constructed for the background.

Solving ud-KdV through 'IST'

Theorem 3 :

[Nakata J. Phys. A: Math. Gen. 42 (2009) 412001]

A solution $\widetilde{U}_\ell^t$ to ud-KdV can be 'dressed' by the transformation:

$$\boxed{\widetilde{T}_\ell^t \mapsto T_\ell^t = \tfrac{1}{2}\max\left[\min[1,\omega]\,\ell - \omega t - c + 2\widetilde{T}_\ell^{t+1},\ -\min[1,\omega]\,\ell + \omega t + c + 2\widetilde{T}_\ell^{t-1}\right]}$$

This yields a map $\widetilde{U}_\ell^t = \widetilde{T}_{\ell+1}^t + \widetilde{T}_\ell^{t+1} - \widetilde{T}_\ell^t - \widetilde{T}_{\ell+1}^{t+1} \ \mapsto\ U_\ell^t = T_{\ell+1}^t + T_\ell^{t+1} - T_\ell^t - T_{\ell+1}^{t+1}$
which adds a speed ω soliton to $\widetilde{U}_\ell^t$ (provided there are no faster solitons).

Solving ud-KdV through 'IST'

Theorem 3 : **[Nakata J. Phys. A: Math. Gen. 42 (2009) 412001]**

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This yields a map $\widetilde{U}_\ell^t = \widetilde{T}_{\ell+1}^t + \widetilde{T}_\ell^{t+1} - \widetilde{T}_\ell^t - \widetilde{T}_{\ell+1}^{t+1} \ \mapsto\ U_\ell^t = T_{\ell+1}^t + T_\ell^{t+1} - T_\ell^t - T_{\ell+1}^{t+1}$ which adds a speed ω soliton to $\widetilde{U}_\ell^t$ (provided there are no faster solitons).

- This dressing procedure yields an explicit solution, for all ℓ and t (!)

  Starting from the background $\widehat{T}_\ell^t = \dfrac{1}{2}\sum_{k=-\infty}^{+\infty} \widehat{U}_k^0 |\ell - k - t|$,solitons are inserted one by one, in reverse order, i.e.: $\omega^{(N)} \to \omega^{(N-1)} \to \cdots \to \omega^{(1)}$, with phases $\mathbf{c}$ given by the normalization coefficients $\alpha_0^{(j)}$: $\mathbf{c}^{(j)} = -\alpha_0^{(j)}$.

Conclusions (1)

- $^{\exists}$a solution method for a CA version of the KdV equation, similar to the inverse scattering method for the continuous KdV equation.

  In a sense, the dynamics exhibited by the BBS is ‘as rich’ as that of its discrete or continuous counterparts.

- Essential ingredients for solving the Cauchy problem are : the soliton speeds, the insertion points of the solitons in the dressing, and the background on which the solitons are superimposed.

  This information is completely determined by the scattering data.

- In fact, in the case of the Takahashi-Satsuma ‘Box & Ball’ system, we can link this method to combinatorial techniques that yield a linearization of the BBS time-evolution in terms of action-angle variables, which turn out to be equivalent to the scattering data.

Combinatorics and the Cauchy problem for the BBS

- The Cauchy problem for the BBS was first solved for periodic boundary conditions (using the inverse scattering technique for the discrete Toda lattice and by taking an appropriate ultradiscrete limit.)

  [**T. Kimijima & T. Tokihiro Inv. Probl. 18 (2002) 1705**]

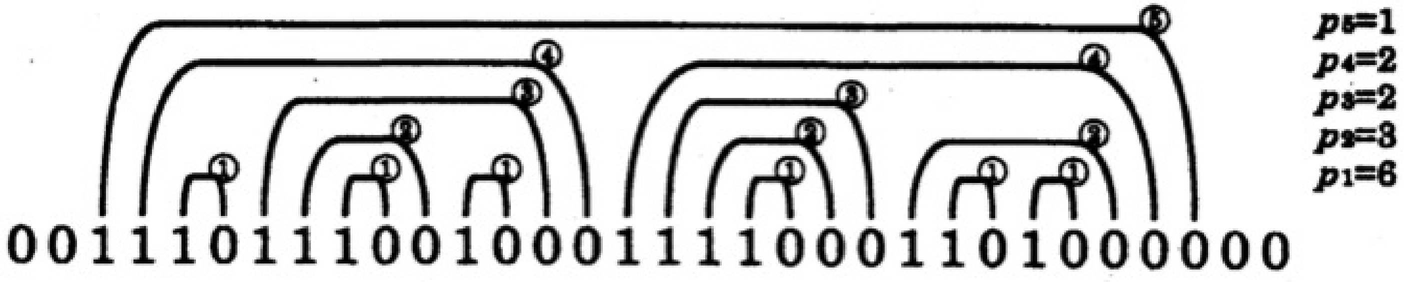
- For non-periodic boundary conditions, the Cauchy problem was first solved by applying a procedure called "10 elimination"

  [**J. Mada et al. J. Phys. A 41 (2008) 175207**]

BBS: 10 elimination

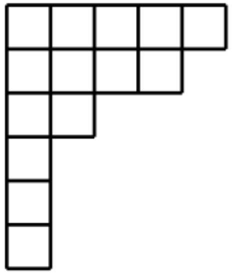
- Connect all 10 pairs in the BBS state by arcs. Then, neglecting all connected pairs, connect all new 10 pairs and keep on repeating this process.

  Fact: interchanging 1s and 0s in every arc amounts to one time-step in the BBS !



[J. Mada et al., RIMS Kōkyūroku 1541 (2007) 15]

- The number of arcs at each stage can be recorded in a Young diagram, which encodes the speeds ($\omega_j := \omega^{(N+1-j)}$) of the solitons in the initial state



$\rightarrow \omega_6 = 5$
$\rightarrow \omega_5 = 4$
$\rightarrow \omega_4 = 2$
$\rightarrow \omega_3 = 1$
$\rightarrow \omega_2 = 1$
$\rightarrow \omega_1 = 1$

- This Young diagram is invariant w.r.t. the BBS time evolution, but does not uniquely characterize a BBS state (as it does not give the soliton positions).

Combinatorics and the Cauchy problem for the BBS

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- The Cauchy problem for the BBS can also be solved by linearizing the evolution by means of action-angle variables [**A. Kuniba et al., Nucl. Phys. B 747 (2006) 354–397**], using a Kerov-Kirillov-Reshetikhin (KKR) type bijection between BBS states and so-called “rigged configurations” (which yield the action-angle variables).

- This approach is intimately related to 10 elimination. [**A.N. Kirillov & R. Sakamoto, Lett. Math. Phys. 89 (2009) 51**].

BBS: 10 elimination

- Our version (based on [**T. Takagi, SIGMA 6 (2010) 027**] of this method is as follows:

$$\begin{array}{ccccccccccccccccccccccccc}
 & 0 & 1 & & & & 5 & & 7 & & & & 11 & & 13 & 14 & 15 & & 17 & 18 & & & 21 & & \\
\cdots & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots
\end{array}$$

BBS: 10 elimination

- Our version (based on [**T. Takagi, SIGMA 6 (2010) 027**] of this method is as follows:

$$\cdots\ \overset{0}{0}\ \overset{1}{1}\ 1\ 1\ 1\ \overset{5}{1}\ 0\ \overset{7}{0}\ 1\ 1\ 1\ \overset{11}{1}\ 0\ \overset{13}{0}\ \overset{14}{1}\ \overset{15}{0}\ 1\ \overset{17}{1}\ \overset{18}{0}\ 1\ 1\ \overset{21}{1}\ 0\ \cdots \qquad ⑤$$

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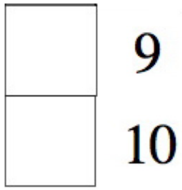
$$\cdots\ \overset{0}{0}\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ \overset{9}{0}\,\Big|\,\overset{10}{1}\,\Big|\,1\ 1\ 0\cdots\quad ③$$

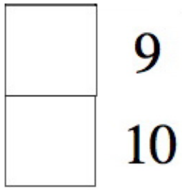


BBS: 10 elimination

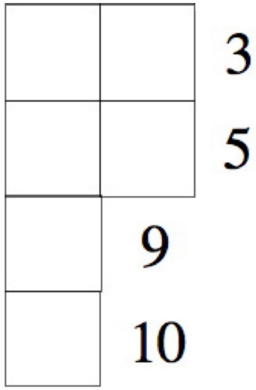
- Our version (based on **[T. Takagi, SIGMA 6 (2010) 027]** of this method is as follows:

$$\begin{array}{ccccccccccccccccccccccccc}
 & \color{blue}{0} & \color{blue}{1} & & & & \color{blue}{5} & & \color{blue}{7} & & & & \color{blue}{11} & & \color{blue}{13} & \color{blue}{14} & \color{blue}{15} & & \color{blue}{17} & \color{blue}{18} & & & \color{blue}{21} & & \\
\cdots & 0 & 1 & 1 & 1 & 1 & \color{red}{1} & \color{red}{0} & 0 & 1 & 1 & 1 & \color{red}{1} & \color{red}{0} & 0 & \color{green}{1} & \color{green}{0} & 1 & \color{green}{1} & \color{green}{0} & 1 & 1 & \color{red}{1} & \color{red}{0} & \cdots
\end{array} \quad ⑤$$

$$\begin{array}{ccccccccccc|c|cccc}
 & \color{blue}{0} & & & & & & & & & \color{green}{9} & \color{green}{10} & & & & \\
\cdots & 0 & 1 & 1 & 1 & \color{red}{1} & \color{red}{0} & 1 & 1 & \color{red}{1} & \color{red}{0} & 1 & 1 & \color{red}{1} & \color{red}{0} & \cdots
\end{array} \quad ③$$



$$\begin{array}{ccccc|cc|cccc}
 & \color{blue}{0} & & & \color{green}{3} & & \color{green}{5} & & & & \\
\cdots & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots
\end{array} \quad ①$$



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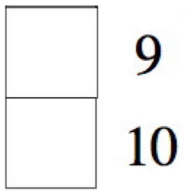
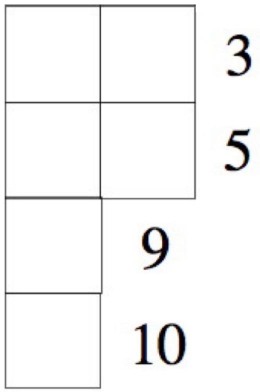
$$\cdots\ \overset{0}{0}\ \overset{1}{1}\ 1\ 1\ 1\ \overset{5}{1}\ 0\ \overset{7}{0}\ 1\ 1\ 1\ \overset{11}{1}\ 0\ \overset{13}{0}\ \overset{14}{1}\ \overset{15}{0}\ 1\ \overset{17}{1}\ \overset{18}{0}\ 1\ 1\ \overset{21}{1}\ 0\ \cdots \quad \textcircled{5}$$

$$\cdots\ \overset{0}{0}\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ \overset{9}{0}\ \Big|\ \overset{10}{1}\ \Big|\ 1\ 1\ 0\cdots \quad \textcircled{3}$$



$$\cdots\ \overset{0}{0}\ 1\ 1\ \overset{3}{1}\ \Big|\ 1\ \overset{5}{1}\ \Big|\ 1\ 1\ 0\cdots \quad \textcircled{1}$$



BBS: 10 elimination

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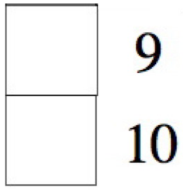
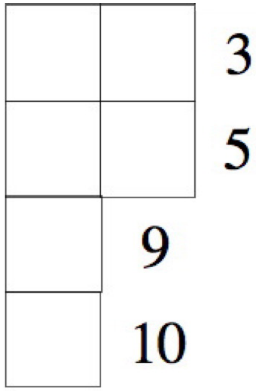
$$\cdots\ \overset{0}{0}\ \overset{1}{1}\ 1\ 1\ 1\ \overset{5}{1}\ 0\ \overset{7}{0}\ 1\ 1\ 1\ \overset{11}{1}\ 0\ \overset{13}{0}\ \overset{14}{1}\ \overset{15}{0}\ 1\ \overset{17}{1}\ \overset{18}{0}\ 1\ 1\ \overset{21}{1}\ 0\ \cdots\ \textcircled{5}$$

$$\cdots\ \overset{0}{0}\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ \overset{9}{0}\ \big|\ \overset{10}{1}\ \big|\ 1\ 1\ 0\cdots\ \textcircled{3}$$



$$\cdots\ \overset{0}{0}\ 1\ 1\ \overset{3}{1}\ \big|\ 1\ \overset{5}{1}\ \big|\ 1\ 1\ 0\cdots\ \textcircled{1}$$

$$\cdots\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0\cdots\ \textcircled{1}$$

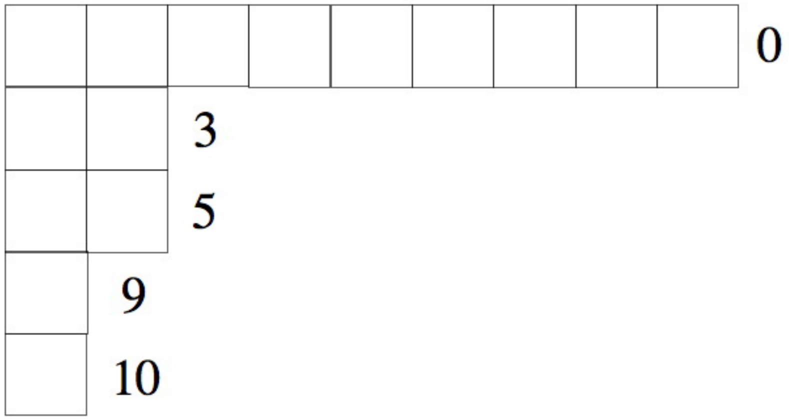


$\vdots$ another 6 times

$$\cdots\ \overset{0}{0}\ \big|\ 0\cdots\ \textcircled{0}$$

BBS: 10 elimination

- This procedure yields a Young diagram (essentially that obtained by 10 elimination) which gives the speeds/lengths of the solitons contained in the initial state – here, 9, 2 (twice) and 1 (twice) – and a ‘rigging’ of the solitons, 10, 9, 5, 3 and 0, which yields the rigged configuration :



- Because of the rigging, there is a one-to-one correspondence between the rigged configuration and a state of the BBS !
- This is a realization of the KKR bijection, in the case of the Takahashi-Satsuma BBS.

BBS: 10 elimination

- Such a rigged configuration offers a linearization of the BBS evolution in which the soliton speeds are the action variables and the riggings the angle variables.

t=0: $\cdots\overset{0}{0}\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\cdots$

t=1: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\cdots$

BBS: 10 elimination

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0

t=0: ···0 1 1 1 1 1 0 0 1 1 1 1 0 0 1 0 1 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0···

t=1: ···0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 0···

10-elim: ···0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 0···

10 11

···0 0 0 0 0 0 1 0 0 0 1 | 0 | 0 0 1 1 1 1 1 1 1 1 0···

BBS: 10 elimination

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0

t=0: ···0 1 1 1 1 1 0 0 1 1 1 1 0 0 1 0 1 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0···

t=1: ···0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 0···

10-elim: ···0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 0···

10 11

···0 0 0 0 0 0 1 0 0 0 1 | 0 | 0 0 1 1 1 1 1 1 1 1 0···

···0 0 0 0 0 0 1 0 0 0 1 0 0 0 1 1 1 1 1 1 1 1 0···

BBS: 10 elimination

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t=0: $\cdots \overset{0}{0}\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

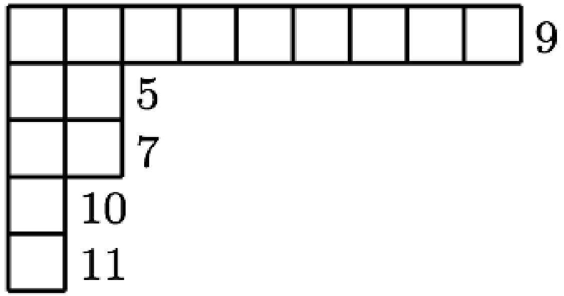
t=1: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$

10-elim: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$

$\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1 \overset{10}{|} 0 \overset{11}{|} 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$

$\cdots 0\ 0\ 0\ 0\ 0\ 0 \overset{5}{|} 0\ 0 \overset{7}{|} 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$

which leads to: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \overset{9}{|} 0 \cdots$



BBS: 10 elimination

- Such a rigged configuration offers a linearization of the BBS evolution in which the soliton speeds are the action variables and the riggings the angle variables.

t=0: $\cdots \overset{0}{0}\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

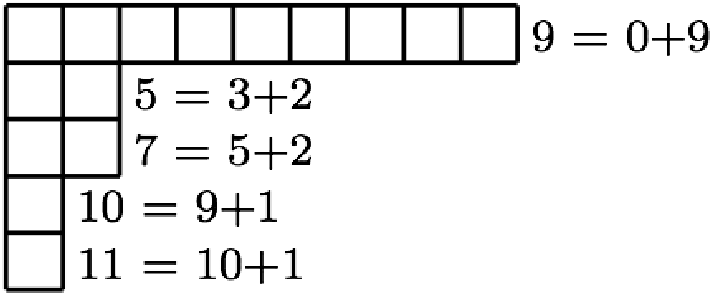
t=1: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$

10-elim: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$

$$\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1 \overset{10}{|} 0 \overset{11}{|} 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$$

$$\cdots 0\ 0\ 0\ 0\ 0\ 0 \overset{5}{|} 0\ 0 \overset{7}{|} 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$$

which leads to: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \overset{9}{|} 0 \cdots$



BBS: 10 elimination

- It can be shown that, for any BBS state, the riggings ϕ_k depend linearly on t:

$$\phi_k(t) = \phi_k(0) + \omega_k\, t,$$

  in terms of the soliton speeds ω_k (which are constant).

- This action-angle type linearization of the BBS was conjectured by Kuniba, Okado, Takagi & Yamada in [**RIMS Kōkyōroku 1302 (2003) 91–107**] and proven (by means of a crystal theoric interpretation) in: [**A. Kuniba et al., Nucl. Phys. B 740 (2006) 299-327**].

- Our proof is direct and elementary. It mainly relies on the fact that 10-elimination commutes with the time evolution of the BBS, up to a right-shift:

$$\begin{tikzcd}
\cdots 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0 \cdots \arrow[r, "10-\text{elim.}"] \arrow[d, "t \to t+1"'] & \cdots 0 | 0\ 1\ 1 || 1\ 0\ 0\ 0\ 0 \cdots \arrow[d, "t \to t+1"', "\ell \to \ell+1"] \\
\cdots 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0 \cdots \arrow[r, "10-\text{elim.}"] & \cdots 0\ 0 | 0\ 0\ 0 || 0\ 1\ 1\ 1\ 0 \cdots
\end{tikzcd}$$

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- Our proof is direct and elementary. It mainly relies on the fact that 10-elimination commutes with the time evolution of the BBS, up to a right-shift:

- We can also establish a relation between the rigings and the normalization coefficients, obtained from the IST-scheme for the BBS:

$$\boxed{1 + \phi_k(t) + (N-k)\,\omega_k + \sum_{\ell=1}^{k-1} \omega_\ell = -\alpha_k(t)}$$

(where ω_k is the k^{th} slowest soliton and α_k its corresponding normalization coefficient)

BBS: 10 elimination

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- Our proof is direct and elementary. It mainly relies on the fact that 10-elimination commutes with the time evolution of the BBS, up to a right-shift:

- Our proof can easily be extended to the case of the Takahashi-Matsukidaira BBS with a carrier (with finite capacity M) and box capacity $L = 1$.

BBS with carrier

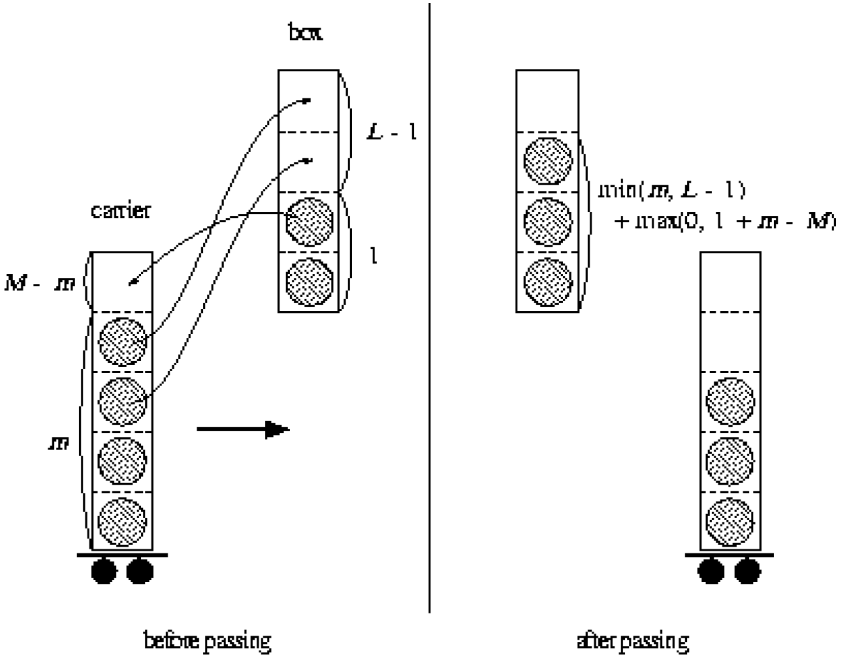
[Takahashi-Matsukidaira J. Phys. A 30 (1997) L733-L739]

$$(T_t\, u, T_\ell\, v) =: R_{\mu\lambda}(u, v) : \text{Yang-Baxter map}$$

mKdV: $\boxed{T_t\, u = v\,\dfrac{1+\mu u v}{1+\kappa u v}, \quad T_\ell\, v = \dfrac{u\, v}{T_1 u}}$

$$\downarrow \text{ud-lim}$$

$$\begin{cases} T_t\, U = V + \max[0, U + V - M] \\ \qquad\qquad - \max[0,\, U + V - L\,] \\ T_\ell\, V = V + U - T_t\, U \end{cases}$$



- This is the combinatorial $R: \; B_L \times B_M \to B_M \times B_L$, for $A_1^{(1)}$-type crystals.
- For $L = 1, M = \infty$ this system reduces to the KdV-type BBS:

$$\begin{cases} U_\ell^{t+1} + U_\ell^t = \min[1, V_\ell^t + U_\ell^t] \\ V_{\ell+1}^t + U_\ell^{t+1} = V_\ell^t + U_\ell^t \end{cases}$$

BBS with carrier: evolution rule

$\underline{L = 1, M = 2}$

t=0: $\cdots 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

BBS with carrier: evolution rule

$\underline{L = 1, M = 2}$

t=0: $\cdots 0\ 1\ 1\ \underline{1}\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

BBS with carrier: evolution rule

$\underline{L = 1, M = 2}$

t=0: $\cdots 0\ 1\ 1\ \underline{1}\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

t=1: $\cdots 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0 \cdots$

BBS with carrier: evolution rule

$\underline{L = 1, M = 2}$

t=0: $\cdots 0\ 1\ 1\ \underline{1}\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

t=1: $\cdots 0\ 0\ 0\ 1\ 1\ 0\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0 \cdots$

BBS with carrier: evolution rule

$\underline{L = 1, M = 2}$

t=0: $\cdots 0\ 1\ 1\ \underline{1}\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

t=1: $\cdots 0\ 0\ 0\ 1\ 1\ 0\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0 \cdots$

t=2: $\cdots 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0 \cdots$

t=3: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0 \cdots$

$\Rightarrow$ the maximum speed is 2 $(= M)$, even if some of the solitons are longer !

BBS with carrier: evolution rule

$\underline{L = 1, M = 2}$

t=0: $\cdots 0\ 1\ 1\ \underline{1}\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

t=1: $\cdots 0\ 0\ 0\ 1\ 1\ 0\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0 \cdots$

t=2: $\cdots 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0 \cdots$

t=3: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0 \cdots$

$\Rightarrow$ the maximum speed is 2 $(= M)$, even if some of the solitons are longer !

$\underline{L = 1, M = 3}$

t=0: $\cdots 0\ 1\ 1\ 1\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

t=1: $\cdots 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0 \cdots$

t=2: $\cdots 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \cdots$

$\Rightarrow$ changing the value of M radically changes the evolution !

BBS with carrier: ‘rigged configuration’

$\underline{L = 1, M = 3}$

$$\text{t=0:} \qquad \cdots \overset{0}{0}\ 1\ 1\ 1\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$$

BBS with carrier: ‘rigged configuration’

$\underline{L = 1, M = 3}$

t=0: $\cdots \overset{0}{0}\ 1\ 1\ 1\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

10-elim for $M = 3$: $\cdots 0\ 1\ \overset{2}{1} \,|\, 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0 \cdots$

BBS with carrier: ‘rigged configuration’

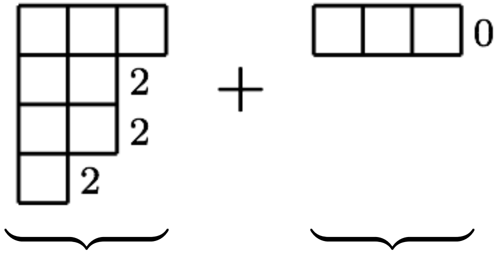
$\underline{L = 1, M = 3}$

t=0: $\cdots \overset{0}{0}\ 1\ 1\ 1\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

10-elim for $M = 3$: $\cdots 0\ 1\ \overset{2}{1} \mid \underline{1}\ 0\ 1\ 0\ 1\ \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 2$: $\cdots 0\ 1\ \overset{2}{\underline{1}} \mid\mid \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 1$: $\cdots 0\ 1\ 1\ 1\ 0 \cdots$



This yields two sets of conserved quantities: soliton speeds + extra soliton content

BBS with carrier: ‘rigged configuration’

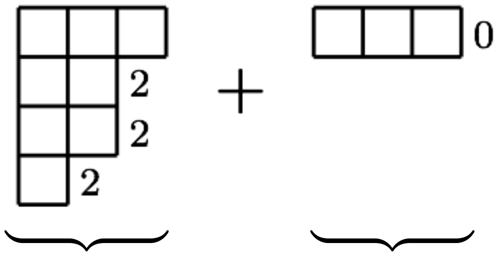
$\underline{L = 1, M = 3}$

t=0: $\cdots \overset{0}{0}\ 1\ 1\ 1\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

10-elim for $M = 3$: $\cdots 0\ 1\ \overset{2}{1} \mid \underline{1}\ 0\ 1\ 0\ 1\ \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 2$: $\cdots 0\ 1\ \overset{2}{\underline{1}} \mid \mid \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 1$: $\cdots 0\ 1\ 1\ 1\ 0 \cdots$



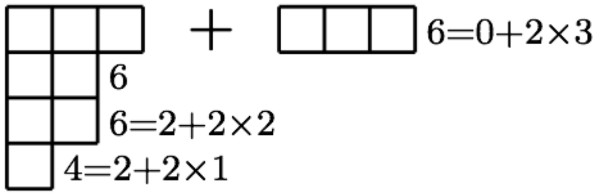
This yields two sets of conserved quantities: soliton speeds + extra soliton content

t=2: $\cdots \overset{0}{0}\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ \underline{1}\ \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 3$: $\cdots 0\ 0\ 0\ 0\ \overset{4}{0} \mid 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 2$: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ \overset{6}{0} \mid \mid 1\ \underline{1}\ \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 1$: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0 \cdots$



The ‘riggings’ evolve linearly, with the speeds of the solitons.

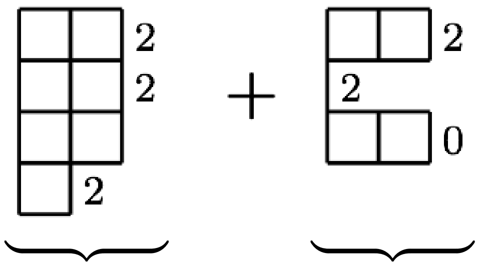
BBS with carrier: ‘rigged configuration’

$\underline{L = 1, M = 2}$

t=0: $\cdots \overset{0}{0}\ 1\ 1\ \underline{1}\ 0\ 1\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \cdots$

10-elim for $M = 2$: $\cdots 0\ 1\ \overset{2}{\underline{1}} \mid \underline{1}\ 0\ 1\ 0\ 1\ \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 1$: $\cdots 0\ 1\ \overset{2}{1} \mid \mid 1\ 1\ 0 \cdots$

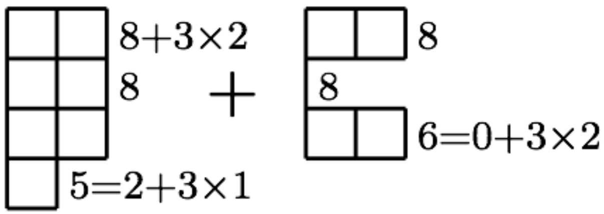


soliton speeds + extra soliton content

t=3: $\cdots \overset{0}{0}\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 2$: $\cdots 0\ 0\ 0\ 0\ 0\ \overset{5}{0} \mid 0\ 1\ \underline{1}\ \underline{1}\ 0\ 1\ 0\ 1\ \underline{1}\ \underline{1}\ 0 \cdots$

10-elim for $M = 1$: $\cdots 0\ 0\ 0\ 0\ 0\ 0\ \overset{6}{0}\ 1\ \overset{8}{1} \mid \mid 1\ 1\ 0 \cdots$



The ‘riggings’ evolve linearly with the speeds of the solitons.

Conclusions (2)

- We have shown that the time evolution of the general $A_1^{(1)}$-type BBS (with box capacity 1), can be linearized in terms of action angle variables. These can be represented (uniquely) by a rigged configuration (Young diagram + rigging) giving the soliton speeds + a 'rigged composition' for the extra soliton content.

- We believe it should be possible to extend these results to the case the $A_n^{(1)}$-type BBS with arbitrary carrier (at the very least, for box capacity 1).

- We also believe that it is possible to extend these results to the case where $L > 1$. (But this is a much harder problem !)

- However, a generalization to general L for the $A_1^{(1)}$-type BBS with infinite carrier capacity would give a combinatorial interpretation of the scattering data in the IST scheme for the Takahashi-Satsuma BBS over the rationals !