Ultradiscrete inverse scattering and an elementary linearization of the Takahashi-Satsuma box-ball system

Ralph WILLOX (the University of Tokyo)
Saburo KAKEI (Rikkyo University)
Satoshi TSUJIMOTO (Kyoto University)
Jonathan NIMMO (University of Glasgow)

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Overview of the talk

• We give a (brief) review of a solution method for a cellular automaton version of the KdV equation, reminiscent of the well-known IST scheme for the continuous KdV equation. [Joint work with A. Ramani and B. Grammaticos]

• We relate these results, in the case of the so-called Takahashi-Satsuma soliton cellular automaton (or ‘Box & Ball’ system), to certain simple combinatorial objects (rigged configurations) that offer a linearization of the BBS time evolution in terms of action-angle variables.

• These same techniques can be used, almost without modification, to provide action-angle variables for the Takahashi-Matsukidaira ‘BBS with carrier’, which is a CA version of the mKdV equation.
Overview of the talk

• (Brief) review of a solution method for a CA version of the KdV equation, reminiscent of the IST scheme for the continuous KdV equation.

The ‘ultradiscrete’ KdV equation

\[ U_{\ell}^{t+1} = \min \left[ 1 - U_{\ell}^t, \sum_{k=-\infty}^{\ell-1} (U_k^t - U_{k+1}^t) \right] \quad (U : \mathbb{Z}^2 \to \mathbb{R}) \]

with boundary conditions \( U_{\ell}^t = 0 \) for \( \ell \ll -1 (\forall t > 0) \) and initial conditions \( U_{\ell}^0 \in \mathbb{R} \) with finite support, i.e.: \( |\ell| \gg 1 : U_{\ell}^0 = 0 \).

• A solitonic system, with all information evolving from left to right.
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• A solitonic system, with all information evolving from left to right.

• Can be obtained from a suitable discretization of KdV by a special limiting procedure: the ultradiscrete limit. [Tokihiro et al. PRL 76 (1996) 3247]

(this is most easily seen on its ‘Yang-Baxter’ form:

\[
\begin{align*}
U_{\ell+1}^{t+1} + U_{\ell}^t &= \min[1, V_{\ell}^t + U_{\ell}^t] \\
V_{\ell+1}^{t+1} + U_{\ell}^t &= V_{\ell}^t + U_{\ell}^t
\end{align*}
\]
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The ‘ultradiscrete’ KdV equation

\[
U^{t+1}_\ell = \min \left[ 1 - U^t_\ell, \sum_{k=\infty}^{\ell-1} (U^t_k - U^{t+1}_k) \right] \quad (U : \mathbb{Z}^2 \to \mathbb{R})
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with boundary conditions \( U^t_\ell = 0 \) for \( \ell \ll -1 \) (\( \forall t > 0 \)) and initial conditions \( U^0_\ell \in \mathbb{R} \) with finite support, i.e.: \(|\ell| \gg 1: U^0_\ell = 0\).  

• A solitonic system, with all information evolving from left to right.
• Can be obtained from a suitable discretization of KdV by a special limiting procedure: the ultradiscrete limit. [Tokihiro et al. PRL 76 (1996) 3247]
• If \( U^0_\ell \in \{0, 1\} \), the ud-KdV evolution is closed on this set and the system contains only solitons: the Takahashi-Satsuma Box&Ball system.
The Box&Ball system

\[ U_{\ell}^{t+1} = \min \left[ 1 - U_{\ell}^t, \sum_{k=-\infty}^{\ell-1} (U_k^t - U_{k+1}^t) \right], \text{ with } U_\ell^0 \in \{0, 1\} \]
The Box&Ball system

\[ U_{\ell}^{t+1} = \min \left[ 1 - U_{\ell}^{t}, \sum_{k=-\infty}^{\ell-1} (U_{k}^{t} - U_{k}^{t+1}) \right], \text{ with } U_{\ell}^{0} \in \{0, 1\} \]

- All initial conditions decompose into solitons (i.e. into sequences of 1s that move with a speed equal to their length) and in fact, its Cauchy problem can be solved exactly.

- Soliton interactions give rise to pair-wise additive phase-shifts: when two solitons interact, the slower one is retarded by an amount twice its own speed, while the faster one is advanced by that amount.
Overview of the talk

• (Brief) review of a solution method for a CA version of the KdV equation, reminiscent of IST for the continuous KdV equation.

• Can one solve the Cauchy problem for the ud-KdV equation over the reals?
IST : continuous vs. ultradiscrete
**IST : continuous vs. ultradiscrete**

KdV: \( u_t + u_{3x} + 6uu_x = 0 \)

Lax pair for KdV:
\[
\begin{aligned}
\frac{\partial^2}{\partial x^2} \phi + u \phi &= \lambda^2 \phi \\
\frac{\partial}{\partial t} \phi &= -(4 \frac{\partial^3}{\partial x^3} \phi + 6u \frac{\partial}{\partial x} \phi + 3u_x \phi)
\end{aligned}
\]

- Suitable initial conditions \( u(x,0) \in L_1^1 := \left\{ p(\xi) \text{ measurable} \mid \int_{-\infty}^{+\infty} |p(\xi)|(1 + |\xi|)d\xi < \infty \right\} \)
  have a **finite and simple discrete spectrum**, when taken as potentials in (\( \ast \)).

- Asymptotically, a generic initial condition separates into a right-moving **solitonic** part (\( \approx \text{discrete spectrum} \)) and a **non-solitonic** remainder consisting of modulated dispersive wave trains and collisionless shock waves (\( \approx \text{continuous spectrum} \)).
**IST : continuous vs. ultradiscrete**

**KdV:**

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\end{aligned}
\]

- The contribution from the discrete spectrum is characterized by the eigenvalues \( \lambda_j \) and the right and left normalization coefficients, \( c^r_j \) and \( c^\ell_j \), for the eigenfunctions for the corresponding bound states.

- The phase-shift \( s_j \) the \( j^{th} \) soliton undergoes as \( t \) runs from \(-\infty\) to \(+\infty\) is given by

\[
s_j = \frac{1}{2\lambda_j} \log \left[ \left( \frac{c^r_j c^\ell_j}{2\lambda_j} \right)^2 \prod_{k=1}^{j-1} \left( \frac{\lambda_j - \lambda_k}{\lambda_j + \lambda_k} \right)^4 \right]
\]

for \( \lambda_1 > \lambda_2 > \cdots > \lambda_N > 0 \).

[Ablowitz & Kodama 1982 ; Ablowitz & Segur 1977]
**IST : continuous vs. ultradiscrete**

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Lax pair for KdV:

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\end{aligned}
\]

- The contribution from the continuous spectrum to the phase-shifts of the solitons can be represented in terms of the (right) reflection coefficient, \( b_r(\kappa) \), which is also part of the scattering data required in the IST scheme for KdV.

- The phase-shift \( s_j \) the \( j^{th} \) soliton undergoes as \( t \) runs from \(-\infty \) to \(+\infty \) is given by

\[
s_j = \frac{1}{\lambda_j} \sum_{k=1}^{j-1} \log \frac{\lambda_k - \lambda_j}{\lambda_k + \lambda_j} - \frac{1}{\lambda_j} \sum_{k=j+1}^{N} \log \frac{\lambda_j - \lambda_k}{\lambda_j + \lambda_k} + \frac{1}{\pi} \int_{0}^{\infty} \frac{\log(1 - |b_r(\kappa)|^2)}{\kappa^2 + \lambda_j^2} \, d\kappa
\]

for \( \lambda_1 > \lambda_2 > \cdots > \lambda_N > 0 \). [P.C. Schuur, Lect. Notes Math. 1232, Springer-Verlag (1986)]
IST: continuous vs. ultradiscrete
### IST: continuous vs. ultradiscrete

| \[ \pi \] | \[ 1 - \pi \] | \[ 1 \] | \[ 1 \] | \[ 1 \] | \[ 2\pi - 6 \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \[ 1 - \pi \] | \[ 7 - 2\pi \] | \[ 1 \] | \[ 1 \] | \[ 1 \] | \[ 4\pi - 12 \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] |
| \[ 1 - \pi \] | \[ 13 - 4\pi \] | \[ 1 \] | \[ 1 \] | \[ 1 \] | \[ 6\pi - 18 \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] | \[ \cdots \] |

**background:** speed 1  
**soliton with speed** \[ 2\pi - 1 \]
IST: continuous vs. ultradiscrete

\[
\begin{array}{cccccccccccc}
\pi & . & . & . & . & . & . & . & . & . & . & . \\
1 - \pi & 1 & 1 & 1 & 1 & 2\pi - 6 & . & . & . & . & . & . \\
1 - \pi & . & . & . & 7 - 2\pi & 1 & 1 & 1 & 1 & 4\pi - 12 & . & . & . & . \\
1 - \pi & . & . & . & . & . & . & . & . & 13 - 4\pi & 1 & 1 & 1 & 1 & 6\pi - 18 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
1 & 1 & 1 & . & -\pi & . & . & . & . & . & . & . & . & . & . \\
1 & 1 & 1 & -\pi & . & . & . & . & . & . & . & . & . & . & . \\
-2 & \pi + 1 & 4 - 2\pi & . & . & . & . & . & . & . & . & . & . & . & . \\
-\pi & 2\pi - 3 & 6 - 2\pi & . & . & . & . & . & . & . & . & . & . & . & . \\
-\pi & 2\pi - 5 & 1 & 7 - 2\pi & . & . & . & . & . & . & . & . & . & . & . \\
-\pi & 2\pi - 6 & 1 & 1 & 7 - 2\pi & . & . & . & . & . & . & . & . & . & . \\
-\pi & . & . & . & 2\pi - 6 & 1 & 1 & 7 - 2\pi & . & . & . & . & . & . & . \\
\end{array}
\]

background soliton with speed 3
Phase-shifts for ultradiscrete solitons & background

- Phase-shifts for the fast solitons ($\Delta_\omega$) and background ($\Delta_{bg}$) are given by:

$$\Delta_{\omega>1} = 2 \left( \sum_{\omega'<\omega} \omega' - \sum_{\omega'>\omega} \omega + \sum_{\ell \in \{bg\}} U_{\ell}^{(bg)} \right), \quad \Delta_{bg} = -2 \left( \sum_{\omega'>1} 1 \right)$$

(slow (= speed 1) solitons can be taken to be part of the background)
Phase-shifts for ultradiscrete solitons & background

- Phase-shifts for the fast solitons ($\Delta_\omega$) and background ($\Delta_{bg}$) are given by:

$$\Delta_{\omega>1} = 2\left(\sum_{\omega' < \omega} \omega' - \sum_{\omega' > \omega} \omega + \sum_{\ell \in \{bg\}} U_{\ell}^{(bg)}\right), \quad \Delta_{bg} = -2 \left(\sum_{\omega' > 1} 1\right)$$

(slow (= speed 1) solitons can be taken to be part of the background)

phase shift: -2       phase shift: $-2\pi$
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• (Brief) review of a solution method for a CA version of the KdV equation, reminiscent of IST for the continuous KdV equation.

• Can one solve the Cauchy problem for the ud-KdV equation over the reals?

• Step 1: relate the ud-KdV equation to a linear system and define the discrete spectrum, bound states etc. of a given initial condition.
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- Step 2: use these bound states to undress the initial condition, leaving only a soliton-less ‘background’, the Cauchy problem for which is trivial.
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• Step 3: solve the Cauchy problem for the original initial condition by dressing the explicit solution constructed for the background.
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- **Step 1:** relate the ud-KdV equation to a linear system and define the discrete spectrum, bound states etc. of a given initial condition.
A linear system for ud-KdV  

\[ U^{t+1}_\ell = \min\left[1 - U^t_\ell, \sum_{k=-\infty}^{\ell-1} (U^t_k - U^{t+1}_k)\right] \]

\[
\begin{cases}
\max[\Phi^t_{\ell+1} - \kappa, \Phi^t_{\ell-1}] = \Phi^t_\ell + \max[-U^t_\ell, U^t_{\ell-1} - 1] \\
\max[\Phi^{t+1}_{\ell+1} - \kappa, \Phi^{t+1}_{\ell-1}] = \Phi^{t+1}_\ell + \max[U^t_\ell - 1, -U^t_{\ell-1}] \\
\max[\Phi^t_\ell + \kappa - \omega, \Phi^{t+1}_\ell + U^t_\ell + \kappa - 1] = \Phi^{t+1}_{\ell+1} \\
\max[\Phi^{t+1}_{\ell+1}, \Phi^{t+1}_{\ell+1} + U^t_\ell - 1] = \Phi^t_\ell 
\end{cases}
\]

for some constants \( \kappa, \omega \geq 0 \)

• This system is ‘linear’ in the semi-field \( \mathbb{R} \cup \{\infty\}_{\max,+} \), in the sense that its solution \( \Phi^t_\ell \) is only defined up to an additive constant : \( \Phi^t_\ell \rightarrow \Phi^t_\ell + c^t \).

\[
\begin{align*}
\max[\Phi_{\ell+1}^t - \kappa, \Phi_{\ell-1}^t] &= \Phi_{\ell}^t + \max[-U_{\ell}^t, U_{\ell-1}^t - 1] \\
\max[\Phi_{\ell+1}^{t+1} - \kappa, \Phi_{\ell-1}^{t+1}] &= \Phi_{\ell}^{t+1} + \max[U_{\ell}^t - 1, -U_{\ell-1}^t] \\
\max[\Phi_{\ell}^t + \kappa - \omega, \Phi_{\ell}^{t+1} + U_{\ell}^t + \kappa - 1] &= \Phi_{\ell+1}^{t+1} \\
\max[\Phi_{\ell+1}^{t+1}, \Phi_{\ell+1}^t + U_{\ell}^t - 1] &= \Phi_{\ell}^t
\end{align*}
\]

for some constants $\kappa, \omega \geq 0$

- Consider the above system at $t = 0$, with a ‘potential’ $U_{\ell}^0$ given by some initial condition for ud-KdV (over the reals, with finite support).
Solving ud-KdV through ‘IST’ [Wilcox et al. Contemporary Mathematics 580 (2012)]

\[
\begin{align*}
\max [\Phi_{\ell+1}^0 - \kappa, \Phi_{\ell-1}^0] &= \Phi_{\ell}^0 + \max [-U_{\ell}^0, U_{\ell-1}^0 - 1] \\
\max [\Phi_{\ell+1}^1 - \kappa, \Phi_{\ell-1}^1] &= \Phi_{\ell}^1 + \max [U_{\ell}^0 - 1, - U_{\ell-1}^0] \\
\max [\Phi_{\ell}^0 + \kappa - \omega, \Phi_{\ell}^1 + U_{\ell}^0 + \kappa - 1] &= \Phi_{\ell+1}^1 \\
\max [\Phi_{\ell+1}^1, \Phi_{\ell+1}^0 + U_{\ell}^0 - 1] &= \Phi_{\ell}^0
\end{align*}
\]

for some constants \( \kappa, \omega \geq 0 \)

- Consider the above system at \( t = 0 \), with a ‘potential’ \( U_{\ell}^0 \) given by some initial condition for ud-KdV (over the reals, with finite support).

- The system has solutions with special asymptotics: \( \begin{align*}
\ell \sim -\infty: & \quad \alpha_t + \kappa \ell \\
\ell \sim +\infty: & \quad 0
\end{align*} \)

  for \( \alpha_t = \alpha_0 - \omega t \) (linear!), and which obey the dispersion relation: \( \kappa = \min [1, \omega] \).
Solving ud-KdV through ‘IST’

Definition of a bound state:

If, for $U^t_\ell$ with finite support, the ultradiscrete linear system has a solution $\Phi^t_\ell$ for some positive $\kappa$ and $\omega$, such that

$$N_\Phi := \max_{\ell \in \mathbb{Z}} \left[ \Phi^t_\ell + \Phi^t_{\ell-1} - \kappa \ell + \omega t \right] < +\infty,$$

we say that the potential $U^t_\ell$ has a bound state.

- $N_\Phi$ is invariant under the ud-KdV time evolution.
- It can be shown that, if there is a bound state for some $U^0_\ell$, the associated $\omega$ is the speed of the fastest soliton contained in that initial condition.
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$$N_\Phi := \max_{\ell \in \mathbb{Z}} [\Phi^t_{\ell} + \Phi^t_{\ell-1} - \kappa \ell + \omega t] < +\infty,$$

we say that the potential $U^t_{\ell}$ has a bound state.

- The quantity $\Omega^t_{\ell} := \Phi^t_{\ell} + \Phi^t_{\ell-1} - \kappa \ell + \omega t$ is an analogue of the squared eigenfunction for KdV.

- It has the asymptotics:
  $$\begin{align*}
  \ell \sim -\infty & : \quad 2\alpha_0 + \kappa(\ell - 1) - \omega t \\
  \ell \sim +\infty & : \quad -\kappa \ell + \omega t 
  \end{align*}$$
Solving ud-KdV through ‘IST’

**Theorem 1:**

\[ \forall \ell : \ U_\ell + U_{\ell+1} \leq 0 \quad \rightarrow \quad \text{pure background: no bound state exists} \]

\[ \exists \ell : \ 0 < U_\ell + U_{\ell+1} \leq \mu \leq 1 \quad \rightarrow \quad \text{bound state(s) exist with } \omega = \kappa = \mu \leq 1 \]

\[ (\mu: \text{maximal}) \]

\[ \exists \ell : \ U_\ell + U_{\ell+1} > 1 \quad \rightarrow \quad \text{bound state(s) exist with } \kappa = 1, \ \omega > 1 \]

- \( \omega \) is obtained uniquely by solving the system for \( \Phi^0_\ell, \Phi^1_\ell \), as \( \omega = \alpha_0 - \alpha_1 \), and at most one bound state can be found for any initial condition.

- Generically, there are no unique functions \( \Phi^0_\ell, \Phi^1_\ell \) for this bound state.

- However, \( \Phi^0_\ell, \Phi^1_\ell \) can be found algorithmically.
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• Step 1: relate the ud-KdV equation to a linear system and define the discrete spectrum, bound states etc. of a given initial condition.

• **Step 2:** use these bound states to *undress* the initial condition, leaving only a soliton-less ‘background’, the Cauchy problem for which is trivial.
Solving ud-KdV through ‘IST’

Theorem 2:

Define the following transformation of $U_\ell$, in terms of the functions $\Phi^0_\ell, \Phi^1_\ell$ that correspond to a bound state $\omega$ for that $U_\ell$:

$$U_\ell \mapsto \tilde{U}_\ell : \quad \tilde{U}_\ell = U_\ell + \Phi^0_{\ell+1} + \Phi^1_{\ell} - \Phi^0_{\ell} - \Phi^1_{\ell+1}$$

This transformation corresponds to an undressing of the potential $U_\ell$:

- The ‘mass’ of the potential $U_\ell$ decreases by $\omega$:
  $$\sum_{\ell=-\infty}^{+\infty} \tilde{U}^t_\ell = \sum_{\ell=-\infty}^{+\infty} U^t_\ell - \omega$$

- the region where large values of the sum $U_\ell + U_{\ell+1}$ occur shrinks under the undressing and $\tilde{U}_\ell$ corresponds to an initial value for ud-KdV in which a speed $\omega$ soliton was eliminated.
Solving ud-KdV through ‘IST’

Theorem 2:

Define the following transformation of $U_\ell$, in terms of the functions $\Phi_\ell^0, \Phi_\ell^1$ that correspond to a bound state $\omega$ for that $U_\ell$:

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This transformation corresponds to an *undressing* of the potential $U_\ell$:

- The ‘mass’ of the potential $U_\ell$ decreases by $\omega$:
  $$\sum_{\ell=-\infty}^{+\infty} \tilde{U}_\ell^t = \sum_{\ell=-\infty}^{+\infty} U_\ell^t - \omega$$

- the region where large values of the sum $U_\ell + U_{\ell+1}$ occur shrinks under the undressing and $\tilde{U}_\ell$ corresponds to an initial value for ud-KdV in which a speed $\omega$ soliton was eliminated.

- Hence, a finite iteration leads to a potential without bound states, i.e. a background without solitons (when considered as an initial condition for ud-KdV).
## Example of an undressing

\[
U_\ell \mapsto \tilde{U}_\ell
\]

\[
\begin{array}{cccccccc}
U_\ell : & \cdot & -1 & \pi/5 & -2 & \cdot & \cdot & \pi/3 & 1 & 1 & \pi/7 & \cdot \\
\Phi^0_\ell : & \alpha_0 -10 & \alpha_0 -9 & \alpha_0 -7 & \pi/3 - 6 & \pi/3 - 3 & \pi/3 - 2 & \pi/3 - 1 & \cdot & \cdot & \cdot & \cdot \\
\Phi^1_\ell : & \alpha_1 -10 & \alpha_1 -9 & \alpha_1 -8 & \alpha_1 -6 & -10\pi/21 - 6 & -10\pi/21 - 3 & -10\pi/21 - 2 & -\pi/7 - 2 & -\pi/7 - 1 & -\pi/7 & \cdot \\
\tilde{U}_\ell : & \cdot & \cdot & -1 & \pi/5 & -2 & \cdot & 1 - \pi/3 & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

where \( \alpha_0 = 8\pi/15 \) and \( \alpha_1 = -29\pi/105 - 1 \), and thus \( \omega := \alpha_0 - \alpha_1 = 1 + 17\pi/21 \).
Example of an undressing

\[ U_\ell \mapsto \tilde{U}_\ell \]

\[ U_\ell : \quad \cdot \quad -1 \quad \pi/5 \quad -2 \quad \cdot 
\]
\[ \Phi^0_\ell : \quad \alpha_0 - 10 \quad \alpha_0 - 9 \quad \alpha_0 - 7 \quad \pi/3 - 6 \quad \pi/3 - 3 \quad \pi/3 - 2 \quad \pi/3 - 1 \quad \cdot \quad \cdot \quad \cdot \]
\[ \Phi^1_\ell : \quad \alpha_1 - 10 \quad \alpha_1 - 9 \quad \alpha_1 - 8 \quad \alpha_1 - 6 \quad -10\pi/21 - 6 \quad -10\pi/21 - 3 \quad -10\pi/21 - 2 \quad -\pi/7 - 2 \quad -\pi/7 - 1 \quad -\pi/7 \quad \cdot \]
\[ \tilde{U}_\ell : \quad \cdot \quad \cdot \quad -1 \quad \pi/5 \quad -2 \quad \cdot 
\]
\[ \quad 1 - \pi/3 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \]

where \( \alpha_0 = 8\pi/15 \) and \( \alpha_1 = -29\pi/105 - 1 \), and thus \( \omega := \alpha_0 - \alpha_1 = 1 + 17\pi/21 \).

Compare this with the time evolution of the initial value \( U_\ell \) for udKdV :

\[ \cdot \quad -1 \quad \pi/5 \quad -2 \quad \cdot 
\]
\[ \cdot \quad \cdot \quad -1 \quad \pi/5 \quad -2 \quad 1 - \pi/3 \quad \cdot 
\]
\[ \cdot \quad \cdot \quad \cdot \quad -1 \quad \pi/5 \quad -2 \quad 1 - \pi/3 \quad \cdot 
\]
\[ \cdot \quad \cdot \quad \cdot \quad \cdot \quad 3 - 20\pi/21 \quad 1 \quad 1 \quad 1 \quad 37\pi/21 - 5 \quad \cdot 
\]

result of undressing : \( \tilde{U}_\ell \quad \text{soliton with speed } 1 + 17\pi/21 \)
Solving ud-KdV through ‘IST’

• Iteration of the undressing $U_\ell \mapsto \tilde{U}_\ell$ yields an (ordered) list of data:
\[
\left\{ (\omega^{(1)}, \alpha_0^{(1)}), (\omega^{(2)}, \alpha_0^{(2)}), \ldots, (\omega^{(N)}, \alpha_0^{(N)}) \right\}, \quad \omega^{(1)} \geq \omega^{(2)} \geq \cdots \geq \omega^{(N)}
\]
where $N$ = the \# of eliminated bound states.

• Ultimately, one obtains a background $\hat{U}_\ell$ which is free of bound states and which, as an initial value for ud-KdV, evolves undeformed at speed 1.

• The set $\left\{ \left[ (\omega^{(1)}, \alpha_0^{(1)}), (\omega^{(2)}, \alpha_0^{(2)}), \ldots, (\omega^{(N)}, \alpha_0^{(N)}) \right], \quad \hat{U}_\ell \right\}$ constitutes the ultradiscrete scattering data.
Solving ud-KdV through ‘IST’

- Iteration of the undressing $U_\ell \mapsto \tilde{U}_\ell$ yields an (ordered) list of data:

$$\left[ (\omega^{(1)}, \alpha^{(1)}_0), (\omega^{(2)}, \alpha^{(2)}_0), \ldots, (\omega^{(N)}, \alpha^{(N)}_0) \right], \quad \omega^{(1)} \geq \omega^{(2)} \geq \cdots \geq \omega^{(N)}$$

where $N = \text{the \# of eliminated bound states.}$

- Ultimately, one obtains a background $\hat{U}_\ell$ which is free of bound states and which, as an initial value for ud-KdV, evolves undeformed at speed 1.

- The set $\left\{ \left[ (\omega^{(1)}, \alpha^{(1)}_0), (\omega^{(2)}, \alpha^{(2)}_0), \ldots, (\omega^{(N)}, \alpha^{(N)}_0) \right], \quad \hat{U}_\ell \right\}$ constitutes the ultradiscrete scattering data.

This is in fact the ultradiscrete analogue of a famous theorem due to P. Deift & E. Trubowitz [Comm. Pure. Appl. Math. 12 (1979) 121-151] stating that the fastest soliton in any given initial state can be removed by Darboux transformation, without perturbing the rest of the spectrum.
Solving ud-KdV through ‘IST’

- Iteration of the undressing $U_\ell \mapsto \tilde{U}_\ell$ yields an (ordered) list of data:
  $$\left[ (\omega^{(1)}, \alpha_0^{(1)}), (\omega^{(2)}, \alpha_0^{(2)}), \ldots, (\omega^{(N)}, \alpha_0^{(N)}) \right], \quad \omega^{(1)} \geq \omega^{(2)} \geq \cdots \geq \omega^{(N)}$$
  where $N$ = the # of eliminated bound states.

- Ultimately, one obtains a background $\hat{U}_\ell$ which is free of bound states and which, as an initial value for ud-KdV, evolves undeformed at speed 1.

- The set $\left\{ \left[ (\omega^{(1)}, \alpha_0^{(1)}), (\omega^{(2)}, \alpha_0^{(2)}), \ldots, (\omega^{(N)}, \alpha_0^{(N)}) \right], \; \hat{U}_\ell \right\}$ constitutes the ultradiscrete scattering data.

- In the absence of faster solitons, the evolution of a background $\hat{U}_\ell$ can be described explicitly, for all times $t$, as: [Hirota Stud. Appl. Math. 122 (2009) 361]

\[
U_\ell^t = T_{\ell+1}^t + T_{\ell}^{t+1} - T_{\ell}^t - T_{\ell+1}^{t+1}, \quad T_{\ell}^t = \frac{1}{2} \sum_{k=-\infty}^{+\infty} \hat{U}_k |\ell - k - t|,
\]
Overview of the talk

- (Brief) review of a solution method for a CA version of the KdV equation, reminiscent of IST for the continuous KdV equation.

- Can one solve the Cauchy problem for the ud-KdV equation over the reals?

- Step 1: relate the ud-KdV equation to a linear system and define the discrete spectrum, bound states etc. of a given initial condition.

- Step 2: use these bound states to undress the initial condition, leaving only a soliton-less ‘background’, the Cauchy problem for which is trivial.

- Step 3: solve the Cauchy problem for the original initial condition by dressing the explicit solution constructed for the background.
Solving ud-KdV through ‘IST’

Theorem 3:

A solution $\tilde{U}_t^\ell$ to ud-KdV can be ‘dressed’ by the transformation:

$$\tilde{T}_t^\ell \mapsto T_t^\ell = \frac{1}{2} \max \left[ \min[1, \omega] \ell - \omega t - c + 2\tilde{T}_t^{\ell+1}, \ - \min[1, \omega] \ell + \omega t + c + 2\tilde{T}_t^{\ell-1} \right]$$

This yields a map $\tilde{U}_t^\ell = \tilde{T}_t^{\ell+1} + \tilde{T}_t^{\ell+1} - \tilde{T}_t^{\ell} - \tilde{T}_t^{\ell+1} \quad \mapsto \quad U_t^\ell = T_t^{\ell+1} + T_t^{\ell+1} - T_t^{\ell} - T_t^{\ell+1}$

which adds a speed $\omega$ soliton to $\tilde{U}_t^\ell$ (provided there are no faster solitons).
Solving ud-KdV through ‘IST’

Theorem 3:

A solution $\tilde{U}_\ell^t$ to ud-KdV can be ‘dressed’ by the transformation:

$$\tilde{T}_\ell^t \mapsto T^t_\ell = \frac{1}{2} \max \left[ \min[1,\omega] \ell - \omega t - c + 2\tilde{T}_{\ell+1}^{t+1}, - \min[1,\omega] \ell + \omega t + c + 2\tilde{T}_{\ell-1}^{t-1} \right]$$

This yields a map $\tilde{U}_\ell^t = \tilde{T}_{\ell+1}^{t+1} - \tilde{T}_\ell^{t+1} + \tilde{T}_\ell^{t-1} - \tilde{T}_{\ell+1}^{t-1} \mapsto U_\ell^t = T_{\ell+1}^{t+1} + T_{\ell-1}^{t-1} - T_\ell^t - T_{\ell+1}^{t+1}$

which adds a speed $\omega$ soliton to $\tilde{U}_\ell^t$ (provided there are no faster solitons).

- This dressing procedure yields an explicit solution, for all $\ell$ and $t$ (!)

Starting from the background $\hat{T}_\ell^t = \frac{1}{2} \sum_{k=-\infty}^{+\infty} \hat{U}_k^0 |\ell - k - t|$, solitons are inserted one by one, in reverse order, i.e.: $\omega^{(N)} \rightarrow \omega^{(N-1)} \rightarrow \cdots \rightarrow \omega^{(1)}$, with phases $c$ given by the normalization coefficients $\alpha_0^{(j)}$: $c^{(j)} = -\alpha_0^{(j)}$. 

Conclusions (1)

• A solution method for a CA version of the KdV equation, similar to the inverse scattering method for the continuous KdV equation.

In a sense, the dynamics exhibited by the BBS is ‘as rich’ as that of its discrete or continuous counterparts.

• Essential ingredients for solving the Cauchy problem are: the soliton speeds, the insertion points of the solitons in the dressing, and the background on which the solitons are superimposed.

This information is completely determined by the scattering data.

• In fact, in the case of the Takahashi-Satsuma ‘Box & Ball’ system, we can link this method to combinatorial techniques that yield a linearization of the BBS time-evolution in terms of action-angle variables, which turn out to be equivalent to the scattering data.
Combinatorics and the Cauchy problem for the BBS

- The Cauchy problem for the BBS was first solved for periodic boundary conditions (using the inverse scattering technique for the discrete Toda lattice and by taking an appropriate ultradiscrete limit.)


- For non-periodic boundary conditions, the Cauchy problem was first solved by applying a procedure called “10 elimination”

BBS: 10 elimination

- Connect all 10 pairs in the BBS state by arcs. Then, neglecting all connected pairs, connect all new 10 pairs and keep on repeating this process.
  Fact: interchanging 1s and 0s in every arc amounts to one time-step in the BBS!

![Diagram of BBS state](image)

[J. Mada et al., RIMS Kōkyūroku 1541 (2007) 15]

- The number of arcs at each stage can be recorded in a Young diagram, which encodes the speeds ($\omega_j := \omega^{(N+1-j)}$) of the solitons in the initial state.

- This Young diagram is invariant w.r.t. the BBS time evolution, but does not uniquely characterize a BBS state (as it does not give the soliton positions).
Combinatorics and the Cauchy problem for the BBS

- The Cauchy problem for the BBS was first solved for periodic boundary conditions (using the inverse scattering technique for the discrete Toda lattice and by taking an appropriate ultradiscrete limit.)

- For non-periodic boundary conditions, the Cauchy problem was first solved by applying a procedure called “10 elimination”

- The Cauchy problem for the BBS can also be solved by linearizing the evolution by means of action-angle variables [A. Kuniba et al., Nucl. Phys. B 747 (2006) 354–397], using a Kerov-Kirillov-Reshetikhin (KKR) type bijection between BBS states and so-called “rigged configurations” (which yield the action-angle variables).

BBS: 10 elimination

- Our version (based on [T. Takagi, SIGMA 6 (2010) 027] of this method is as follows:

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 5 & 7 & 11 & 13 & 14 & 15 & 17 & 18 & 21 \\
\ldots & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \ldots
\end{array}
\]
BBS: 10 elimination

- Our version (based on [T. Takagi, SIGMA 6 (2010) 027] of this method is as follows:

\[
\begin{align*}
0 & 1 & 5 & 7 & 11 & 13 & 14 & 15 & 17 & 18 & 21 \\
\vdots & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \ldots
\end{align*}
\]
BBS: 10 elimination

- Our version (based on [T. Takagi, SIGMA 6 (2010) 027] of this method is as follows:

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots \\
\cdots & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots \\
\cdots & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots \end{array}
\]
BBS: 10 elimination

- Our version (based on [T. Takagi, SIGMA 6 (2010) 027]) of this method is as follows:

\[\begin{array}{cccccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots
\end{array}\]

\[\begin{array}{cccccccccccccccccc}
0 & \cdots & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots
\end{array}\]

\[\begin{array}{cccccccccccccccccc}
0 & \cdots & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots
\end{array}\]

\[\begin{array}{cccccccccccccccccc}
0 & \cdots & 0 & 1 & 1 & 1 & 1 & 0 & \cdots
\end{array}\]
BBS: 10 elimination

- Our version (based on [T. Takagi, SIGMA 6 (2010) 027]) of this method is as follows:

\[
\begin{array}{cccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \cdots & 5 & 1
\end{array}
\]

\[
\begin{array}{cccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots & 3 & 1
\end{array}
\]

\[
\begin{array}{cccccccccccccc}
0 & 3 & 5 & 1 & 1 & 1 & 1 & 0 & \cdots & 1 & 1
\end{array}
\]
BBS: 10 elimination

- Our version (based on [T. Takagi, SIGMA 6 (2010) 027] of this method is as follows:

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 5 \\
\vdots & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & \cdots & 3 \\
0 & 3 & 1 & 1 & 1 & 1 & 0 & \cdots & 1 \\
\vdots & 0 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{array}
\]

\[\text{\vdots \ another 6 times}\]

\[\begin{array}{cccc}
3 & 5 & 9 & 10 \\
\end{array}\]

\[\begin{array}{cccc}
0 & 0 & \cdots & 0
\end{array}\]
**BBS: 10 elimination**

- This procedure yields a Young diagram (essentially that obtained by 10 elimination) which gives the speeds/lengths of the solitons contained in the initial state – here, 9, 2 (twice) and 1 (twice) – and a ‘rigging’ of the solitons, 10, 9, 5, 3 and 0, which yields the **rigged configuration**:

```
  0
  3
  5
  9
 10
```

- Because of the rigging, there is a one-to-one correspondence between the rigged configuration and a state of the BBS!

- This is a realization of the **KKR bijection**, in the case of the Takahashi-Satsuma BBS.
BBS: 10 elimination

- Such a rigged configuration offers a linearization of the BBS evolution in which the soliton speeds are the action variables and the riggings the angle variables.

\[
\begin{align*}
t=0: & \quad \cdots 0 1 1 1 1 1 1 0 0 1 1 1 1 0 0 1 0 1 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 \\t=1: & \quad \cdots 0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 0 \\
\end{align*}
\]
BBS: 10 elimination

- Such a rigged configuration offers a linearization of the BBS evolution in which the soliton speeds are the action variables and the riggings the angle variables.

\[
\begin{align*}
t=0: & \quad \cdots 0 1 1 1 1 1 0 0 1 1 1 1 0 0 1 0 1 1 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 \cdots \\
& \quad \cdots 0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 1 1 0 \cdots \\
10\text{-elim}: & \quad \cdots 0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 1 1 0 \cdots \\
& \quad \cdots 0 0 0 0 0 0 1 0 0 0 1 1 0 0 0 1 0 0 1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 \cdots
\end{align*}
\]
**BBS: 10 elimination**

- Such a rigged configuration offers a linearization of the BBS evolution in which the soliton speeds are the action variables and the riggings the angle variables.

\[ t=0: \quad \cdots 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \cdots \]

\[ t=1: \quad \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ \cdots \]

10-elim:

\[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ \cdots \]

\[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ \underbrace{0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0} \]

\[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ \underbrace{0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0} \]

\[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ \underbrace{0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1} \]

\[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ \underbrace{0 \ 0 \ 1 \ 1 \ 1 \ 1} \]
BBS: 10 elimination

• Such a rigged configuration offers a linearization of the BBS evolution in which
  the soliton speeds are the action variables and the riggings the angle variables.

\[
\begin{align*}
\text{t=0: } & \cdots 0 1 1 1 1 1 0 0 1 1 1 1 0 0 1 0 1 1 1 0 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 \cdots \\
\text{t=1: } & \cdots 0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 0 \cdots \\
\text{10-elim: } & \cdots 0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 1 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1 0 \cdots \\
& \cdots 0 0 0 0 0 0 1 0 0 0 1 0 0 1 0 0 1 1 1 1 1 1 1 1 1 0 \cdots \\
& \cdots 0 0 0 0 0 0 1 0 0 0 1 0 0 1 1 1 1 1 1 1 1 0 \cdots \\
& \cdots 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 0 \cdots \\
\end{align*}
\]

which leads to: \( \cdots 0 0 0 0 0 0 0 0 0 0 0 0 \cdots \)
BBS: 10 elimination

- Such a rigged configuration offers a linearization of the BBS evolution in which the soliton speeds are the action variables and the riggings the angle variables.

\[
\begin{align*}
t=0: & \quad \cdots 01111100111100101101110000000000000\cdots \\
t=1: & \quad \cdots 00000011000011010100100011111111110\cdots \\
10\text{-elim}: & \quad \cdots 00000011000110100100011111111110\cdots \\
& \quad \cdots 000000010001100011111111110\cdots \\
& \quad \cdots 000000000000000111111110\cdots \\
\end{align*}
\]

which leads to: \( \cdots 000000000000000000000\cdots \)
BBS: 10 elimination

- It can be shown that, for any BBS state, the riggings $\phi_k$ depend linearly on $t$:

$$\phi_k(t) = \phi_k(0) + \omega_k t,$$

in terms of the soliton speeds $\omega_k$ (which are constant).


- Our proof is direct and elementary. It mainly relies on the fact that 10-elimination commutes with the time evolution of the BBS, up to a right-shift:

$$
\cdots 0 1 0 0 1 1 1 0 1 0 1 1 0 0 0 0 0 \cdots \overset{10\text{-elim.}}{\to} \cdots 0 1 1 1 0 0 0 0 0 \cdots
$$

\[t \to t + 1 \downarrow\]

$$
\cdots 0 0 1 0 0 0 0 1 0 1 0 0 1 1 1 1 0 \cdots \overset{10\text{-elim.}}{\to} \cdots 0 0 0 0 0 0 1 1 1 0 \cdots
$$
BBS: 10 elimination

• It can be shown that, for any BBS state, the riggings $\phi_k$ depend linearly on $t$:
  \[ \phi_k(t) = \phi_k(0) + \omega_k t, \]
  in terms of the soliton speeds $\omega_k$ (which are constant).


• Our proof is direct and elementary. It mainly relies on the fact that 10-elimination commutes with the time evolution of the BBS, up to a right-shift:

• We can also establish a relation between the rigings and the normalization coefficients, obtained from the IST-scheme for the BBS:
  \[ 1 + \phi_k(t) + (N - k) \omega_k + \sum_{\ell=1}^{k-1} \omega_\ell = -\alpha_k(t) \]
  (where $\omega_k$ is the $k^{th}$ slowest soliton and $\alpha_k$ its corresponding normalization coefficient)
BBS: 10 elimination

- It can be shown that, for any BBS state, the riggings $\phi_k$ depend linearly on $t$:

$$\phi_k(t) = \phi_k(0) + \omega_k t,$$

in terms of the soliton speeds $\omega_k$ (which are constant).


- Our proof is direct and elementary. It mainly relies on the fact that 10-elimination commutes with the time evolution of the BBS, up to a right-shift:

- Our proof can easily be extended to the case of the Takahashi-Matsukidaira BBS with a carrier (with finite capacity $M$) and box capacity $L = 1$. 
BBS with carrier

\[(T_t u, T_\ell v) =: R_{\mu\lambda}(u, v) : \text{Yang-Baxter map}\]

**mKdV:**

\[
\begin{align*}
T_t u &= v \frac{1 + \mu uv}{1 + \kappa uv}, \\
T_\ell v &= \frac{uv}{T_1 u} \\
\end{align*}
\]

\[
\downarrow \text{ud-lim}
\]

\[
\begin{cases}
T_t U = V + \max[0, U + V - M] \\
- \max[0, U + V - L] \\
T_\ell V = V + U - T_t U
\end{cases}
\]

- This is the combinatorial \( R : B_L \times B_M \to B_M \times B_L \), for \( A_1^{(1)} \)-type crystals.
- For \( L = 1, M = \infty \) this system reduces to the KdV-type BBS:

\[
\begin{cases}
U_{\ell+1}^{t+1} + U_\ell^t = \min[1, V_\ell^t + U_\ell^t] \\
V_{\ell+1}^t + U_\ell^{t+1} = V_\ell^t + U_\ell^t
\end{cases}
\]
BBS with carrier: evolution rule

$L = 1, M = 2$

$t=0$: $\cdots 011101100110011110000000\cdots$
BBS with carrier: evolution rule

$L = 1, M = 2$

$t=0: \quad \cdots 011101100110011110000000000\cdots$
BBS with carrier: evolution rule

$L = 1, M = 2$

\[ t=0: \quad \cdots 0111011001100111000000000\cdots \]

\[ t=1: \quad \cdots 000110111001100111100000000\cdots \]
BBS with carrier: evolution rule

$L = 1, M = 2$

\[ t=0: \quad \cdots 011\_101\_10011001111000000000\cdots \]

\[ t=1: \quad \cdots 0001101\_11001100111110000000\cdots \]
BBS with carrier: evolution rule

$L = 1, M = 2$

$t=0$: $\cdots 0 \_1 \_1 0 \_1 \_1 0 0 1 1 0 0 1 1 \_1 \_1 0 0 0 0 0 0 0 \cdots$

t=1: $\cdots 0 0 0 1 1 0 1 1 1 0 0 1 1 0 0 1 1 1 1 0 0 0 0 0 \cdots$

t=2: $\cdots 0 0 0 0 1 0 1 1 1 1 0 0 1 1 0 0 1 1 1 1 0 0 0 \cdots$

t=3: $\cdots 0 0 0 0 0 0 1 0 0 1 1 1 1 0 0 1 1 0 0 1 1 1 1 0 \cdots$

$\Rightarrow$ the maximum speed is 2 ($= M$), even if some of the solitons are longer!
BBS with carrier: evolution rule

\(L = 1, M = 2\)

\(t=0: \quad \cdots 0111011001111100000000\cdots\)
\(t=1: \quad \cdots 0001101110011001111000000\cdots\)
\(t=2: \quad \cdots 0000010111100110011110000\cdots\)
\(t=3: \quad \cdots 00000010011111001100111110\cdots\)

\(\Rightarrow\) the maximum speed is 2 (= \(M\)), even if some of the solitons are longer!

\(L = 1, M = 3\)

\(t=0: \quad \cdots 0111011001111100000000\cdots\)
\(t=1: \quad \cdots 0000101110011001111110000\cdots\)
\(t=2: \quad \cdots 0000001000110011001111110\cdots\)

\(\Rightarrow\) changing the value of \(M\) radically changes the evolution!
**BBS with carrier: ‘rigged configuration’**

$L = 1, M = 3$

$t=0$: \[011101100110011110000000000\cdots\]
**BBS with carrier: ‘rigged configuration’**

$L = 1, M = 3$

\[
\begin{array}{c}
\begin{array}{cccccccccccccccccccccc}
\end{array}
\end{array}
\]

\[
t=0: \quad \cdots 0 1 1 1 0 1 1 0 0 1 1 0 0 1 1 1 1 0 0 0 0 0 0 0 0 \cdots
\]

10-elim for $M = 3$: \[
\cdots 0 1 \overline{1} 1 0 1 0 1 1 1 0 \cdots
\]
**BBS with carrier: ‘rigged configuration’**

\[ L = 1, M = 3 \]

\[ t=0: \quad \cdots 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \]

10-elim for \( M = 3 \): \[ \cdots 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \cdots \]

10-elim for \( M = 2 \): \[ \cdots 0 \ 1 \ 1 \ 1 \ 1 \ 0 \cdots \]

10-elim for \( M = 1 \): \[ \cdots 0 \ 1 \ 1 \ 1 \ 0 \cdots \]

This yields two sets of conserved quantities:  
- soliton speeds  
- extra soliton content
**BBS with carrier: ‘rigged configuration’**

$\mathcal{L} = 1, M = 3$

$t=0$: \[ \cdots 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots \]

10-elim for $M = 3$: \[ \cdots 0 \ 1 \ 1 \ \underline{1} \ \underline{0} \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \cdots \]

10-elim for $M = 2$: \[ \cdots 0 \ 1 \ \underline{1} \ \underline{1} \ 0 \cdots \]

10-elim for $M = 1$: \[ \cdots 0 \ 1 \ 1 \ 1 \ 0 \cdots \]

This yields two sets of conserved quantities: \soliton speeds + extra soliton content

$t=2$: \[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \cdots \]

10-elim for $M = 3$: \[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \cdots \]

10-elim for $M = 2$: \[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \cdots \]

10-elim for $M = 1$: \[ \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \cdots \]

The ‘riggings’ evolve linearly, with the speeds of the solitons.
**BBS with carrier: ‘rigged configuration’**

\[ L = 1, M = 2 \]

\[
t=0: \quad \cdots 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \cdots
\]

10-elim for \( M = 2 \):
\[
\cdots 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ \cdots
\]

10-elim for \( M = 1 \):
\[
\cdots 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ \cdots
\]

\[
\text{soliton speeds } + \text{ extra soliton content}
\]

\[
t=3: \quad \cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ \cdots
\]

10-elim for \( M = 2 \):
\[
\cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ \cdots
\]

10-elim for \( M = 1 \):
\[
\cdots 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ \cdots
\]

\[
\text{The ‘riggings’ evolve linearly with the speeds of the solitons.}
\]
Conclusions (2)

• We have shown that the time evolution of the general $A_1^{(1)}$-type BBS (with box capacity 1), can be linearized in terms of action angle variables. These can be represented (uniquely) by a rigged configuration (Young diagram + rigging) giving the soliton speeds + a ‘rigged composition’ for the extra soliton content.

• We believe it should be possible to extend these results to the case the $A_n^{(1)}$-type BBS with arbitrary carrier (at the very least, for box capacity 1).

• We also believe that it is possible to extend these results to the case where $L > 1$. (But this is a much harder problem !)

• However, a generalization to general $L$ for the $A_1^{(1)}$-type BBS with infinite carrier capacity would give a combinatorial interpretation of the scattering data in the IST scheme for the Takahashi-Satsuma BBS over the rationals !