LOCAL NORMAL FORMS OF NONCOMMUTATIVE FUNCTIONS

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Abstract. This article describes local normal forms of functions in noncommuting variables, up to equivalence generated by isomorphism of noncommutative Jacobi algebras, extending singularity theory in the style of Arnold's commutative local normal forms into the noncommutative realm. This generalisation unveils many new phenomena, including an ADE classification when the Jacobi ring has dimension zero and, by suitably taking limits, a further ADE classification in dimension one. These are natural generalisations of the simple singularities and those with infinite multiplicity in Arnold's classification. We obtain normal forms away from some exceptional Type E cases. Remarkably these normal forms have no moduli, and the key new feature is that the noncommutative world affords larger families, and there are many more examples of each type.

The first application of noncommutative singularity theory is to the birational geometry of 3-folds. We prove that all local normal forms of Type A and D are geometric, in the sense that each one gives rise to the contraction algebra of some smooth 3-fold flop or divisor-to-curve contraction. The general elephant of the corresponding contraction has matching type, and so this fully classifies contraction algebras of flops of length one and two. In the process, we describe the first and conjecturally only infinite family of length two crepant divisor-to-curve contractions. A further consequence is the classification of Gopakumar–Vafa invariants for length two flops, with noncommutative obstructions forcing gaps in the invariants that can arise.

1. Introduction

Any smooth point on a d-dimensional variety is, up to complete local change in coordinates, the power series ring in d variables. Complete locally, all smooth points look the same. A similar phenomenon applies to more interesting singular points: as observed by Arnold and others in various landmark papers [A1, M3, BGS], if we pass to the completion, there are many fewer singular points than might naively be expected, and they often have surprising connections to other parts of mathematics. The purpose of this paper is to extend both the theory and the key classifications into the noncommutative context. In the process, we uncover surprising algebraic facts with geometric corollaries.

1.1. Noncommutative Singularity Theory.

For $d \geq 1$ consider the noncommutative formal power series ring $\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle = \mathbb{C}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle$, which is the complete local version of the free algebra. From the perspective of this paper, the algebra $\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$ replaces the commutative power series ring $\mathbb{C}[\![x_1, \ldots, x_d]\!]$ from classical singularity theory.

For any $f \in \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$, it is possible to cyclically differentiate f with respect to a variable x_i to obtain an element $\delta_i f$. The collection of such elements generate a closed two-sided ideal $(\!(\delta_1 f, \ldots, \delta_d f)\!)$, the details of which are recalled in §2.2. The resulting quotient

$$\mathcal{J}\mathrm{ac}(f) = \frac{\mathbb{C}\langle\!\langle x_1, \ldots, x_d \rangle\!\rangle}{(\!(\delta_1 f, \ldots, \delta_d f)\!)}$$

is called the *Jacobi algebra* of f, and the element f is called the *potential*.

We will regard f and g as being equivalent if their Jacobi algebras are isomorphic, remarking that in the noncommutative setting, given the hidden dependence on cyclic equivalence, naive versions of the Tjurina algebra do not exist (see 4.2). With the ring $\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$ fixed and the equivalence relation established, the overarching aim of singularity theory remains: to classify all equivalence classes of potentials satisfying numerical criteria, and to develop powerful theory in the situation where classification is not possible.

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Whenever $d > 1$ the algebra $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ is not noetherian, and the exponential explosion in its growth means that factoring by only d elements often results in Jacobi algebras with pathological properties. As in the classical case, pathologies turn out not to matter: the complexity of *some* singularities prevents neither the development of a general theory, nor various classification results for those which satisfy reasonable numerical conditions.

Writing $\mathfrak{J}$ for the Jacobson radical of $\mathcal{J}\mathrm{ac}(f)$, the first and natural restriction to impose on f is to numerically constrain the growth of successive quotients of the chain of ideals

$$\mathcal{J}\mathrm{ac}(f) \supseteq \mathfrak{J} \supseteq \mathfrak{J}^2 \supseteq \dots.$$

This numerical growth, defined in 3.4, is called the $\mathfrak{J}$-dimension, and will be written $\mathrm{Jdim}\,\mathcal{J}\mathrm{ac}(f)$. As explained in 3.6, since $\mathcal{J}\mathrm{ac}(f)$ is a factor of a complete ring, there is no reasonable Gelfand–Kirillov dimension, and so the $\mathfrak{J}$-dimension replaces it.

Alongside the development of a more general theory, one of our motivating problems is to extend Arnold-style classification of germs into the above noncommutative setting.

Problem 1.1. For any finite $n \geq 0$, produce a set of potentials $\mathcal{S}_n$ that realise every Jacobi algebra of $\mathfrak{J}$-dimension n, up to isomorphism.

We furthermore insist that the elements of $\mathcal{S}_n$ should be *normal forms*, namely that if $f, g \in \mathcal{S}_n$ with $f \neq g$, then the resulting Jacobi algebras are not isomorphic. Building on foundational algebraic results of Iyudu–Shkarin [IS], in Appendix A we show that, for small $\mathfrak{J}$-dimension n, 1.1 essentially reduces to a problem in $d \leq 3$ variables.

Below we will focus mainly on the situation $n \leq 1$, which is already highly non-trivial. Exactly as in Arnold, such precise numerical restrictions are often only motivated afterwards, by their answer and by the incredibly rich families that they describe. The restriction $n \leq 1$ is also, happily, the condition needed for the applications to birational geometry. We do however remark that it is not even clear that the set $\mathcal{S}_0$ is countable, never mind $\mathcal{S}_1$, and there is certainly no prima facie reason why ADE should enter.

1.2. **Noncommutative ADE Normal Forms.** We now introduce the ADE families that will turn out to solve 1.1 when $n \leq 1$. The main results regarding what precisely these families classify are stated later, in §1.3.

As remarked by Arnold, it is often not possible to rigorously define a series until *after* it has been classified. In the subsections that follow we will use various different phenomena to explain the ADE names of the families, but it is only after classification that one can make the moves needed to extract this ADE information. As such, the definition of the families below follows the usual pattern of classical singularity theory: their definition comes first, and their justification comes afterwards.

Below, we view the families with $n = 0$ as the noncommutative version of simple singularities à la Arnold, whereas we view the $n = 1$ families as the 'limit' of the $n = 0$ case, and thus the noncommutative version of the tame singularities A_∞ and D_∞ of [BGS].

With the above caveats, for any $d \geq 2$ consider the following families, which below will fully classify the $n = 0$ case in 1.1. The big $\mathcal{O}$ notation is explained in §1.7.

Type	**Name**	**Normal form**	**Conditions**
A	A_n	$z_1^2 + \ldots + z_{d-2}^2 + x^2 + y^n$	$n \geq 2$
D	$D_{n,m}$	$z_1^2 + \ldots + z_{d-2}^2 + xy^2 + x^{2n} + x^{2m-1}$	$n, m \geq 2, m \leq 2n-1$
	$D_{n,\infty}$	$z_1^2 + \ldots + z_{d-2}^2 + xy^2 + x^{2n}$	$n \geq 2$
E	$E_{6,n}$	$z_1^2 + \ldots + z_{d-2}^2 + x^3 + xy^3 + y^n$	$n \geq 4$
		$z_1^2 + \ldots + z_{d-2}^2 + x^3 + \mathcal{O}_4$	(various cases)

It is possible to write Type D in the unified manner $z_1^2 + \ldots + z_{d-2}^2 + xy^2 + x^{2n} + \varepsilon x^{2m-1}$ where ε is either 0 or 1, but often it will be preferable to regard them as two distinct families, both of Type D. In addition to the fact that Type D is larger than in the classical case, what is perhaps much more remarkable is that in Type E there are infinitely many cases: the family $E_{6,n}$ stated, together with various other examples all of the form $x^3 + \mathcal{O}_4$, whose expressions are more complicated, and will be optimised elsewhere [BW2].

Taking the limit $n \to \infty$ of the above forms gives the following, where again all are optimised, except the very last line.

Type	Name	Normal form	Conditions
A	A_∞	$z_1^2 + \ldots + z_{d-2}^2 + x^2$	
D	$D_{\infty,m}$	$z_1^2 + \ldots + z_{d-2}^2 + xy^2 + x^{2m-1}$	$m \geq 2$
	$D_{\infty,\infty}$	$z_1^2 + \ldots + z_{d-2}^2 + xy^2$	
E	$E_{6,\infty}$	$z_1^2 + \ldots + z_{d-2}^2 + x^3 + xy^3$	
		$z_1^2 + \ldots + z_{d-2}^2 + x^3 + \mathcal{O}_4$	

The classical case admits precisely two examples, namely the singularities A_∞ and D_∞ of [BGS]. The above noncommutative families are thus again larger: Type D splits into two and there are infinitely many examples within $D_{\infty,m}$, and Type E is non-empty.

With the benefit of hindsight, there are two reasons why to expect that the $n = 1$ case should be the limits of $n = 0$. First, as Arnold remarks, his simple A_n and D_n families give rise, under limits, to the germs x^2 and xy^2, and the noncommutative families above generalise this passage from the isolated to the non-isolated. Second, in terms of the birational geometry of §1.6 below, contraction algebras should make sense of the feeling that divisor-to-curve contractions are limits of infinite families of flops.

In this paper, we prove that every $\mathcal{J}\mathrm{ac}(f)$ with $\mathrm{Jdim}\,\mathcal{J}\mathrm{ac}(f) \leq 1$ is isomorphic to a normal form in Type A or D above, or has the general form stated for E. We remark that the precise Type E normal forms stated, namely $E_{6,n}$ and $E_{6,\infty}$, are indeed genuine examples with $\mathfrak{J}$-dimension zero and one respectively. However, we refrain from describing the general case here, as we will treat all the exceptional Type E cases together, in a more technical companion paper [BW2].

We now outline our results in more detail, before describing their applications.

1.3. **Main Noncommutative Singularity Theory Results.** Since constants differentiate to zero, and elements with linear terms differentiate to units, we can and do assume that f has only quadratic and higher terms, which we write as $f \in \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle_{\geq 2}$ or equivalently as an explicit sum of its homogeneous pieces

$$f = f_2 + f_3 + f_4 + \ldots$$

Just as in the classical theory, a Splitting Lemma 4.5 identifies a coordinate system which separates variables of the non-degenerate quadratic part from variables of a higher order potential, so that without loss of generality

$$f = x_1^2 + \cdots + x_r^2 + f_{\geq 3}(x_{r+1}, \ldots, x_d)$$

and thus we may turn attention to the potential $f_{\geq 3}$ in, typically, fewer variables. The number $d - r$ is called the *corank*, and as in the classical case there is a more intrinsic way of characterising it (4.3), namely as

$$\mathcal{C}\mathrm{rk}(f) = d - \dim_{\mathbb{C}} \left(\frac{\mathfrak{n}^2 + I}{\mathfrak{n}^2} \right) \tag{1.A}$$

where $\mathfrak{n} = (x_1 \ldots, x_d)$ and $I = \langle\!\langle \delta_1 f, \ldots \delta_d f \rangle\!\rangle$. By the above and A.18 it turns out, in a manner pleasantly reminiscent of classical simple singularities, that the case when $\mathrm{Jdim}\,\mathcal{J}\mathrm{ac}(f) \leq 1$ reduces to that of two variables. We rename the variables $z_1, \ldots, z_{d-2}, x, y$ to emphasise this fact.

The following, a consequence of the Splitting Lemma together with a degree three preparation result then characterises commutative Jacobi algebras in two variables. These are precisely our Type A families in §1.2. Below we adopt the convenient abuse of notation $f \cong g$ to mean $\mathcal{J}\mathrm{ac}(f) \cong \mathcal{J}\mathrm{ac}(g)$.

Proposition 1.2 (5.1, 5.4)**.** *If $f \in \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle_{\geq 2}$, then the following hold.*

(1) $\mathcal{C}\mathrm{rk}(f) \leq 1$ *if and only if*

$$f \cong \begin{cases} z_1^2 + \ldots + z_{d-2}^2 + x^2 \\ z_1^2 + \ldots + z_{d-2}^2 + x^2 + y^n & \textit{for some } n \geq 2. \end{cases}$$

Each member of the bottom family has finite dimensional Jacobi algebra, whereas in the top case the algebra is infinite dimensional, with $\operatorname{Jdim}\mathfrak{Jac}(f) = 1$.

(2) *If* $d = 2$, *i.e.* $f \in \mathbb{C}\langle\langle x, y\rangle\rangle$, *then* $\mathfrak{Jac}(f)$ *is commutative if and only if* $\mathfrak{Crk}(f) \leq 1$.

Thus most Jacobi algebras are strictly not commutative and so new noncommutative invariants are needed to classify them. The equation (1.A) does admit an obvious generalisation, namely the *higher coranks* defined in §4.1, where for $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle_{\geq 3}$ the second corank is

$$\mathfrak{Crk}_2(f) = d^2 - \dim_{\mathbb{C}}\left(\frac{\mathfrak{n}^3 + I}{\mathfrak{n}^3}\right). \tag{1.B}$$

In classifying all f with $\operatorname{Jdim}\mathfrak{Jac}(f) \leq 1$, A.18 together with 1.2 then reduces us to the case where $\mathfrak{Crk}(f) = 2$ and $\mathfrak{Crk}_2(f) = 2, 3$. The lowest case $\mathfrak{Crk}_2(f) = 2$ turns out to be given by the Type D families in the tables of §1.2.

Theorem 1.3 (6.18)**.** *Suppose that* $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle_{\geq 2}$ *with* $\mathfrak{Crk}(f) = 2$ *and* $\mathfrak{Crk}_2(f) = 2$.

(1) *Then either*

$$f \cong \begin{cases} z_1^2 + \ldots + z_{d-2}^2 + xy^2 & & D_{\infty,\infty} \\ z_1^2 + \ldots + z_{d-2}^2 + xy^2 + x^{2m+1} & \textit{with } m \geq 1 & D_{\infty,m} \\ z_1^2 + \ldots + z_{d-2}^2 + xy^2 + x^{2n} & \textit{with } n \geq 2 & D_{n,\infty} \\ z_1^2 + \ldots + z_{d-2}^2 + xy^2 + x^{2n} + x^{2m+1} & \textit{with } 2n-2 \geq m \geq n \geq 2 & D_{n,m} \\ z_1^2 + \ldots + z_{d-2}^2 + xy^2 + x^{2m+1} + x^{2n} & \textit{with } n > m \geq 1 & D_{n,m} \end{cases}$$

These f *all have mutually non-isomorphic Jacobi algebras.*

(2) *Furthermore, those labelled* $D_{\infty,*}$ *satisfy* $\operatorname{Jdim}\mathfrak{Jac}(f) = 1$, *whilst those labelled* $D_{n,*}$ *satisfy* $\operatorname{Jdim}\mathfrak{Jac}(f) = 0$.

It is remarkable that all normal forms are polynomial, and even more remarkable that all coefficients are integers. Indeed, all coefficients equal 1, and there are no moduli.

The last remaining case for which $\operatorname{Jdim}\mathfrak{Jac}(f) \leq 1$ holds is when $\mathfrak{Crk}(f) = 2$ and $\mathfrak{Crk}_2(f) = 3$. After a suitable change in coordinates, all such f have the form

$$f \cong z_1^2 + \ldots + z_{d-2}^2 + x^3 + f_{\geq 4}(x, y).$$

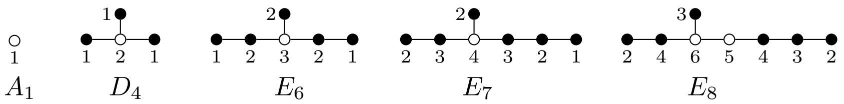
with some extra conditions on $f_{\geq 4}(x, y)$ that ensures $\operatorname{Jdim}\mathfrak{Jac}(f) \leq 1$. We refer to these potentials as Type E. As stated in §1.2, the families described in both Types $E_{6,n}$ and $E_{6,\infty}$ are genuine examples. However there are many others; see [BW2]. Their classification depends, in a rather more subtle manner, on naturally-defined higher coranks (see 4.9). For example, the potential $x^3 + xy^3$ of Type $E_{6,\infty}$ has second corank equal to 3, with all higher coranks equal to 4, while in contrast the potentials f_n of Type $E_{6,n}$ for $n \geq 5$ trim those coranks to

$$\mathfrak{Crk}_2(f_n), \mathfrak{Crk}_3(f_n), \ldots, \mathfrak{Crk}_{n+6}(f_n) = 3, 4, 4, \ldots, 4, 4, 3, 3, 2, 1, 1.$$

In particular $\mathfrak{Jac}(f_n)$ has dimension $4(n+3)$. Controlling normal forms in such situations is both theoretically and computationally more difficult.

1.4. **Extracting ADE.** It turns out that there are two, completely distinct, ways to extract ADE behaviour from the families defined above, and thus explain the ADE naming conventions. In this section we explain the purely algebraic method; the birational geometry method is explained in §1.5 below.

As notation, consider the following ADE Dynkin diagrams, which we also furnish with the information of their highest roots.



(1.C)

To each such Dynkin diagram, there is an associated preprojective algebra Π (see e.g. [CBH]), which is a finite dimensional algebra. The vertices of the Dynkin diagram give rise to idempotents in the corresponding Π. In each diagram in (1.C), let e be the idempotent corresponding to the unique vertex marked $\circ$, except for E_8 when there are two cases: e is either the left $\circ$ or the right $\circ$. From this information, consider the algebra $e\Pi e$.

The following result allows us to associate ADE information directly to the normal forms in §1.2. Nothing in the definition of the families has involved any mention of the preprojective algebra, and aside from our naming conventions, any mention of ADE. A priori, it is not even clear that if $\operatorname{Jdim}\mathfrak{Jac}(f) \leq 1$, then $\mathfrak{Jac}(f)$ admits a non-unit central element. In order to consider all cases $\operatorname{Jdim}\mathfrak{Jac}(f) \leq 1$ together, below we adopt the convention that each ε_i can be either 0 or 1.

Theorem 1.4 (7.5)**.** *Consider the normal forms A_n, $D_{n,m}$, $D_{n,\infty}$, $E_{6,n}$, A_∞, $D_{\infty,m}$, $D_{\infty,\infty}$ and $E_{6,\infty}$ from §1.2. In each case, define an element s as follows*

Type	Normal form	Conditions	s
A	$z_1^2+\ldots+z_{d-2}^2+x^2+\varepsilon_1 y^n$	$n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$	y
D	$z_1^2+\ldots+z_{d-2}^2+xy^2+\varepsilon_2 x^{2n}+\varepsilon_3 x^{2m-1}$	$m, n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$	x^2
E	$z_1^2+\ldots+z_{d-2}^2+x^3+xy^3+\varepsilon_4 y^n$	$n \in \mathbb{N}_{\geq 4}$	$g_{6,n}$

where $g_{6,n}$ is defined in §7. Then the following statements hold.

1. *The element s is central in $\mathfrak{Jac}(f)$, and $\mathfrak{Jac}(f)/(s) \cong e\Pi e$, where Π is the preprojective algebra of Type A_1, D_4, or E_6, and e is the idempotent marked $\circ$.*
2. *In Type A and D, a generic central element g satisfies $\mathfrak{Jac}(f)/(g) \cong e\Pi e$.*

Most of the content in the theorem lies within the second part since generic elements, defined in 7.2, provide an *intrinsic* method of extracting the ADE information. The choice of central element $g_{6,n}$, which is rather involved, works for Type $E_{6,*}$, and there is also strong evidence that generic elements there also quotient to give $e\Pi e$. Establishing this is computationally much harder, and will be addressed elsewhere [BW2]. We remark that all other examples we know within Type E, but which are not explicitly stated above, also factor to some $e\Pi e$ where Π is the preprojective algebra of some Type E Dynkin diagram, and e is one of the idempotents marked $\circ$ in (1.C). All such diagrams, and all such indicated choices of vertex $\circ$ arise.

In the geometric context of §1.6 below, the generic central element g of 1.4 should be thought of as the noncommutative version of Reid's general elephant [R2]. Remarkably, the above theorem neither implies, nor is implied by, Reid's version. In both cases, taking a *generic* central element is essential to avoid pathological behaviour (see 7.6).

1.5. **Geometric Corollaries.** The noncommutative singularity results in §1.3 have immediate applications in birational geometry, primarily to invariants of 3-fold flops and 3-fold crepant divisor-to-curve contractions. We first very briefly recall this specific setting (for more general background see e.g. [KMM]), and then outline our new results.

Given any crepant projective birational morphism $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$, where $\mathcal{R}$ is complete local cDV singularity, there is an associated *contraction algebra* $\mathrm{A}_{\mathrm{con}}$ formed by considering noncommutative deformations of the curves above the unique closed point [DW1, DW3]. This is the finest known curve invariant associated to the contraction. When the contraction is furthermore *simple*, namely the reduced fibre above the origin is $\mathbb{P}^1$, and further $\mathcal{X}$ is smooth, then it is well known [DW1, V1] that $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{Jac}(f)$ for some $f \in \mathbb{C}\langle\langle x, y\rangle\rangle$ (see e.g. [BW, 3.1(2)]).

Since cDV singularities are normal, necessarily $\operatorname{Jdim}\mathrm{A}_{\mathrm{con}} \leq 1$, and there is a natural geometric dichotomy. Indeed, as explained in 8.5, if $\mathrm{A}_{\mathrm{con}}$ is a contraction algebra associated to a crepant $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$ as above, then

1. $\operatorname{Jdim}\mathrm{A}_{\mathrm{con}} = 0$ if and only if $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$ is a flop, and
2. $\operatorname{Jdim}\mathrm{A}_{\mathrm{con}} = 1$ if and only if $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$ is a divisorial contraction to a curve.

The only other fact we will require is that every $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$ has an associated ADE type, since by Reid's general elephant theorem [R2] a generic $g \in \mathfrak{m}$ slices to give an ADE surface singularity $\mathcal{R}/g$. We will say $\mathcal{R}$ has Type D if the generic slice is Type D, etc.

With this in mind, the results in §1.3 have immediate corollaries. The Type A classification of contraction algebras follows from Reid's Pagoda family [R2], which has been known since the early 1980s. The following extends this into Type D, and is more remarkable since in contrast Type D flops have yet to be classified. It furthermore gives the second method to extract ADE information from the normal forms in §1.2.

Theorem 1.5 (8.2, 8.9, 8.13)**.** *With notation as above, the following hold:*

(1) *The only contraction algebras for Type A and D flops are, up to isomorphism, the Jacobi algebras of the potentials in* 1.2 *and* 1.3 *that have* $\mathfrak{J}$*-dimension* 0.
(2) *All Jacobi algebras in* 1.2 *and* 1.3 *arise from geometry.*
(3) *Furthermore, if* $\mathrm{A}_{\mathrm{con}}$ *is the contraction algebra of a Type E flop, then* $\mathrm{A}_{\mathrm{con}}$ *is not isomorphic to the Jacobi algebra of any potential in either* 1.2 *and* 1.3.

We say precisely which potentials in 1.2 and 1.3 correspond to flops, and which correspond to divisorial contractions to curves; full details are given in 6.18. In the process of establishing 1.5 we use the examples of flops given in our previous work [BW], together with their generalisations [vG, Ka].

An immediate corollary of 1.5 is the first complete description of the possible curve-counting Gopakumar–Vafa (GV) invariants that can arise for Type D flops.

Corollary 1.6 (8.10)**.** *Consider* $(a, b) \in \mathbb{N}^2$*. Then* (a, b) *are the GV invariants for a Type D flopping contraction if and only if either*

(1) $(a, b) = (2m + 3, m)$ *for some* $m \geq 1$*, or*
(2) $(a, b) = (2n, b)$ *for some* $n \geq 2$*, with* $b \geq n - 1$.

Further, when $a = 2m+3$ *there are precisely* $m+1$ *distinct contraction algebras realising* (a, b)*, up to isomorphism, whilst for any given* $(2n, b)$ *the contraction algebra is unique.*

The above GV invariants are sketched in Figure 1, where perhaps the most striking aspect is that not all pairs (n_1, n_2) can be realised.

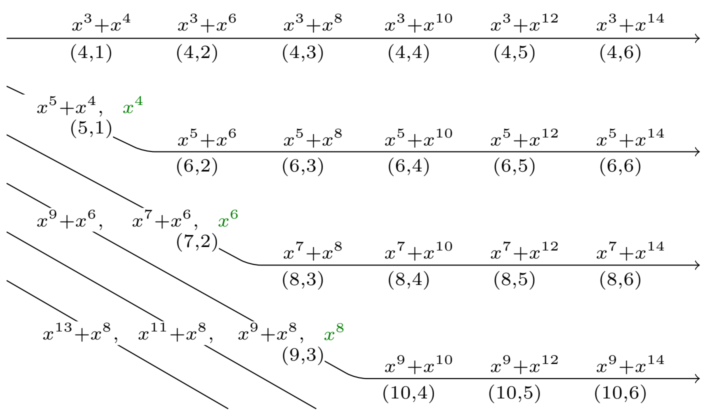


FIGURE 1. List of $p(x)$ for which $xy^2 + p(x)$ is one of the normal forms in $D_{n,m}$ or $D_{n,\infty}$. The pair (n_1, n_2) associated to each $p(x)$ describes the GV invariants of any simple flop having isomorphic contraction algebra.

Corollary 1.7. *There are no simple flopping contractions with GV invariants* $(5, n)$ *with* $n \geq 2$*. Similarly for* $(2m + 1, n)$ *with* $m \geq 2$ *and* $n \neq m - 1$.

Our results also have application to divisorial contractions to a curve. There is an extensive literature [T2, T3, Du] on extremal (K-negative) divisorial contractions in the presence of terminal singularities, but the K-trivial divisorial contractions from smooth varieties considered here are much less studied (see [W] in the Calabi–Yau context).

Proposition 1.8 (8.15, 8.16)**.** *The only contraction algebras for Type A and* D_4 *smooth divisor-to-curve contractions are, up to isomorphism, the Jacobi algebras of the potentials in* 1.2 *and* 1.3 *that have* $\mathfrak{J}$*-dimension* 1.

We furthermore prove that all Jacobi algebras in 1.2 and 1.3 are geometrically realised.

Proposition 1.9 (8.12)**.** *Consider the element of* $\mathbb{C}[\![X,Y,Z,T]\!]$ *defined by*

$$F_m := \begin{cases} Y(X^m+Y)^2+XZ^2-T^2 & \text{if } m\geq 1 \\ Y^3+XZ^2-T^2 & \text{if } m=\infty \end{cases}$$

and set $\mathcal{R}_m = \mathbb{C}[\![X,Y,Z,T]\!]/F_m$. *Then the following statements hold.*

(1) $\mathrm{Sing}(\mathcal{R}_m)^{\mathrm{red}} = (X^M+Y,Z,T)$ *if* $m\geq 1$, *and* (Y,Z,T) *if* $m=\infty$.
(2) *In either case, blowing up this locus gives rise to a crepant Type D divisorial contraction to a curve* $\mathcal{X}_m \to \operatorname{Spec}\mathcal{R}_m$ *where* $\mathcal{X}_m$ *is smooth.*
(3) *The contraction algebra of* $\mathcal{X}_m \to \operatorname{Spec}\mathcal{R}_m$ *is isomorphic to* $\mathcal{J}\mathrm{ac}(xy^2+x^{2m+1})$ *when* $m\geq 1$, *respectively* $\mathcal{J}\mathrm{ac}(xy^2)$ *when* $m=\infty$.

Thus the noncommutative forms $D_{\infty,*}$ are geometrically realised by the F_*. The case $m=\infty$ appeared in [DW4, 2.18], whilst the other infinite family is new. We expect that the families in 1.9 are in fact the classification of Type D divisorial contractions to curves, as this would follow from generalised versions of conjectures in [DW1].

1.6. **The Geometric Realisation Conjecture.** We call $f\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ *geometric* if it arises from geometry, that is, $\mathcal{J}\mathrm{ac}(f)$ isomorphic to the contraction algebra of some $\mathcal{X}\to\operatorname{Spec}\mathcal{R}$ described in §1.5. Based partly on the results in this paper, and partly on extensive computer algebra searches (using the software [BCP, DGPS]), we conjecture the following.

Conjecture 1.10 (The Realisation Conjecture)**.** Every $f\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ whose Jacobi algebra satisfies $\mathrm{Jdim}\,\mathcal{J}\mathrm{ac}(f)\leq 1$ is geometric.

The conjecture being true would imply that every finite dimensional $\mathcal{J}\mathrm{ac}(f)$ is symmetric [A2, 2.6], that is $\mathrm{Hom}_{\mathbb{C}}(\mathcal{J}\mathrm{ac}(f),\mathbb{C})\cong\mathcal{J}\mathrm{ac}(f)$ as bimodules, a property which itself is far from clear. In 2014 our original expectation was that contraction algebras are a strict subset of Jacobi algebras and the task was to recognise them, but since then all computer searches and all papers (e.g. [D]) which have tried to disprove the conjecture have inadvertently ended up giving more evidence for it. This paper is no different.

Corollary 1.11 (8.14)**.** *Conjecture 1.10 is true, except for the one remaining unresolved case when* $f\cong x^3+\mathcal{O}_4$, *where some further analysis is required.*

In the remaining cases, it does now seem likely that all potentials $f\cong x^3+\mathcal{O}_4$ for which $\mathrm{Jdim}\,\mathcal{J}\mathrm{ac}(f)\leq 1$ are isomorphic to contraction algebras of cE_n singularities.

1.7. **Notation and Conventions.** Throughout we work over the complex numbers $\mathbb{C}$, which is necessary for various statements to hold, although any algebraically closed field of characteristic zero would suffice. In addition, we adopt the following notation.

(1) Throughout $d\geq 1$ is fixed to be the number of variables. Set $\mathsf{x}=x_1,\ldots,x_d$, and $\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle=\mathbb{C}\langle\!\langle x_1,\ldots,x_d\rangle\!\rangle$.
(2) Vector space dimension will be written $\dim_{\mathbb{C}} V$.
(3) $\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle_i$ or $\mathbb{C}\langle \mathsf{x}\rangle_i$ will denote the vector subspace of $\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ consisting of homogeneous degree i polynomials. For a formal power series $g\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ we denote the graded (necessarily polynomial) piece of degree i of g by $g_i\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle_i$.
(4) Write $g_{<d}=\sum_{i<d}g_i$ and $g_{>d}=\sum_{i>d}g_i$, with natural self-documenting variations such as $g_{\geq d}$. Thus, for example, $g=g_3+g_4+g_{\geq 5}$ is a power series with no terms in degrees 0, 1 and 2, and no further conditions.
(5) Given $g,h\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$, write $g=h+\mathcal{O}_d$ as a shorthand for $g_{<d}=h_{<d}$.
(6) The previous conventions on degree introduce one typographical difficulty, namely the compatibility with sequences. We will frequently work with sequences $(\mathsf{f}_n)_{n\geq 1}$ of power series $\mathsf{f}_n\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$, and analogously we write $(\mathsf{f}_n)_d$, $(\mathsf{f}_n)_{<d}\in\mathbb{C}\langle \mathsf{x}\rangle$ and $(\mathsf{f}_n)_{>d}\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ for its pieces in the indicated degrees. To scrupulously avoid confusion, we will systematically use Greek font f_n to denote the nth power series in a sequence, and not the nth degree graded piece of a single power series.
(7) The notation §$x.y$ refers to Subsection $x.y$, $(n.m)$ refers to displayed equation $(n.m)$, and $n.m$ refers to statement $n.m$, where the type of statement – Definition, Theorem, and so on – is usually left unspecified.

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2. Formal Automorphisms

This section reviews properties of the noncommutative formal power series $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$, and also constructions of various automorphisms of $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$, mainly following [DWZ, §2]. From the viewpoint of noncommutative singularity theory, it is the construction in §2.6 leading to 3.7(3) that will be used heavily in later sections.

2.1. **Polynomial and Power Series Notation.** As in the introduction, write $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ for formal noncommutative power series in d variables, and further write $\mathbb{C}\langle \mathsf{x} \rangle = \mathbb{C}\langle x_1, \dots, x_d \rangle$ for the free algebra in d variables. For either $f \in \mathbb{C}\langle \mathsf{x} \rangle$ or $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ write f in terms of its homogeneous pieces as

$$f = f_0 + f_1 + f_2 + f_3 + f_4 + \dots,$$

and define the *order* of f to be $\mathrm{ord}(f) = \min\{i \mid f_i \neq 0\}$, where by convention $\mathrm{ord}(0) = \infty$. For any $t \geq 0$ set $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq t} = \{f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle \mid f_i = 0 \text{ if } i < t\} = \{f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle \mid \mathrm{ord}(f) \geq t\}$, and note that this contains the zero element.

2.2. **Complete Completions.** To fix notation, let $\mathfrak{m} = (x_1, \dots, x_d)$ denote the two-sided maximal ideal of the free algebra $\mathbb{C}\langle \mathsf{x} \rangle$. The $\mathfrak{m}$-adic completion of $\mathbb{C}\langle \mathsf{x} \rangle$ is

$$\varprojlim \mathbb{C}\langle \mathsf{x} \rangle / \mathfrak{m}^n$$

which is the set of sequences $(a_n)_{n \geq 1}$ of $a_n \in \mathbb{C}\langle \mathsf{x} \rangle/\mathfrak{m}^n$ that satisfy $a_{n+1} + \mathfrak{m}^n = a_n + \mathfrak{m}^n$ for all n, sometimes called *coherent sequences.*

On the other hand, consider the formal power series ring $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ in noncommutative variables $x_1, \dots, x_d$, with two-sided maximal ideal $\mathfrak{n}$ containing those power series with zero constant term. There is an isomorphism

$$\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle \cong \varprojlim \mathbb{C}\langle \mathsf{x} \rangle / \mathfrak{m}^n$$

which sends a formal power series f to the coherent sequence $(f_{<n} + \mathfrak{m}^n)_{n \geq 1}$. Below we will freely make this identification, and further that the following diagrams for all $i \geq j$ form an inverse limit system

$$\begin{tikzcd} & \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ \mathbb{C}\langle \mathsf{x} \rangle/\mathfrak{m}^i \arrow[rr] & & \mathbb{C}\langle \mathsf{x} \rangle/\mathfrak{m}^j \end{tikzcd}$$

where the map π_i sends $f \mapsto f_{<i} + \mathfrak{m}^i$, and the horizontal map is the natural one.

Given a sequence $(\mathsf{f}_i)_{i \geq 1}$ of elements of $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$, and $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$, recall the following:

- (f_i) *converges to* f if $\forall\, n \geq 1, \exists\, N$ such that $\mathsf{f}_i - f \in \mathfrak{n}^n$ for all $i \geq N$.
- (f_i) is *Cauchy* if $\forall\, n \geq 1, \exists\, N$ such that $\mathsf{f}_i - \mathsf{f}_j \in \mathfrak{n}^n$ for all $i, j \geq N$.

Taking completions of non-noetherian rings in general can be subtle. However, in the situation here, since for all i,

$$\mathfrak{n}^i = \mathrm{Ker}(\pi_i) = \{f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle \mid f_0 = \dots = f_{i-1} = 0\},$$

it is clear that $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ is complete with respect to its $\mathfrak{n}$-adic topology. That is, every Cauchy sequence in $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ converges.

The algebra $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ is a *topological algebra* with basis of the topology given by the ideals $\{\mathfrak{n}^i\}$, where $\mathfrak{n}^i$ is both open and closed. The free algebra $\mathbb{C}\langle \mathsf{x} \rangle$ embeds as a dense subalgebra of $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$, and the ideal $\mathfrak{n}^n$ is the closure of $\mathfrak{m}^n$, or equivalently, $\mathfrak{n}^n$ is the smallest closed ideal that contains all monomials $x_1^{a_1} \dots x_d^{a_d}$ of degree $\sum_{k=1}^d a_k = n$.

2.3. **Formal Automorphisms.** As input, consider a sequence of algebra isomorphisms $(\phi_i\colon \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \to \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i)_{i\geq 1}$ for which

$$\begin{tikzcd} \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r] \arrow[d, "\phi_i"'] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \arrow[d, "\phi_j"] \\ \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \end{tikzcd} \tag{2.A}$$

commutes for all $i \geq j$. Then the universal property for the $\mathfrak{m}$-adic completion lifts these to an algebra automorphism $\phi\colon \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] \arrow[dd, "\phi"] & \\ \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[rr] \arrow[dd, "\phi_i"'] & & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \arrow[dd, "\phi_j"] \\ & \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[rr] & & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \end{tikzcd} \tag{2.B}$$

The following special case will be important later. For any fixed $\mathsf{f}_1, \ldots, \mathsf{f}_d \in \mathfrak{n}^2 \subset \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$, consider the algebra homomorphisms

$$\varphi_i\colon \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \to \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i$$

defined by sending $x_k + \mathfrak{m}^i \mapsto x_k + (\mathsf{f}_k)_{<i} + \mathfrak{m}^i$ for each $1 \leq k \leq d$. On the truncated finite dimensional algebras $\mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i$, clearly each φ_i is an algebra isomorphism, and further since the truncation of a truncation is itself a truncation, (2.A) applied to the φ_i commutes. As a consequence, (2.B) induces an automorphism $\varphi\colon \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$.

Definition 2.1. Given $\mathsf{f}_1, \ldots, \mathsf{f}_d \in \mathfrak{n}^2$, the above $\varphi\colon \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ is called a *unitriangular automorphism*. We will abuse notation slightly and write

$$\begin{aligned} \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle &\to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \\ x_k &\mapsto x_k + \mathsf{f}_k \end{aligned}$$

for φ, since indeed φ is induced by such morphisms on the truncations $\mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i$. For $e \geq 1$ we say that φ has *depth* e provided that $\mathsf{f}_1, \ldots, \mathsf{f}_d \in \mathfrak{n}^{e+1}$.

Lemma 2.2. *With notation as above, the following statements hold.*

(1) *A $\mathbb{C}$-algebra homomorphism $\varphi\colon \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ is a unitriangular automorphism of depth $e \geq 1$ if and only if $\varphi(f)_{\leq e} = f_{\leq e}$ for every $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$.*
(2) *If φ and ψ are unitriangular automorphisms of $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ of depth $e_1 \geq 1$ and $e_2 \geq 1$ respectively, then their composition $\psi \circ \varphi$ is a unitriangular automorphism, of depth* $\min\{e_1, e_2\}$.

Remark 2.3. Any homomorphism $\varphi\colon \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ is continuous. Indeed, $\varphi^{-1}(\mathfrak{n})$ is the kernel of the surjective composition

$$\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \xrightarrow{\varphi} \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \longrightarrow \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle/\mathfrak{n}$$

hence $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle/\varphi^{-1}(\mathfrak{n}) \cong \mathbb{C}$ and so $\varphi^{-1}(\mathfrak{n}) = \mathfrak{n}$ since $\mathfrak{n}$ is the unique maximal ideal. In particular, in the language of [Wa, 5.10], any algebra automorphism of $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ is automatically a topological isomorphism, since its inverse is automatically continuous.

2.4. **Limits of unitriangular automorphisms.** Under specific situations, it is possible to build a sequence of automorphisms $\varphi^1, \varphi^2, \ldots$ of $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$, and take their limit.

For this, consider any d sequences $(\mathsf{g}_i^1)_{i\geq 1}, \ldots, (\mathsf{g}_i^d)_{i\geq 1}$, where each $\mathsf{g}_i^k \in \mathfrak{n}^{i+1}$. By 2.1 these give rise to a sequence of unitriangular automorphisms $\varphi^1, \varphi^2, \ldots$ where

$$\begin{aligned} \varphi^i\colon \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle &\to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \\ x_k &\mapsto x_k + \mathsf{g}_i^k. \end{aligned}$$

Again, the above are induced from the corresponding maps $x_k + \mathfrak{m}^j \mapsto x_k + (\mathsf{g}_i^k)_{<j} + \mathfrak{m}^j$ on the truncations $\mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j$, and where each φ^i has depth i. To ease the subscripts in the notation below, we will also write φ^i for these morphisms viewed on the truncations.

Given this abuse of notation, for all $i \geq j \geq 1$ we claim that the following diagram commutes, where if $i, j = 1$ then by convention the vertical maps are taken to be the identity.

$$\begin{tikzcd} \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r] \arrow[d, "\varphi^{i-1}\circ\cdots\circ\varphi^1"'] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \arrow[d, "\varphi^{j-1}\circ\cdots\circ\varphi^1"] \\ \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \end{tikzcd} \tag{2.C}$$

To see this, note that since each $\mathsf{g}_i^k \in \mathfrak{n}^{i+1}$, it follows (in the case $i > j$) that the bottom square in the following diagram commutes:

$$\begin{tikzcd} \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r] \arrow[d, "\varphi^1"'] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \arrow[d, "\varphi^1"] \\ \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r] \arrow[d, "\varphi^2"'] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \arrow[d, "\varphi^2"] \\ \vdots \arrow[d, "\varphi^{j-1}"'] & \vdots \arrow[d, "\varphi^{j-1}"] \\ \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r] \arrow[d, "\varphi^{i-1}\circ\cdots\circ\varphi^j"'] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \arrow[d, "\mathrm{Id}"] \\ \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^j \end{tikzcd}$$

Since we are abusing notation, the higher squares commute simply since the truncation of a truncation is itself a truncation. Thus all squares commute, establishing (2.C).

Setting $\vartheta_i := \varphi^{i-1} \circ \cdots \circ \varphi^1 \colon \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \to \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i$, again with the convention that $\vartheta_1 = \mathrm{Id}$, then each ϑ_i is an automorphism since each φ^t is. Thus (2.A) induces, through (2.B), an automorphism of $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ such that for all $i \geq j$ the following diagram commutes.

$$\begin{tikzcd} \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \arrow[r, dotted, "\exists"] \arrow[d, "\pi_i"'] & \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \arrow[d, "\pi_i"] \\ \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \arrow[r, "\vartheta_i"] & \mathbb{C}\langle \mathsf{x}\rangle/\mathfrak{m}^i \end{tikzcd} \tag{2.D}$$

Write $\varprojlim \varphi^n \cdots \varphi^1$ for the induced automorphism.

Lemma 2.4. *With notation and assumptions as directly above, for any $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ the sequence $(\varphi^n \cdots \varphi^1(f))_{n\geq 1}$ has limit $\varprojlim \varphi^n \cdots \varphi^1(f)$.*

Proof. Set $F = \varprojlim \varphi^n \cdots \varphi^1$, then it suffices to prove that for all $n \geq 1$, there exists N such that $\varphi^t \cdots \varphi^1(f) - F(f) \in \mathfrak{n}^n$ for all $t \geq N$. This follows since for all $i > n$

$$F(f) + \mathfrak{n}^n \overset{(2.D)}{=} \varphi^{n-1} \cdots \varphi^1(f) + \mathfrak{n}^n \overset{(2.C)}{=} \varphi^{i-1} \cdots \varphi^1(f) + \mathfrak{n}^n. \qquad \square$$

2.5. **Closure and Cyclic Permutation.**

Definition 2.5. For any subset $\mathcal{S} \subset \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$, its *closure* is defined to be

$$\overline{\mathcal{S}} = \bigcap_{i=0}^{\infty} (\mathcal{S} + \mathfrak{n}^i).$$

That is, $b \in \overline{\mathcal{S}}$ if and only if for all $i \geq 0$, there exists $s_i \in \mathcal{S}$ such that $b - s_i \in \mathfrak{n}^i$.

Notation 2.6. For $\mathcal{A} := \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$, consider $\{\mathcal{A}, \mathcal{A}\}$, the commutator vector space of $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$. That is, elements of $\{\mathcal{A}, \mathcal{A}\}$ are finite sums

$$\sum_{i=1}^{n} \lambda_i (a_i b_i - b_i a_i)$$

for elements $a_i, b_i \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ and $\lambda_i \in \mathbb{C}$. Write $\{\{\mathcal{A}, \mathcal{A}\}\}$ for the closure of the commutator vector space $\{\mathcal{A}, \mathcal{A}\}$. Note that $\{\{\mathcal{A}, \mathcal{A}\}\}$ is only a vector space, not an ideal.

Definition 2.7. Two elements $f, g \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ are called cyclically equivalent, or f is said to cyclically permute to g, if $f - g \in \{\{\mathcal{A}, \mathcal{A}\}\}$. We write $f \sim g$ in this case.

Remark 2.8. This notion of cyclic equivalence applied a pair of polynomials is finite and elementary: it is generated over $\mathbb{C}$ by commutators $[m_1, m_2]$ of monomials $m_i \in \mathbb{C}\langle \mathsf{x} \rangle$. With that in mind, 2.7 is then the natural notion for formal power series, as $f \sim g$ means precisely that $f_d \sim g_d$ in every degree d, and no more: the closure merely handles the possibility that f and g may differ by infinitely many such operations.

2.6. **Chasing into Higher Degrees.** The following will be one of our main techniques for producing normal forms of potentials in $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$. The basic idea is to start with a given f, then produce an infinite sequence of automorphisms which chase terms into higher and higher degrees. Taking limits then gives a single automorphism which takes f to the desired normal form. The subtle point is that, at each stage, the automorphisms in (2) below only give the desired elements up to cyclic permutation. As such, the content in the following is that, with care, limits interact well with cyclic permutation.

Theorem 2.9. *Let $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$, and set $\mathsf{f}_1 = f$. Suppose that there exist elements $\mathsf{f}_2, \mathsf{f}_3, \dots$ and automorphisms $\varphi^1, \varphi^2, \dots$ such that*

(1) *Every φ^i is unitriangular, of depth of $\geq i$, and*
(2) *$\varphi^i(\mathsf{f}_i) - \mathsf{f}_{i+1} \in \{\{\mathcal{A}, \mathcal{A}\}\} \cap \mathfrak{n}^{i+1}$, for all $i \geq 1$.*

Then $\lim \mathsf{f}_i$ exists, and there exists an automorphism F such that $F(f) \sim \lim \mathsf{f}_i$.

Proof. The proof follows the strategy used in [DWZ, 4.7], but as the axiomatics are different here, we give the full proof. By §2.4 there is an automorphism $F := \varprojlim \varphi^n \cdots \varphi^1$.

Since the depth of φ^i is $\geq i$, by 2.2(1) $\varphi^i(\mathsf{f}_i)$ differs from f_i only in degrees $> i$. By (2), $\varphi^i(\mathsf{f}_i)$ differs from f_{i+1} only in degrees $> i$. Hence f_{i+1} differs from f_i only in degrees $> i$, from which it easily follows that (f_n) is a Cauchy sequence. Since Cauchy sequences converge in $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$, the limit $\lim \mathsf{f}_i$ exists.

Set $c_i = \varphi^i(\mathsf{f}_i) - \mathsf{f}_{i+1} \in \{\{\mathcal{A}, \mathcal{A}\}\} \cap \mathfrak{n}^{i+1}$. Since $f = \mathsf{f}_1$, it is easy to see that

$$\begin{aligned} \varphi^n \cdots \varphi^1(f) &= \mathsf{f}_{n+1} + \sum_{t=1}^{n} \varphi^n \cdots \varphi^{t+1}(c_t) \\ &= \mathsf{f}_{n+1} + \varphi^n \cdots \varphi^1 \left(\sum_{t=1}^{n} (\varphi^t \cdots \varphi^1)^{-1}(c_t) \right). \end{aligned} \tag{2.E}$$

By 2.4 the left hand side has limit $F(f)$. The first part of the right hand side has limit $\lim \mathsf{f}_i$, which exists by above. We next claim that the rightmost term has limit $F(g)$, where g is the limit of the sequence $(\sum_{t=1}^{n} (\varphi^t \cdots \varphi^1)^{-1}(c_t))_{n \geq 1}$.

First, g exists, since by (2) $c_i \in \mathfrak{n}^{i+1}$, and so since automorphisms preserve the maximal ideal, $(\varphi^t \cdots \varphi^1)^{-1}(c_t) \in \mathfrak{n}^{t+1}$ for all t. It follows easily that the sequence $\left(\sum_{t=1}^{n} (\varphi^t \cdots \varphi^1)^{-1}(c_t)\right)_{n \geq 1}$ is Cauchy, and so its limit g exists in $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$. Given this, the fact that the sequence $\left(\varphi^n \cdots \varphi^1 (\sum_{t=1}^{n} (\varphi^t \cdots \varphi^1)^{-1}(c_t))\right)_{n \geq 1}$ has limit $F(g)$ follows, since for all $i > n$

$$\begin{aligned} F(g) + \mathfrak{n}^{n+1} &= \varphi^n \cdots \varphi^1(\pi_{n+1}(g)) + \mathfrak{n}^{n+1} && \text{(by (2.D))} \\ &= \varphi^n \cdots \varphi^1 \left(\sum_{t=1}^{n} (\varphi^t \cdots \varphi^1)^{-1}(c_t) + \mathfrak{n}^{n+1} \right) + \mathfrak{n}^{n+1} \\ & && \text{(since } (\varphi^t \cdots \varphi^1)^{-1}(c_t) \in \mathfrak{n}^{t+1}) \\ &= \varphi^n \cdots \varphi^1 \left(\sum_{t=1}^{i} (\varphi^t \cdots \varphi^1)^{-1}(c_t) + \mathfrak{n}^{n+1} \right) + \mathfrak{n}^{n+1} && \text{(add zero)} \\ &= \varphi^i \cdots \varphi^1 \left(\sum_{t=1}^{i} (\varphi^t \cdots \varphi^1)^{-1}(c_t) \right) + \mathfrak{n}^{n+1}. && \text{(by (2.C))} \end{aligned}$$

Combining with (2.E) and taking limits it follows that

$$F(f)=\lim \mathsf{f}_i+F(g). \tag{2.F}$$

Now, it is easy to check that automorphisms preserve $\{\!\{\mathcal{A},\mathcal{A}\}\!\}$, so each term in the sequence $\big(\varphi^n\cdots\varphi^1(\sum_{t=1}^n(\varphi^t\cdots\varphi^1)^{-1}(c_t))\big)_{n\geq 1}$ belongs to $\{\!\{\mathcal{A},\mathcal{A}\}\!\}$. But since $\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ is complete, every Cauchy sequence within a closed set has limit in that closed set. It follows that the limit $g\in\{\!\{\mathcal{A},\mathcal{A}\}\!\}$. One final application of the fact that automorphisms preserve $\{\!\{\mathcal{A},\mathcal{A}\}\!\}$ shows that $F(g)\in\{\!\{\mathcal{A},\mathcal{A}\}\!\}$, and so $F(f)\sim\lim\mathsf{f}_i$. □

2.7. **Elementary Properties of Closed Ideals.** We finish this section with some technical results on closed ideals that are used throughout §6–§8.

Notation 2.10. When I is an ideal, write $(\!(I)\!)$ for its closure (in the sense of 2.5), which is again an ideal since the ring operations are continuous. Note that $(\!(I)\!)$ need not be finitely generated, even if I is.

For a finite set of elements $\mathcal{S}$ in $\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$, consider the closed ideal $(\!(\mathcal{S})\!)=(\!(s\mid s\in\mathcal{S})\!)$.

Lemma 2.11. *Let $\mathcal{S}$ be a finite subset of elements in $\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$, and $f_1,\ldots,f_s\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$. Then the following statements hold.*

1. $(\!(f_1,\ldots,f_s)\!)=(\!(f_1u_1,\ldots,f_su_s)\!)$ *for any units $u_1,\ldots,u_s\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$.*
2. $(\!(f+(\!(\mathcal{S})\!)\,)\!)=(\!(f,s\mid s\in\mathcal{S})\!)/(\!(\mathcal{S})\!)$ *in $\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle/(\!(\mathcal{S})\!)$.*
3. *If $\psi\colon\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle/(\!(\mathcal{S})\!)\to\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle/(\!(\mathcal{S})\!)$ is a topological isomorphism which sends $f+(\!(\mathcal{S})\!)\mapsto g+(\!(\mathcal{S})\!)$, for two elements $f,g\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$, then there is an induced topological isomorphism*
$$\frac{\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle}{(\!(f,s\mid s\in\mathcal{S})\!)}\xrightarrow{\cong}\frac{\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle}{(\!(g,s\mid s\in\mathcal{S})\!)}.$$

Proof. (1) $(\!(f_1,\ldots,f_s)\!)$ is the smallest closed ideal containing all f_i. Since $f_i=(f_iu_i)u_i^{-1}\in(\!(f_1u_1,\ldots,f_su_s)\!)$ for each i, by minimality $(\!(f_1,\ldots,f_s)\!)\subseteq(\!(f_1u_1,\ldots,f_su_s)\!)$. Repeating the same argument to $f_iu_i\in(\!(f_1,\ldots,f_s)\!)$, the converse inclusion also holds.
(2) Certainly $(\!(f+(\!(\mathcal{S})\!)\,)\!)=I/(\!(\mathcal{S})\!)$ for some ideal I, given it is an ideal of the quotient. This ideal I is closed by [Wa, 5.2], since the map to the quotient is continuous, and hence the inverse image of a closed set is closed. This closed ideal I contains both f and $\mathcal{S}$, and so $(\!(f,s\mid s\in\mathcal{S})\!)\subseteq I$.

On the other hand $(\!(f+(\!(\mathcal{S})\!)\,)\!)$ is the smallest closed ideal containing $f+(\!(\mathcal{S})\!)$. Setting $A=\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$, $J=(\!(\mathcal{S})\!)$, and $H=(\!(f,s\mid s\in\mathcal{S})\!)$, the third isomorphism theorem for topological rings [Wa, 5.13] asserts that there is a topological isomorphism

$$(A/J)/(H/J)\cong A/H.$$

In particular, A/H is Hausdorff, since H is closed in A, by [Wa, 5.7(1)] applied to A. This being the case, H/J is closed in A/J, by [Wa, 5.7(1)] applied to A/J. Hence $(\!(f,s\mid s\in\mathcal{S})\!)/(\!(\mathcal{S})\!)$ is a closed ideal, which clearly contains $f+(\!(\mathcal{S})\!)$. By minimality $(\!(f+(\!(\mathcal{S})\!)\,)\!)=I/(\!(\mathcal{S})\!)\subseteq(\!(f,s\mid s\in\mathcal{S})\!)/(\!(\mathcal{S})\!)$ and thus $I\subseteq(\!(f,s\mid s\in\mathcal{S})\!)$. Combining inclusions, the required equality holds.
(3) Since ψ is a continuous isomorphism, the closed ideal generated by $f+(\!(\mathcal{S})\!)$ corresponds to the closed ideal generated by $g+(\!(\mathcal{S})\!)$. Thus there is a topological isomorphism

$$\frac{\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle/(\!(\mathcal{S})\!)}{(\!(f+(\!(\mathcal{S})\!)\,)\!)}\longrightarrow\frac{\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle/(\!(\mathcal{S})\!)}{(\!(g+(\!(\mathcal{S})\!)\,)\!)}.$$

Now by (2) we have $(\!(f+(\!(\mathcal{S})\!)\,)\!)=(\!(f,s\mid s\in\mathcal{S})\!)/(\!(\mathcal{S})\!)$, and likewise for g. The statement follows by the third isomorphism theorem for topological rings [Wa, 5.13]. □

3. Jacobi Algebras

3.1. **Differentiation.** Consider the $\mathbb{C}$-linear maps $\eth_i\colon\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle\to\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ which simply 'strike off' the leftmost x_i of each monomial, in other words act on monomials via the rule

$$\eth_i(m)=\begin{cases}n & \text{if } m=x_in\\ 0 & \text{otherwise.}\end{cases} \tag{3.A}$$

The $\mathbb{C}$-linear symmetrisation map sym: $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ on monomials sends

$$x_{i_1}\dots x_{i_t} \mapsto \sum_{j=1}^{t} x_{i_j}x_{i_{j+1}}\dots x_{i_t}\cdot x_{i_1}\dots x_{i_{j-1}}.$$

Combining these two gives the cyclic derivatives. These are the $\mathbb{C}$-linear maps $\delta_i\colon \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle \to \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ which on monomials send

$$x_{i_1}\dots x_{i_t} \mapsto \partial_i \operatorname{sym}(x_{i_1}\dots x_{i_t}) = \sum_{j=1}^{t} \partial_i(x_{i_j}x_{i_{j+1}}\dots x_{i_t}\cdot x_{i_1}\dots x_{i_{j-1}}). \tag{3.B}$$

Definition 3.1. For $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$, the Jacobi algebra is defined to be

$$\mathfrak{Jac}(f) = \frac{\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle}{\langle\langle \delta_1 f,\dots,\delta_d f\rangle\rangle}$$

where $\langle\langle \delta_1 f,\dots,\delta_d f\rangle\rangle =: \langle\langle \delta f\rangle\rangle$ is the closure of the two-sided ideal $(\delta_1 f,\dots,\delta_d f)$.

In general, the quotient of a complete topological ring by a closed ideal is always separated, but it need not be complete.

Notation 3.2. For any ring R, write $\mathfrak{J}(R)$ for its Jacobson radical. If I is any ideal of R contained in $\mathfrak{J}(R)$, then $\mathfrak{J}(R/I) = \mathfrak{J}(R)/I$ (see e.g. [L, 4.6]).

(1) It is clear that $\mathfrak{J}(\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle) = \mathfrak{n}$. If $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle_{\geq 2}$, then $\langle\langle \delta f\rangle\rangle$ is contained in $\mathfrak{n}$, and so $\mathfrak{J}(\mathfrak{Jac}(f)) = \mathfrak{n}/\langle\langle \delta f\rangle\rangle$, and furthermore $\mathfrak{J}(\mathfrak{Jac}(\mathfrak{f}))^n = \big(\mathfrak{n}^n + \langle\langle \delta f\rangle\rangle\big)/\langle\langle \delta f\rangle\rangle$ for $n \geq 2$.

(2) The topology on $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ is an ideal topology generated by powers of $\mathfrak{n}$, so the natural quotient topology on the quotient $\mathfrak{Jac}(f)$ is induced by powers of the image of $\mathfrak{n}$ in the quotient [Wa, 5.5]. Thus, by (1), provided $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle_{\geq 2}$ then the topology on both $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ and $\mathfrak{Jac}(f)$ is the radical-adic topology. Since $\langle\langle \delta f\rangle\rangle$ is closed, $\mathfrak{Jac}(f)$ is Hausdorff [Wa, 5.7(1)]. Under extra assumptions it is also complete; see 8.4(3).

Remark 3.3. A (polynomial or) power series $f = \sum f_i \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ is called *cyclically symmetric* if $\operatorname{sym}(f_i) = i f_i$ for each graded piece $f_i \in \mathbb{C}\langle \mathsf{x}\rangle$. It is possible to phrase the whole paper using only cyclically symmetric potentials, however this becomes notationally unmanageable in §4–§6, since the property of being cyclically symmetric is not preserved under change variables. Thus from the viewpoint of NC singularity theory, it is much more natural to work with plain old elements of $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$. There are times when passing to cyclically symmetric potentials is convenient, but this is confined entirely to §A.2.

3.2. **Dimension.** Being a quotient of formal noncommutative power series, determining which dimension to use for $\mathfrak{Jac}(f)$ is a subtle point.

Definition 3.4. For $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle_{\geq 2}$, we say that $\mathfrak{Jac}(f)$ has *polynomial growth* (*with respect to* $\mathfrak{J}$) if there exist $c, r \in \mathbb{R}$ such that $\dim \mathfrak{Jac}(f)/\mathfrak{J}^n \leq cn^r$ for all $n \in \mathbb{N}$. In the case that $\mathfrak{Jac}(f)$ has polynomial growth, then the $\mathfrak{J}$-*dimension of* $\mathfrak{Jac}(f)$ is the degree of that growth, precisely

$$\operatorname{Jdim} \mathfrak{Jac}(f) := \inf\{r \in \mathbb{R} \mid \text{for some } c \in \mathbb{R},\ \dim \mathfrak{Jac}(f)/\mathfrak{J}^n \leq cn^r \text{ for every } n \in \mathbb{N}\},$$

and $\operatorname{Jdim} \mathfrak{Jac}(f) = \infty$ otherwise.

The $\mathfrak{J}$-dimension is analogous to the usual dimension of a commutative noetherian local ring $(A, \mathfrak{m})$, defined as the degree of the characteristic polynomial $\chi_{\mathfrak{m}}(n) = \ell(A/\mathfrak{m}^n)$, where, in that context, the dimension is necessarily an integer [AM, 11.4, 11.14].

Lemma 3.5. *If* $\operatorname{Jdim} \mathfrak{Jac}(f) \leq 1$*, then either* $\operatorname{Jdim} \mathfrak{Jac}(f) = 0$ *or* $\operatorname{Jdim} \mathfrak{Jac}(f) = 1$*.*

Proof. Certainly $\mathfrak{Jac}(f)/\mathfrak{J}^{n+1} \twoheadrightarrow \mathfrak{Jac}(f)/\mathfrak{J}^n$ for all $n \geq 1$, with equality if and only if $\mathfrak{J}^{n+1} = \mathfrak{J}^n$. If each such map has nontrivial kernel, then $\dim \mathfrak{Jac}(f)/\mathfrak{J}^n \geq n$ and so $\operatorname{Jdim} \mathfrak{Jac}(f) \geq 1$. Otherwise, by Nakayama's Lemma, $\mathfrak{J}^n = 0$ for some n, hence $\mathfrak{n}^n \subset \langle\langle \delta f\rangle\rangle$ and so $\dim_{\mathbb{C}} \mathfrak{Jac}(f) \leq \dim \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle/\mathfrak{n}^n = 2^n - 1$ and $\operatorname{Jdim} \mathfrak{Jac}(f) = 0$. $\square$

Remark 3.6. The $\mathfrak{J}$-dimension is used throughout, since it is better suited to the complete local situation than the GK dimension [KL]. Indeed, it is well-known that the GK dimension does not behave well with respect to completions. For example, $\operatorname{GKdim} \mathbb{C}[\![x]\!] = \infty$

whereas $\operatorname{Jdim}\mathbb{C}[\![x]\!] = 1$. Compare [AB, §3.4], and in particular [AB, §5.6]. Furthermore, $\operatorname{Jdim}\mathfrak{Jac}(f) = 0$ if and only if $\dim_{\mathbb{C}}\mathfrak{Jac}(f) < \infty$, a property which does not hold for GK dimension since $\mathfrak{Jac}(f)$ is not finitely generated.

3.3. **Equivalences and Isomorphisms.** In what follows, recognising and producing isomorphisms of Jacobi algebras will be key. The following techniques will be used extensively. The first is trivial, but worth recording since it gives great flexibility in proofs; the second two are more substantial with Part (2) being [DWZ, 3.7], and Part (3) following from (2), together with 2.9. Recall the notation $f \sim g$ from 2.7.

Summary 3.7. Suppose that $f \in \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$.

(1) If f cyclically permutes to g, so $f \sim g$, then $\mathfrak{Jac}(f) \cong \mathfrak{Jac}(g)$.
(2) If $\varphi \in \operatorname{Aut}\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$ then $\mathfrak{Jac}(f) \cong \mathfrak{Jac}(g)$, where $g = \varphi(f)$.
(3) Set $\mathsf{f}_1 = f$. If there exist $\mathsf{f}_2, \mathsf{f}_3, \dots$ and automorphisms $\varphi^1, \varphi^2, \dots$ such that
   (a) every φ^i is unitriangular of depth of $\geq i$, and
   (b) $\varphi^i(\mathsf{f}_i) - \mathsf{f}_{i+1} \in \{\!\{\mathcal{A}, \mathcal{A}\}\!\} \cap \mathfrak{n}^{i+1}$, for all $i \geq 1$,
   then the sequence $(\mathsf{f}_i)_{i\geq 1}$ converges and $\mathfrak{Jac}(f) \cong \mathfrak{Jac}(g)$ where $g = \lim \mathsf{f}_i$.

Lemma 3.8. *Let $f \in \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$, and $m \in \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$ be a monomial. Then the following hold.*

(1) $\operatorname{sym}(m) \sim \deg(m)m$.
(2) *If f contains λm, then* $f \sim f + \lambda\left(\frac{1}{\deg m}\operatorname{sym}(m) - m\right)$.
(3) *Let h be the sum of terms of f whose monomials appear in* $\operatorname{sym}(m)$. *Then*
$$f \sim f - h + \alpha \operatorname{sym}(m) \sim f - h + \alpha \deg(m)\, m,$$
for some $\alpha \in \mathbb{C}$.

Proof. Writing $m = m_1 m_2 \dots m_r$, where each m_i is a variable $x_{j(i)}$, we have
$$\begin{aligned} rm - \operatorname{sym}(m) &= (r-1)m_1 \dots m_r - m_2 \dots m_r m_1 \\ &\qquad - m_3 \dots m_r m_1 m_2 - \dots - m_r m_1 \dots m_{r-1} \\ &= [m_1, m_2 \dots m_r] + [m_1 m_2, m_3 \dots m_r] + \dots + [m_1 \dots m_{r-1}, m_r] \end{aligned}$$
and (1) follows. (2) follows at once from (1). The final claim (3) follows by applying (2) to each monomial of h in turn. □

Below it will be convenient to work with the following three equivalence relations.

Definition 3.9. For elements $f, g \in \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$, (recall and) define

(1) $f \sim g$ if $f - g \in \{\!\{\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle, \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle\}\!\}$ (see 2.7).
(2) $f \simeq g$ if there is an equality of ideals $(\!(\delta_1 f, \dots, \delta_d f)\!) = (\!(\delta_1 g, \dots, \delta_d g)\!)$.
(3) $f \cong g$ if there is an isomorphism of algebras $\mathfrak{Jac}(f) \cong \mathfrak{Jac}(g)$.

Clearly $f \sim g$ implies $f \simeq g$ implies $f \cong g$, but the converse implications do not hold. The relation $\sim$ is additive by definition, but $\simeq$ is not: $x^2 + y^3 \simeq x^2 + 2y^3$ but $x^2 \not\simeq x^2 + y^3$.

The Jacobi isomorphism relation $\cong$ is the equivalence relation that we will classify up to, but the others help understand the structure of the various arguments. For example, by 2.9, the symmetrisation relation $\sim$ behaves well in limits. It appears to permit creation from the void, in the sense that $0 \sim xy - yx$, but of course this form has all derivatives zero, so does not contribute to Jacobi ideals. The relation $\simeq$ is useful for cancelling high order terms in potentials (see e.g. the proof of 6.5), whereas $\cong$ is most suited to, and is often a by-product of, analytic changes in coordinates.

4. NC Singularity Theory 101

4.1. **Corank and the Splitting Lemma.** The closed vector subspace of commutators $\{\!\{\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle, \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle\}\!\}$ generates the much larger closed ideal of commutators, and the quotient of $\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$ by this ideal is the ring of commutative power series $\mathbb{C}[\![x_1, \dots, x_d]\!]$. The quotient, or 'abelianisation', map $\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle \to \mathbb{C}[\![x_1, \dots, x_d]\!]$ written $g \mapsto g^{\mathrm{ab}}$ simply takes the expression for g to the same expression in the commutative ring.

Lemma 4.1. *With notation as above, the following hold:*

(1) *The abelianisation map* $\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle \to \mathbb{C}[\![x_1, \dots, x_d]\!]$ *is continuous and surjective.*

(2) *For any* $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$*, the map* $f \mapsto f^{\mathrm{ab}}$ *descends to a surjection*

$$\mathfrak{J}\mathrm{ac}(f) \twoheadrightarrow \frac{\mathbb{C}[\![x_1,\ldots,x_d]\!]}{(\!(\partial f^{\mathrm{ab}}/\partial x_i \mid i=1,\ldots,n)\!)}. \tag{4.A}$$

Proof. (1) At the level of ideals, $\mathfrak{n}^k \twoheadrightarrow \mathfrak{n}^k_{\mathrm{ab}}$ for every $k \geq 0$, since abelianisation is a ring homomorphism mapping each x_i to x_i.
(2) Since $(\delta_i f)^{\mathrm{ab}} = \partial f^{\mathrm{ab}}/\partial x_i$, where $\partial/\partial x_i$ is the usual differentiation of commutative functions, surjectivity at the level of (unclosed) Jacobian ideals follows. Since the abelianisation map is continuous and surjective by (1), this passes to their closures, as claimed. ☐

Below we will consider

$$\mathfrak{J}\mathrm{ac}(f)^{\mathrm{ab}} := \frac{\mathbb{C}[\![x_1,\ldots,x_d]\!]}{(\!(\partial f^{\mathrm{ab}}/\partial x_i \mid i=1,\ldots,n)\!)} = \frac{\mathbb{C}[\![x_1,\ldots,x_d]\!]}{(\partial f^{\mathrm{ab}}/\partial x_i \mid i=1,\ldots,n)}$$

where, since $\mathbb{C}[\![x_1,\ldots,x_d]\!]$ is commutative noetherian, all ideals are closed [M1, 8.1(1)].

Remark 4.2. In classical singularity theory, for $g \in \mathbb{C}[\![x_1,\ldots,x_d]\!]$ both the Milnor algebra $\mathbb{C}[\![x_1,\ldots,x_d]\!]/(\delta_1 g,\ldots,\delta_d g)$ and the Tjurina algebra $\mathbb{C}[\![x_1,\ldots,x_d]\!]/(g,\delta_1 g,\ldots,\delta_d g)$ are defined, and play a major role. In the noncommutative setting, the analogous Tjurina algebra is not well defined on $\sim$ classes. For example, the potentials $0 \sim xy-yx$ determine the same Jacobi algebra, but their naively-defined Tjurina algebras are $\mathbb{C}\langle\langle x,y\rangle\rangle$ and $\mathbb{C}[\![x,y]\!]$ respectively. To have any hope of classifying elements in the completed free algebra, some identification is required, and for us identifying $\sim$ classes is essential for applications. Compare [HZ], where the lack of a noncommutative Tjurina algebra motivates the use of Hochschild classes to generalise Saito's theorem on homogeneous potentials.

Definition 4.3. For $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle_{\geq 2}$, the *corank of* f is defined to be

$$\mathfrak{Crk}(f) = \dim_{\mathbb{C}}\left(\frac{\mathfrak{J}}{\mathfrak{J}^2}\right)$$

where $\mathfrak{J}$ is the Jacobson radical of $\mathfrak{J}\mathrm{ac}(f)$.

Remark 4.4. Clearly $0 \leq \mathfrak{Crk}(f) \leq d$. Since $\mathfrak{J}/\mathfrak{J}^2 \cong (\mathfrak{n}+I)/(\mathfrak{n}^2+I)$, where $I = (\!(\delta f)\!)$, the exactness of the sequence of $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle/\mathfrak{n} = \mathbb{C}$-vector spaces

$$0 \longrightarrow \frac{\mathfrak{n}^2+I}{\mathfrak{n}^2} \longrightarrow \frac{\mathfrak{n}}{\mathfrak{n}^2} \longrightarrow \frac{\mathfrak{n}+I}{\mathfrak{n}^2+I} \longrightarrow 0$$

shows that $\mathfrak{Crk}(f) = d - \dim_{\mathbb{C}}\left(\frac{\mathfrak{n}^2+I}{\mathfrak{n}^2}\right)$, so that the corank is determined by the linear conditions imposed by derivatives, and is therefore uniquely determined by f_2.

Theorem 4.5 (Splitting Lemma). *Let* $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$*. Then* $f \cong x_1^2 + \cdots + x_r^2 + g$ *for some* $g \in \mathbb{C}\langle\langle x_{r+1},\ldots,x_d\rangle\rangle_{\geq 3}$*, where* $d-r = \mathfrak{Crk}(f)$*. In particular,*

$$\mathfrak{J}\mathrm{ac}(f) \cong \frac{\mathbb{C}\langle\langle x_{r+1},\ldots,x_d\rangle\rangle}{(\!(\delta_{x_{r+1}} g,\ldots,\delta_{x_d} g)\!)}.$$

Proof. This is [DWZ, 4.5] for the d-loop quiver. Since $\operatorname{ord} g \geq 3$, the derivatives of g impose no linear conditions, so necessarily $d-r = \mathfrak{Crk}(f)$. ☐

4.2. **Golod–Shafarevich–Vinberg.** The classical approach to growth of algebras comes from the Golod–Shafarevich theorem [GS], adapted by Vinberg [V3] to power series; see also [E2]. This result constrains f to achieve $\operatorname{Jdim}\mathfrak{J}\mathrm{ac}(f) < \infty$, and we develop a stronger version in 4.7 adapted to Jacobi algebras.

Theorem 4.6 (Golod–Shafarevich, Vinberg). *Let* $I = (g_1,\ldots,g_s) \subset \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ *be an ideal, set* $r_i = \operatorname{ord} g_i$ *for each* $i = 1,\ldots,s$*, and write* $h = 1 - dt + t^{r_1} + \cdots + t^{r_s} \in \mathbb{R}[\![t]\!]$*. If the coefficients of* $(1-t)/h$ *are non negative, then* $\dim_{\mathbb{C}} \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle/(\!(I)\!) = \infty$*.*

In most cases where the result applies, one can in fact show exponential growth. The Golod–Shafarevich–Vinberg estimates readily show that $\operatorname{Jdim}\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle/(\!(g_1,\ldots,g_d)\!) = \infty$ in the following cases:

(1) $d=2$ with either $r_1 \geq 3$, $r_2 \geq 8$, or $r_1 \geq 4$, $r_2 \geq 5$.
(2) $d=3$ with either $r_1 \geq 2$, $r_2, r_3 \geq 3$, or $r_1 = r_2 = 2$, $r_3 \geq 5$.

(3) $d \geq 4$ with $r_i \geq 2$ for every i.

For example, in the case $d = 4$, it is sufficient to observe the exponential growth of

$$(1 - 4t + 4t^2)^{-1} = 1 + 3t + 8t^2 + 20t^3 + 48t^4 + \cdots + (1+k)2^k t^k + \cdots$$

as this bounds the growth of the algebra from below in the case of an order 3 potential with four order 2 derivatives.

Setting aside quadratic terms by the Splitting Lemma, this then puts constraints on the motivating problem 1.1. Indeed, if $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq 3}$ and $\operatorname{Jdim} \mathfrak{Jac}(f) < \infty$, then either

(1) $d = 2$, $\operatorname{ord} f \leq 5$ and $f_{\leq 5} \not\sim \ell^5$ for a linear form $\ell = \ell(x_1, x_2)$, or
(2) $d = 3$, $\operatorname{ord} f = 3$, and $f_3 \not\sim \ell^3$ for a linear form $\ell = \ell(x_1, x_2, x_3)$.

It turns out that these estimates can be substantially improved, but this requires much more work. Iyudu and collaborators [ISm, IS] introduce several new ideas that exploit the Jacobi structure; in Appendix A we extend their techniques into the power series context, and establish the following. Recall that $\mathsf{x} = x_1, \ldots, x_d$.

Theorem 4.7 (A.18)**.** *Suppose that $d = 2$ and $k \geq 4$, or $d \geq 3$ and $k \geq 3$. If $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ has order k, then* $\operatorname{Jdim} \mathfrak{Jac}(f) \geq 3$.

Remark 4.8. Together with the Splitting Lemma, the above 4.7 reduces the classification of those f satisfying $\operatorname{Jdim} \mathfrak{Jac}(f) \leq 1$ to the case of two variables ($d = 2$).

4.3. **Higher Coranks.** Higher-degree versions of the corank exist, and contain more detailed information about Jacobi algebras.

Definition 4.9. Let $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq 2}$. For $m \geq 1$, the *mth corank of f* is defined to be

$$\mathfrak{Crk}_m(f) = \dim_{\mathbb{C}} \left(\frac{\mathfrak{J}^m}{\mathfrak{J}^{m+1}} \right),$$

where $\mathfrak{J}$ is the Jacobson radical of $\mathfrak{Jac}(f)$. We also define $\mathfrak{Crk}_0(f) = \dim_{\mathbb{C}} \mathfrak{Jac}(f)/\mathfrak{J} = 1$ and note that $\mathfrak{Crk}_1(f) = \mathfrak{Crk}(f)$.

Remark 4.10. Since $\mathfrak{J}^m/\mathfrak{J}^{m+1} \cong (\mathfrak{n}^m + I)/(\mathfrak{n}^{m+1} + I)$, the exactness of the sequence of $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle/\mathfrak{n} = \mathbb{C}$-vector spaces

$$0 \to \frac{\mathfrak{n}^m \cap I}{\mathfrak{n}^{m+1} \cap I} \cong \frac{(\mathfrak{n}^m \cap I) + \mathfrak{n}^{m+1}}{\mathfrak{n}^{m+1}} \to \frac{\mathfrak{n}^m}{\mathfrak{n}^{m+1}} \to \frac{\mathfrak{n}^m + I}{\mathfrak{n}^{m+1} + I} \to 0$$

shows that $\mathfrak{Crk}_m(f) = d^m - \dim_{\mathbb{C}} \left(\frac{\mathfrak{n}^m \cap I}{\mathfrak{n}^{m+1} \cap I} \right)$; compare [V3, (4)]. Thus the mth corank is determined by the conditions imposed on the leading terms of elements of the Jacobian ideal of order exactly m. In particular, $0 \leq \mathfrak{Crk}_m(f) \leq d^m$. If $\operatorname{ord}(f) \geq m+1$, then

$$\frac{(\mathfrak{n}^m \cap I) + \mathfrak{n}^{m+1}}{\mathfrak{n}^{m+1}} \cong \frac{\mathfrak{n}^{m+1} + I}{\mathfrak{n}^{m+1}}$$

matching (1.A), (1.B), and 4.4.

By definition, the $\mathfrak{J}$-dimension is the growth of the sum of coranks. Calculating the mth corank is not necessarily straightforward: essentially it amounts to calculating a Gröbner basis of the Jacobian ideal with a local monomial order to at least order m.

To study Jacobi algebras $\mathfrak{Jac}(f)$ of $\mathfrak{J}$-dimension ≤ 1, 4.7 constrains the number of variables to $d \leq 2$ and $k = \operatorname{ord}(f) \leq 3$. The corank controls the rank of f_2. The main case is when $f_2 = 0$, when it is clear that $2 \leq \mathfrak{Crk}_2(f) \leq 4$. The two derivatives $\delta_x f_3$ and $\delta_y f_3$ are linearly independent when $\mathfrak{Crk}_2(f) = 2$ and they are dependent when $\mathfrak{Crk}_2(f) = 3$. The case $\mathfrak{Crk}_2(f) = 4$ holds only when $f_3 = 0$, which is ruled out by 4.7.

This provides a numerical characterisation of the ADE types. The first Type A case is when $\mathfrak{Crk}(f) = 0$, in which case $\mathfrak{Jac}(f) \cong \mathbb{C}$. In addition to this, if $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ has $\operatorname{Jdim} \mathfrak{Jac}(f) \leq 1$, we say f has Type A, D or E according to the following table.

Type	$\mathfrak{Crk}(f)$	$\mathfrak{Crk}_2(f)$
A	1	1
D	2	2
E	2	3

The higher coranks provide much more detail. In Type A they provide enough information to classify up to isomorphism, however in Type D this is not true.

Example 4.11. Consider the families $D_{n,\infty}$ and $D_{n,m}$ from the introduction. The higher coranks are given by the following table.

Type	f	**Conditions**	$\mathfrak{Crk}_i(f),\ i=0,1,\dots$
$D_{n,\infty}$	xy^2+x^{2n}	$n\geq 2$	
$D_{n,m}$	$xy^2+x^{2n}+x^{2m+1}$	$2n-2\geq m\geq n\geq 2$	$1,\overbrace{2,\dots,2}^{2n-1},\overbrace{1,\dots,1}^{2n-2}$
$D_{n,m}$	$xy^2+x^{2m+1}+x^{2n}$	$n>m\geq 1$	$1,\overbrace{2,\dots,2}^{2n-1},\overbrace{1,\dots,1}^{2m-1}$

In particular $\dim_{\mathbb{C}}\mathfrak{Jac}(f)=6n-3$ in the first families, which is independent of m, whilst $\dim_{\mathbb{C}}\mathfrak{Jac}(f)=4n+2m-2$ in the final family.

4.4. **Linear Changes in Coordinates and Discriminants.** In light of 4.8, from §6 onwards we work in two non-commuting variables x and y.

The following is an immediate consequence of the Splitting Lemma and abelianisation.

Lemma 4.12. *Let $f\in\mathbb{C}\langle\langle x,y\rangle\rangle_{\geq 2}$ with $f_2=ax^2+b_1xy+b_2yx+cy^2\not\sim 0$. Set $b=b_1+b_2$ and consider the discriminant $\Delta=b^2-4ac$. Then $f\cong g$, for some $g\in\mathbb{C}\langle\langle x,y\rangle\rangle_{\geq 2}$ with*

$$g_2=\begin{cases}x^2+y^2 & \text{if } \Delta\neq 0\\ x^2 & \text{if } \Delta=0.\end{cases}$$

As for the quadratic forms above, up to $\sim$ we may commute variables appearing in cubic forms in $\mathbb{C}\langle\langle x,y\rangle\rangle$, and we use this to simplify the statement of the following lemma, writing bx^2y rather than $b_1x^2y+b_2xyx+\dots$, and so on. Note that, in general, cyclic equivalence no longer simulates commutativity in higher degree, as $xyxy\not\sim x^2y^2$.

Lemma 4.13. [I] *Let $f\in\mathbb{C}\langle\langle x,y\rangle\rangle_{\geq 3}$ with $f_3\sim ax^3+bx^2y+cxy^2+dy^3$ for some $a,b,c,d\in\mathbb{C}$, not all zero. Let $\Delta=-27a^2d^2+18abcd-4ac^3-4b^3d+b^2c^2\in\mathbb{C}$ be the cubic discriminant. Then $f\cong g$, for some $g\in\mathbb{C}\langle\langle x,y\rangle\rangle_{\geq 3}$ with*

$$g_3=\begin{cases}x^3+y^3 & \text{if } \Delta\neq 0\\ x^2y & \text{if } \Delta=0 \text{ and } \begin{cases}a(b^2-3ac)\neq 0 \text{ or}\\ (c^2-3bd)d\neq 0 \text{ or}\\ a=d=0\end{cases}\\ x^3 & \text{otherwise.}\end{cases}$$

Thus these three leading cubic normal forms are characterised by whether f_3^{ab} has three, two or one distinct factors respectively.

Proof. Consider the linear automorphism of $\mathbb{C}[x,y]$

$$x\mapsto \alpha x+\beta y,\qquad y\mapsto\gamma x+\delta y\qquad\text{for } \alpha,\beta,\gamma,\delta\in\mathbb{C}\tag{4.B}$$

that maps $(f_3)^{\mathrm{ab}}\in\mathbb{C}[x,y]$ to one of the normal forms x^3+y^3, xy^2 or x^3. The choice of normal form is determined by the cubic determinant. The additional conditions on the coefficients in the statement are simply that $p\neq 0$ in the depressed form after completing the cube $x^3+pxy^2+qy^3$, in which $\Delta=-4p^3-27q^2$, and accounting for the fact that $a=0$ or $d=0$ or both are possible.

Let φ be the linear automorphism of $\mathbb{C}\langle\langle x,y\rangle\rangle$ defined by the same formula (4.B) and $g_3\in\mathbb{C}\langle\langle x,y\rangle\rangle$ be the corresponding cubic normal form. Then $\varphi(f_3)\sim g_3$, although they are not equal, so $\varphi(f)\sim g_3+\varphi(f_{\geq 4})$. Thus $f\cong\varphi(f)\cong g_3+\varphi(f_{\geq 4})$, as claimed. ☐

5. Type A and Commutativity

This section considers the most elementary situation, namely $f\in\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle_{\geq 2}$ with large quadratic part. Normal forms are established in §5.1. Together with the linear coordinate changes from §4.4, this proves in §5.2 that for any $f\in\mathbb{C}\langle\langle x,y\rangle\rangle_{\geq 2}$, the algebra $\mathfrak{Jac}(f)$ is commutative if and only if f has corank at most 1. This fact is used in later sections.

5.1. **Normal Forms of Type A.** It is notationally convenient to identify $y = x_d$ and work in the ring $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle = \mathbb{C}\langle\langle x_1, \ldots, x_{d-1}, y\rangle\rangle$.

Theorem 5.1. *If $f \in \mathbb{C}\langle\langle \mathsf{x}\rangle\rangle_{\geq 2}$ with $\mathcal{C}\mathrm{rk}(f) \leq 1$, then there is a unique polynomial g of the form $g = x_1^2 + \cdots + x_{d-1}^2 + \varepsilon y^n$ for some $n \geq 2$ and $\varepsilon \in \{0, 1\}$ such that $f \cong g$.*

(1) *If $\varepsilon = 1$, then $\mathcal{J}\mathrm{ac}(f)$ is commutative with $\dim_{\mathbb{C}} \mathcal{J}\mathrm{ac}(f) = n - 1$.*
(2) *If $\varepsilon = 0$, then $\mathcal{J}\mathrm{ac}(f)$ is commutative with $\mathrm{Jdim}\, \mathcal{J}\mathrm{ac}(f) = 1$.*

Proof. By the Splitting Lemma 4.5, there is $\mathsf{f} \in \mathbb{C}\langle\langle x_1, \ldots, x_{d-1}, y\rangle\rangle$ with $f \cong \mathsf{f}$ and either

$$\mathsf{f} = \begin{cases} x_1^2 + \cdots + x_{d-1}^2 + y^2 & \text{if } \mathcal{C}\mathrm{rk}(f) = 0 \\ x_1^2 + \cdots + x_{d-1}^2 + q(y) & \text{if } \mathcal{C}\mathrm{rk}(f) = 1 \end{cases}$$

for some $q \in \mathbb{C}[\![y]\!]$ with $\mathrm{ord}(q) \geq 3$.

If q is zero we are done, else after pulling out the lowest term, we can write $q = y^n \mathsf{u}$ for some $\mathsf{u} = c_n + c_{n+1} y + \ldots \in \mathbb{C}[\![y]\!]$ with $c_n \neq 0$. The homomorphism $\mathbb{C}\langle\langle x_1, \ldots, x_{d-1}, y\rangle\rangle \to \mathbb{C}\langle\langle x_1, \ldots, x_{d-1}, y\rangle\rangle$ which sends $x_k \mapsto x_k$ and $y \mapsto y \sqrt[n]{\mathsf{u}}$ is an automorphism. Since $\sqrt[n]{\mathsf{u}}$ is a power series only in y, it commutes with y, and so this automorphism sends $\sum x_i^2 + y^n$ to $\sum x_i^2 + y^n \mathsf{u} = \mathsf{f}$. Hence $\mathcal{J}\mathrm{ac}(\sum x_i^2 + y^n) \cong \mathcal{J}\mathrm{ac}(\mathsf{f}) \cong \mathcal{J}\mathrm{ac}(f)$, as required.

Parts (1)–(2) are obvious, since $\mathcal{J}\mathrm{ac}(\sum x_i^2 + y^n) \cong \mathbb{C}[\![y]\!]/(y^{n-1})$ and $\mathcal{J}\mathrm{ac}(\sum x_i^2) \cong \mathbb{C}[\![y]\!]$, and uniqueness then follows since $\sum x_i^2 + y^{n_1} \cong \sum x_i^2 + y^{n_2}$ if and only if $n_1 = n_2$. $\square$

Recall from the introduction our geometric applications in the setting of cDV singularities. These correspond to Jacobi algebras in two non-commutating variables, so we set $d = 2$ and write the variables as x, y.

Corollary 5.2. *Every $\mathcal{J}\mathrm{ac}(f)$, where $f \in \mathbb{C}\langle\langle x, y\rangle\rangle_{\geq 2}$ with $f_2 \neq 0$, is geometric.*

Proof. Consider $\mathcal{R} = \mathbb{C}[\![x, y, z]\!]^{\frac{1}{2}(1,1,0)}$, and its unique crepant resolution $\mathcal{X} \to \mathrm{Spec}\, \mathcal{R}$. This has contraction algebra $\mathbb{C}[\![y]\!] \cong \mathcal{J}\mathrm{ac}(x^2)$, realising the second case in 5.1. On the other hand, by [DW1, 3.10] the Type A m-Pagoda flop (with $m \geq 1$) has contraction algebra $\mathbb{C}[\![y]\!]/y^m \cong \mathcal{J}\mathrm{ac}(x^2 + y^{m+1})$, which realises the infinite family in 5.1. $\square$

Example 5.3. Consider $f = x^2 + \frac{2}{3}(xy^2 + yxy + y^2x) \in \mathbb{C}\langle\langle x, y\rangle\rangle$. This has 3-dimensional Jacobi algebra

$$\mathcal{J}\mathrm{ac}(f) \cong \mathbb{C}\langle\langle x, y\rangle\rangle/(\!(x + y^2, xy + yx)\!) \cong \mathbb{C}[y]/y^3$$

so, by 5.1 or Reid's Pagoda [R2], f gives the same Jacobi algebra as $g = x^2 + y^4$. Commutatively, one would see this by completing the square, but that automorphism does not work directly in the noncommutative context: $x \mapsto x - \frac{2}{3}y^2$ gives $f \mapsto x^2 + \frac{2}{3}yxy - \frac{8}{9}y^4$, and we cannot attack the yxy term by coordinate changes that preserve $f_2 = x^2$. But $f \sim x^2 + xy^2 + y^2x$, which then allows us to complete the square (and a scalar on y) to conclude. This exemplifies the way $\sim$ helps to navigate the Jacobi isomorphism classes.

5.2. **Commutativity.** The following characterisation of commutative Jacobi algebras in $d = 2$ variables will be used later.

Proposition 5.4. *$f \in \mathbb{C}\langle\langle x, y\rangle\rangle_{\geq 2}$, then $\mathcal{J}\mathrm{ac}(f)$ is commutative if and only if $\mathcal{C}\mathrm{rk}(f) \leq 1$.*

Proof. ($\Leftarrow$) is clear from 5.1. For ($\Rightarrow$), we prove the contrapositive. If $\mathcal{C}\mathrm{rk}(f) \not\leq 1$, then $f_2 = 0$, and we need to prove that $\mathcal{J}\mathrm{ac}(f)$ is not commutative. For this, it suffices to exhibit a factor that is not commutative. By 4.13 without loss of generality we can assume that f equals

$$x^3 + y^3 + \mathcal{O}_4, \quad xy^2 + \mathcal{O}_4, \quad x^3 + \mathcal{O}_4 \quad \text{or} \quad \mathcal{O}_4.$$

Write $\mathcal{M}_3$ for the set of all noncommutative monomials of degree 3, then factor by the ideal $(\!(\delta_x f, \delta_y f, \mathcal{M}_3)\!)/(\!(\delta_x f, \delta_y f)\!)$ in $\mathcal{J}\mathrm{ac}(f)$. But in the four cases above, by differentiating then using the third isomorphism theorem it follows that $\mathcal{J}\mathrm{ac}(f)$ is one of

$$\frac{\mathbb{C}\langle\langle x, y\rangle\rangle}{(\!(x^2, y^2, \mathcal{M}_3)\!)}, \quad \frac{\mathbb{C}\langle\langle x, y\rangle\rangle}{(\!(y^2, xy + yx, \mathcal{M}_3)\!)}, \quad \frac{\mathbb{C}\langle\langle x, y\rangle\rangle}{(\!(x^2, \mathcal{M}_3)\!)}, \quad \text{or} \quad \frac{\mathbb{C}\langle\langle x, y\rangle\rangle}{(\!(\mathcal{M}_3)\!)}.$$

None of these factors is commutative, and so $\mathcal{J}\mathrm{ac}(f)$ is not commutative. $\square$

6. Type D normal forms

This section considers the next case, namely those $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq 2}$ with $\mathcal{C}\mathrm{rk}(f) = 2$ and $\mathcal{C}\mathrm{rk}_2(f) = 2$. Reducing to two variables by the Splitting Lemma 4.5, the assumption $\mathcal{C}\mathrm{rk}_2(f) = 2$ is then equivalent to the first two cases in 4.13, namely those $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ with $f_3 \neq 0$ for which f_3^{ab} has either two or three distinct linear factors. Full normal forms are obtained in both situations, and are then merged into a unified form in §6.4. These are the Type D normal forms in the tables in §1.2.

Throughout this section, it will be convenient to adopt the following language.

Definition 6.1. We say that a monomial $m \in \mathbb{C}\langle\langle x, y \rangle\rangle$ *contains* x^2 if $m \sim nx^2$ for some monomial n, else m *does not contain* x^2. Similarly, an element $f \in \mathbb{C}\langle\langle x, y \rangle\rangle$ contains x^2 if for some (nonzero) term λm of f, the monomial m contains x^2, else f does not contain x^2. We also use the analogous expressions for y^2.

6.1. **Abelianized Cubic with Three Factors.** This subsection considers the case of 4.13 where f_3^{ab} has three distinct factors, that is $f \cong x^3 + y^3 + \mathcal{O}_4$, and in 6.5 and 6.6 provides two different, but equivalent, normal forms.

Recall from 2.2(1) that a unitriangular automorphism φ of $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ has depth $e \geq 1$ if and only if $\varphi(f)_{\leq e} = f_{\leq e}$ for all $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$.

Lemma 6.2. *Fix $t \geq 4$, and let $f = x^3 + y^3 + f_4 + \cdots + f_t + \mathcal{O}_{t+1}$. For any $h_1, h_2 \in \mathbb{C}\langle\langle x, y \rangle\rangle$ with $\mathrm{ord}(h_i) \geq t-2$, there is a unitriangular automorphism ψ of depth $\geq t-3$ such that*

$$\psi(f) \sim x^3 + y^3 + f_4 + \cdots + f_{t-1} + \left(f_t - h_1 x^2 - h_2 y^2\right) + \mathcal{O}_{t+1}$$

and $\psi(f) - (x^3 + y^3 + f_4 + \cdots + f_{t-1}) \in \mathfrak{n}^t$.

Proof. Consider the unitriangular automorphism ψ which sends $x \mapsto x - \frac{1}{3}h_1$, $y \mapsto y - \frac{1}{3}h_2$. The result follows since

$$\psi(x^3 + y^3) \sim x^3 - h_1 x^2 + \tfrac{1}{3}h_1^2 x - \tfrac{1}{27}h_1^3 + y^3 - h_2 y^2 + \tfrac{1}{3}h_2^2 y - \tfrac{1}{27}h_2^3,$$

and $\psi(m) \equiv m \mod \mathfrak{n}^{t+1}$ whenever $\deg(m) \geq 4$. ∎

With the preparatory lemma in place, the strategy is to first find a standard power series form of each potential, and then distill that down to a polynomial normal form.

Proposition 6.3. *Suppose that $f = f_3 + \mathcal{O}_4$ where f_3^{ab} has three distinct factors. Then $f \cong x^3 + y^3 + p(xy)$ for some power series $p(z) \in \mathbb{C}[\![z]\!]$ with $\mathrm{ord}(p) \geq 2$.*

Recall the Conventions 1.7 on denoting graded pieces of sequence elements: we denote sequence elements $\mathsf{f}_n \in \mathbb{C}\langle\langle x, y \rangle\rangle$ in Greek font, and we write $(\mathsf{f}_n)_t$ for its degree t piece, and $(\mathsf{f}_n)_{<t}$ and $(\mathsf{f}_n)_{>t}$ for sub- and super-degree t portions respectively.

Proof. We construct a sequence of power series $\mathsf{f}_1, \mathsf{f}_2, \ldots$ and unitriangular automorphisms $\varphi^1, \varphi^2, \ldots$ inductively, with each f_t having the form of the target power series $x^3 + y^3 + p(xy)$ in small degree. Summary 3.7(3) then constructs $\mathsf{f} = \lim \mathsf{f}_i$ of the required form with $\mathcal{J}\mathrm{ac}(f) \cong \mathcal{J}\mathrm{ac}(\mathsf{f})$.

By 4.13, $f \cong g$ where $g_3 = x^3 + y^3$. After grouping together terms containing x^2 or y^2 and cyclically permuting, we may write

$$g \sim x^3 + y^3 + \mathsf{h}_2 \cdot x^2 + \mathsf{h}_2' \cdot y^2 + \mu_4 (xy)^2 + \mathcal{O}_5$$

for $\mathsf{h}_2, \mathsf{h}_2' \in \mathbb{C}\langle\langle x, y \rangle\rangle_2$ and $\mu_4 \in \mathbb{C}$.

Hence we begin the induction by setting

$$\mathsf{f}_1 = x^3 + y^3 + \mathsf{h}_2 \cdot x^2 + \mathsf{h}_2' \cdot y^2 + \mu_4 (xy)^2 + g_{\geq 5}$$

and note that $\mathsf{f}_1 \sim g \cong f$. Thus f_1 is in the desired form in degrees ≤ 3 and has its degree 4 piece prepared in standard form for further analysis.

For the inductive step more generally, we may suppose that $\mathsf{f}_t \in \mathbb{C}\langle\langle x, y \rangle\rangle$ has been constructed of the form

$$\mathsf{f}_t = \left(x^3 + y^3 + \mathsf{p}_{t+2}(xy)\right) + \left(\mathsf{h}_{t+1} \cdot x^2 + \mathsf{h}_{t+1}' \cdot y^2 + \mu_{t+3} (xy)^{\lfloor (t+3)/2 \rfloor}\right) + \mathcal{O}_{t+4}$$

with $\mathsf{p}_3 = 0$ and by convention $\mu_{t+3} = 0$ for even t, where

(1) $(\mathsf{f}_t)_{\leq t+2} = x^3+y^3+\mathsf{p}_{t+2}(xy)$, for some polynomial $\mathsf{p}_{t+2} \in \mathbb{C}[z]_{\geq 2}$ of degree $\leq (t+2)/2$, where the polynomials $\mathsf{p}_3, \dots, \mathsf{p}_{t+2}$ satisfy $\mathsf{p}_{i+1} = \mathsf{p}_i$ for even i and $\mathsf{p}_{i+1} - \mathsf{p}_i = \upmu_{i+1} z^{(i+1)/2}$ for odd i, and
(2) $(\mathsf{f}_t)_{t+3} = \mathsf{h}_{t+1} \cdot x^2 + \mathsf{h}'_{t+1} \cdot y^2 + \upmu_{t+3}(xy)^{\lfloor (t+3)/2 \rfloor}$ for some homogeneous forms h_{t+1}, h'_{t+1} of degree $t+1$.

Applying 6.2 with $h_1 = \mathsf{h}_{t+1}$ and $h_2 = \mathsf{h}'_{t+1}$, there exists a unitriangular $\upvarphi^t$ of depth $\geq t$ such that

$$\upvarphi^t(\mathsf{f}_t) \sim \big(x^3+y^3+\mathsf{p}_{t+2}(xy)\big) + \upmu_{t+3}(xy)^{\lfloor (t+3)/2 \rfloor} + \mathcal{O}_{t+4}.$$

In degree $t+4$, again grouping together the terms containing x^2 or y^2 and cyclically permuting, we may write

$$\upvarphi^t(\mathsf{f}_t)_{t+4} \sim \mathsf{h}_{t+2} \cdot x^2 + \mathsf{h}'_{t+2} \cdot y^2 + \upmu_{t+4}(xy)^{\lfloor (t+4)/2 \rfloor}$$

for homogeneous forms h_{t+2}, h'_{t+2} of degree $t+2$ and some $\upmu_{t+4} \in \mathbb{C}$, where again $\upmu_{t+4} = 0$ for odd t. Thus, after setting $\mathsf{p}_{t+3}(xy) = \mathsf{p}_{t+2}(xy) + \upmu_{t+3}(xy)^{\lfloor (t+3)/2 \rfloor}$, define

$$\mathsf{f}_{t+1} = x^3+y^3+\mathsf{p}_{t+3}(xy) + \big(\mathsf{h}_{t+2} \cdot x^2 + \mathsf{h}'_{t+2} \cdot y^2 + \upmu_{t+4}(xy)^{\lfloor (t+4)/2 \rfloor}\big) + \upvarphi^t(\mathsf{f}_t)_{\geq t+5}.$$

Note that $\upvarphi^t(\mathsf{f}_t) \sim \mathsf{f}_{t+1}$, and $\upvarphi^t(\mathsf{f}_t) - \mathsf{f}_{t+1} \in \mathfrak{n}^{t+3} \subset \mathfrak{n}^{t+1}$ using the last statement of 6.2.

Thus we have constructed a sequence of power series $\mathsf{f}_1, \mathsf{f}_2, \dots$ and unitriangular automorphisms $\upvarphi^1, \upvarphi^2, \dots$ to which 3.7(3) applies. For $s \geq 3$ either s is even, in which case $\mathsf{p}_{s+1} = \mathsf{p}_s$, or s is odd, in which case $\mathsf{p}_{s+1} = \mathsf{p}_s + \upmu_{s+1} z^{(s+1)/2}$, thus it is clear that $p := \lim \mathsf{p}_s = \sum_{s=2}^{\infty} \upmu_{2s} z^s$. Further, $\mathsf{f} = \lim \mathsf{f}_i = x^3+y^3+p(xy)$, since the difference $(\mathsf{f}_i - (x^3+y^3+p(xy)))_{i\geq 1}$ converges to zero.

It follows from 3.7(3) that $\mathcal{J}\mathrm{ac}(f) \cong \mathcal{J}\mathrm{ac}(x^3+y^3+p(xy))$, as required. □

The next step is to replace the power series $p(xy)$ by its leading term, without changing the Jacobi algebra.

Lemma 6.4. *If $f = x^3+y^3+p(xy) \in \mathbb{C}\langle\langle x,y\rangle\rangle$ for some $0 \neq p(z) \in \mathbb{C}[\![z]\!]$ for which $s = \mathrm{ord}(p) \geq 2$, then the following statements hold.*
(1) $yx^2, xy^2 \in \langle\langle \updelta_x f, \updelta_y f\rangle\rangle$.
(2) $x^2y, y^2x \in \langle\langle \updelta_x f, \updelta_y f\rangle\rangle$.
(3) $(xy)^s x, (yx)^s y \in \langle\langle \updelta_x f, \updelta_y f\rangle\rangle$.

Proof. (1) Write $J_f = (\updelta_x f, \updelta_y f)$, so that $\langle\langle \updelta_x f, \updelta_y f\rangle\rangle$ is the closure of J_f. Differentiating and pulling out the lowest terms, write

$$\updelta_x f = 3x^2 + y(xy)^{s-1}q(xy) \qquad \text{and} \qquad \updelta_y f = 3y^2 + q(xy)(xy)^{s-1}x \tag{6.A}$$

for some $q = \uplambda_0 + \uplambda_1 z + \uplambda_2 z^2 + \dots \in \mathbb{C}[\![z]\!]$ with $\uplambda_0 \neq 0$. Writing $A \equiv B$ for $A - B \in J_f$, then in particular

$$x^2 \equiv -\tfrac{1}{3}y(xy)^{s-1}q(xy) \qquad \text{and} \qquad y^2 \equiv -\tfrac{1}{3}q(xy)(xy)^{s-1}x. \tag{6.B}$$

Substituting for x^2 or y^2 at each step, we see that

$$\begin{aligned}
y \cdot x^2 &\equiv \tfrac{-1}{3}y^2 \cdot (xy)^{s-1}q(xy) && \in \mathfrak{n}^{2s}\\
&\equiv \tfrac{+1}{3^2}q(xy)(xy)^{s-1}x \cdot (xy)^{s-1}q(xy) &&\\
&= \tfrac{+1}{3^2}q(xy)(xy)^{s-1} \cdot x^2 \cdot y(xy)^{s-2}q(xy) && \in \mathfrak{n}^{4s-3}\\
&\equiv \tfrac{-1}{3^3}q(xy)(xy)^{s-1} \cdot y(xy)^{s-1}q(xy) \cdot y(xy)^{s-2}q(xy) &&\\
&= \tfrac{-1}{3^3}q(xy)(xy)^{s-2}x \cdot y^2 \cdot (xy)^{s-1}q(xy)y(xy)^{s-2}q(xy) && \in \mathfrak{n}^{6s-6}\\
&= \dots
\end{aligned}$$

At each substitution, the resulting power series has order $2s-3 \geq 1$ higher than the previous one. It follows from the above that for all $t \geq 2s$ there exists $n_t \in \mathfrak{n}^t$ such that $yx^2 - n_t \in J_f$. Hence $yx^2 \in \bigcap_{t\geq 2s}(J_f + \mathfrak{n}^t)$, which is precisely the closure $\langle\langle \updelta_x f, \updelta_y f\rangle\rangle$. By symmetry in x and y, the analogous statement $xy^2 \in \langle\langle \updelta_x f, \updelta_y f\rangle\rangle$ also follows.
(2) This follows in an analogous way: start by writing

$$\updelta_x f = 3x^2 + r(yx)(yx)^{s-1}y \qquad \text{and} \qquad \updelta_y f = 3y^2 + x(yx)^{s-1}r(yx)$$

then consider $x^2 \cdot y \equiv \frac{-1}{3} r(yx)(yx)^{s-1}y^2$, etc.
(3) Now write $A \equiv B$ for $A - B \in \langle\!\langle \delta_x f, \delta_y f \rangle\!\rangle$. Separating off the lowest term of $q(xy)$ in (6.A), we may write

$$\delta_y f = 3y^2 + \lambda_0 (xy)^{s-1}x + (q(xy) - \lambda_0)(xy)^{s-1}x$$

and so $\lambda_0 (xy)^{s-1}x \equiv -3y^2 - (q(xy) - \lambda_0)(xy)^{s-1}x$ where $q(xy) - \lambda_0 \in \mathfrak{n}^2$. Then

$$\begin{aligned}
\lambda_0 (xy)^s x &= \left(\lambda_0 (xy)^{s-1}x\right) yx \\
&\equiv \left(-3y^2 - (q(xy) - \lambda_0)(xy)^{s-1}x\right)(yx) \\
&= -3y \cdot y^2 \cdot x - (q(xy) - \lambda_0)(xy)^s x \\
&\equiv y \cdot q(xy)(xy)^{s-1}x \cdot x - (q(xy) - \lambda_0)(xy)^s x && \text{(by (6.B))} \\
&\equiv -(q(xy) - \lambda_0)(xy)^s x && (yx^2 \equiv 0 \text{ by (1)})
\end{aligned}$$

The $\lambda_0 (xy)^s x$ on each side cancel, showing that $q(xy)(xy)^s x \in \langle\!\langle \delta_x f, \delta_y f \rangle\!\rangle$. Since $q(xy)$ is a unit, it follows that $(xy)^s x \in \langle\!\langle \delta_x f, \delta_y f \rangle\!\rangle$. Again, appealing to symmetry in x and y proves the final statement. □

Proposition 6.5. *Suppose that $f = x^3 + y^3 + p(xy)$ where $p(z) \in \mathbb{C}[\![z]\!]$ with $s = \operatorname{ord}(p) \geq 2$. Then*

$$f \cong \begin{cases} x^3 + y^3 & \text{when } p = 0 \\ x^3 + y^3 + (xy)^s & \text{when } p \neq 0. \end{cases}$$

Furthermore

1. *$\operatorname{Jdim} \mathfrak{Jac}(f) \leq 1$, with equality if and only if $p = 0$.*
2. *If $p \neq 0$, then $\dim_{\mathbb{C}} \mathfrak{Jac}(f) = 4s$, and $\dim_{\mathbb{C}} \mathfrak{Jac}(f)^{\mathrm{ab}} = 4$.*

Therefore the expressions $x^3 + y^3 + (xy)^s$ with $s \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ form a set of normal forms.

Proof. For the first statement, if $p = 0$ we are done, so suppose $p \neq 0$. Continuing the notation in the proof of 6.4 above, after differentiating and pulling out the lowest terms, we may write

$$\delta_x f = 3x^2 + y(xy)^{s-1}q(xy) \qquad \text{and} \qquad \delta_y f = 3y^2 + q(xy)(xy)^{s-1}x \tag{6.C}$$

for some $q = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \cdots \in \mathbb{C}[\![z]\!]$ with $\lambda_0 \neq 0$. Set $g = x^3 + y^3 + (\lambda_0/s)(xy)^s$. Now 6.4 applies equally well to both f and g, hence both $(xy)^s x$ and $(yx)^s y$ belong to *both* the Jacobi ideals associated to f and g. Consequently

$$\begin{aligned}
\langle\!\langle \delta_x f, \delta_y f \rangle\!\rangle &= \langle\!\langle \delta_x f, \delta_y f, (xy)^s x, (yx)^s y \rangle\!\rangle \\
&= \langle\!\langle \delta_x g, \delta_y g, (xy)^s x, (yx)^s y \rangle\!\rangle && \text{(cancel higher terms from } \delta_x f \text{ and } \delta_y f) \\
&= \langle\!\langle \delta_x g, \delta_y g \rangle\!\rangle.
\end{aligned}$$

It follows that $f \simeq g$. The coordinate change $x \mapsto ax$, $y \mapsto ay$ for $a = \sqrt[2s-3]{s/\lambda_0}$ then normalises the constant factor $\lambda_0/s \neq 0$, as required.
(1) Consider the case $p = 0$. As in 3.2, if $\mathfrak{J}$ is the Jacobson radical of $\mathfrak{Jac}(f)$, then

$$\frac{\mathfrak{J}^d}{\mathfrak{J}^{d+1}} = \frac{\mathfrak{n}^d + \langle\!\langle x^2, y^2 \rangle\!\rangle}{\mathfrak{n}^{d+1} + \langle\!\langle x^2, y^2 \rangle\!\rangle}.$$

This is always a two-dimensional vector space, since if d is even it has basis $(xy)^{d/2}$ and $(yx)^{d/2}$, whilst if d is odd it has basis $x(yx)^{(d-1)/2}$ and $y(xy)^{(d-1)/2}$. It follows that $\mathfrak{Jac}(f)$ is an infinite-dimensional $\mathbb{C}$-algebra, with $\operatorname{Jdim} \mathfrak{Jac}(f) = 1$.

When $p \neq 0$, 6.4 shows at once that $\mathfrak{Jac}(f)$ is finite dimensional: indeed any monomial of degree $t \geq 2s + 1$ either contains one of the monomials listed in 6.4(1–3) or is x^t or y^t. But by (6.C) $x^t = x^{t-4} \cdot x^2 \cdot x^2$ and $y^t = y^2 \cdot y^2 \cdot y^{t-4}$ is equivalent, modulo $\langle\!\langle \delta_x f, \delta_y f \rangle\!\rangle$, to a monomial that contains one of those listed, and thus is equivalent to zero. Consequently, the entire graded piece of degree t is zero, and so $\mathfrak{Jac}(f)$ is finite dimensional.
(2) We compute a suitable Gröbner basis with a local graded monomial order, that is, the leading terms are those of lowest order (there is no need here for lexicographical considerations to break ties). The proof of part (1) above introduces a key simplifying

factor: since all monomials of degree $t = 2s + 1$ lie in the closed Jacobian ideal, it is sufficient to work in the quotient

$$\mathcal{J}\mathrm{ac}(f) \cong \frac{\mathbb{C}\langle x, y\rangle}{(\delta_x f, \delta_y f, \mathcal{M}_t)}$$

where $\mathcal{M}_t$ for the set of all noncommutative monomials of degree t, where now closure is no longer an issue.

We compile a Gröbner basis $\{g_1, g_2, \dots\}$ of $(\delta_x f, \delta_y f, \mathcal{M}^t)$ starting with

$$g_1 = 3x^2 + sy(xy)^{s-1} \quad \text{and} \quad g_2 = 3y^2 + sx(yx)^{s-1}$$

which have leading terms $3x^2$ and $3y^2$ respectively, as $s \geq 2$. The computation proceeds by resolving non-trivial overlaps among leading terms. The leading term of g_1 overlaps non-trivially with itself to produce

$$g_3 = xg_1 - g_1 x = s(xy)^s - s(yx)^s$$

which does not reduce further modulo existing leading terms. The analogous overlap $yg_2 - g_2 y$ gives the same g_3, and all other overlaps have order $\geq t$, so are zero in the ideal. Hence the Gröbner basis is $\{g_1, g_2, g_3\}$.

This Gröbner basis provides a monomial basis for the quotient as follows. Among all monomials of degree $< t$, any of the form $m_1 x^2 m_2$, for monomials m_1, m_2, reduce modulo g_1 to zero at once, and similarly with y^2. Thus a basis consists of all other monomials, with the single relation g_3. That is, in each pair of degrees $2e, 2e+1$ for $1 \leq e \leq s-1$ there are 4 monomials

$$(xy)^e, (yx)^e, \ (xy)^e x, (yx)^e y,$$

and then $1, x, y$ in degrees ≤ 1 and finally $(xy)^s \equiv (yx)^s$ in degree $2s$. Summing up, this basis has size $4s$, as claimed.

In the abelianisation, if $s > 2$ we may rewrite the derivatives as $x^2(\text{unit})$ and $y^2(\text{unit})$. Hence $\mathcal{J}\mathrm{ac}(f)^{\mathrm{ab}} \cong \mathbb{C}[\![x, y]\!]/(x^2, y^2)$, which is four dimensional. When $s = 2$, it is also easy to verify that $\mathcal{J}\mathrm{ac}(f)^{\mathrm{ab}} \cong \mathbb{C}[\![x, y]\!]/(x^2, y^2)$, so in all cases the dimension is four. □

In order to state a unified theorem with the xy^2 case in §6.4 below, it is convenient to mildly change basis. This is rather cheap, largely because there are no moduli.

Corollary 6.6. *Suppose that $f \in \mathbb{C}\langle\!\langle x, y\rangle\!\rangle_{\geq 3}$ where f_3^{ab} has three roots. Then either*

$$f \cong \begin{cases} xy^2 + x^3 \\ xy^2 + x^3 + x^{2n} & \text{for some } n \geq 2. \end{cases}$$

Furthermore, the above are normal forms.

Proof. Each of the forms f listed satisfies the condition that f_3^{ab} has three roots. Thus there exists some g from the list in 6.5 with $g \cong h$. Furthermore, $\dim_{\mathbb{C}} \mathcal{J}\mathrm{ac}(xy^2 + x^3) = \infty$, whereas $\dim_{\mathbb{C}} \mathcal{J}\mathrm{ac}(xy^2 + x^3 + x^{2n}) = 4n$ (see e.g. [vG, §5] and [Ka, §5], or 4.11), and so all options are uniquely covered. Since the g listed in 6.5 are normal forms, it follows that the f listed here are normal forms. □

6.2. **Isomorphisms on the Quantum Plane.** The following, which may be of independent interest, is one of the key reduction steps that will be used in §6.3.

Lemma 6.7. *For any units $v, w \in \mathbb{C}[\![x^2]\!]$, the unitriangular automorphism φ of $\mathbb{C}\langle\!\langle x, y\rangle\!\rangle$ sending $x \mapsto xv$, $y \mapsto yw$ descends to a topological isomorphism*

$$\frac{\mathbb{C}\langle\!\langle x, y\rangle\!\rangle}{(\!(xy + yx)\!)} \xrightarrow{\cong} \frac{\mathbb{C}\langle\!\langle x, y\rangle\!\rangle}{(\!(xy + yx)\!)}.$$

Proof. The inverse of φ, as an automorphism of $\mathbb{C}\langle\!\langle x, y\rangle\!\rangle$, is clearly given by the unitriangular automorphism $\psi\colon x \mapsto xv^{-1}$, $y \mapsto yw^{-1}$. Set $I = (\!(xy + yx)\!)$, then since φ is a topological isomorphism by 2.3, we just need to prove that $\varphi(I) = I$.

Now x commutes with v and w, being power series in x^2, and also $vw = wv$. But, modulo $I = (\!(xy + yx)\!)$, y commutes with x^2, thus since the ideal is closed y commutes with both v and w. It follows that

$$\varphi(xy + yx) = xvyw + ywxv \equiv (xy + yx)vw \equiv 0 \mod I. \tag{6.D}$$

Since φ is a continuous isomorphism, and I is the smallest closed ideal containing $xy+yx$, $\varphi(I)$ is the smallest closed ideal containing $\varphi(xy+yx)$. But by (6.D) $\varphi(xy+yx)$ also belongs to the closed ideal I, so by minimality $\varphi(I) \subseteq I$.

Since v^{-1} and w^{-1} are also units in $\mathbb{C}[\![x^2]\!]$, exactly the same logic applied to ψ shows that $\psi(I) \subseteq I$. Applying φ to this inclusion, we see that $I = \varphi\psi(I) \subseteq \varphi(I)$. Combining inclusions gives $\varphi(I) = I$. $\square$

6.3. **Abelianized Cubic with Two Factors.** This subsection considers the case of 4.13 where f_3^{ab} has two distinct factors, that is $f \cong x^2y + \mathcal{O}_4$, and in 6.17 provides normal forms. This is substantially harder than in §6.1.

Lemma 6.8. *Fix $t \geq 4$, and let $f = xy^2 + f_4 + \cdots + f_t + \mathcal{O}_{t+1}$. For any $h \in \mathbb{C}\langle\!\langle x,y\rangle\!\rangle$ with* $\operatorname{ord}(h) = t-2$*, the unitriangular automorphism $x \mapsto x-h$, $y \mapsto y$ sends*

$$f \mapsto xy^2 + f_4 + \cdots + f_{t-1} + \big(f_t - hy^2\big) + \mathcal{O}_{t+1}.$$

Proof. Write ψ for the stated automorphism. The result follows since $\psi(xy^2) = xy^2 - hy^2$ and $\psi(m) \equiv m \mod \mathfrak{n}^{t+1}$ whenever $\deg(m) \geq 4$. $\square$

The next lemma is much less elementary.

Lemma 6.9. *Fix $t \geq 4$, and let $f = xy^2 + f_4 + \cdots + f_t + \mathcal{O}_{t+1}$, where furthermore*

$$f_t = \mathsf{h}_{t-2} \cdot y^2 + \sum \lambda_a x^{a_1} y \dots x^{a_r} y + \alpha x^t \tag{6.E}$$

for some homogeneous form h_{t-2} of degree $t-2$, each $a_i \geq 1$, $r \geq 1$ and $r + \sum a_i = t$, $\alpha \in \mathbb{C}$ and each $\lambda_a = \lambda_{a_1 \cdots a_r} \in \mathbb{C}$. Then there exists a unitriangular automorphism φ of depth $\geq t-3$ such that

$$\varphi(f) = xy^2 + f_4 + \cdots + f_{t-1} + (g_t + \alpha x^t) + \mathcal{O}_{t+1}.$$

where $g_t \in \mathbb{C}\langle\!\langle x,y\rangle\!\rangle_t$ satisfies $g_t \sim 0$.

Example 6.10. It is worth considering an example to make the notation of both the statement and proof more transparent. Consider

$$f = xy^2 + \Big(\lambda_{51} x^5 yxy + \lambda_{42} x^4 yx^2 y + \lambda_{33} x^3 yx^3 y\Big) + f_{\geq 9}$$

which has f_8 of the form (6.E). Applying $\varphi_1 \colon x \mapsto x$ and $y \mapsto y - \lambda_{33} x^3 yx^2$, where we cancelled xy from the right of the target λ_{33} term to obtain the subtracted term, gives

$$\varphi_1(f) = xy^2 + \lambda_{51} x^5 yxy + (\lambda_{42} - \lambda_{33}) x^4 yx^2 y + g_1 + \mathcal{O}_9$$

where $g_1 = \lambda_{33}(x^3yx^3y - xyx^3yx^2) \sim 0$. Ignoring g_1, the summation in degree 8 symbolically now has only two terms, which is progress.

An analogous automorphism φ_2 sending $x \mapsto x$ and $y \mapsto -(\lambda_{42} - \lambda_{33})x^4yx$, where we cancelled xy from the right of the next target term, gives

$$\varphi_2\varphi_1(f) = xy^2 + (\lambda_{51} - \lambda_{42} + \lambda_{33}) x^5 yxy + g_2 + \mathcal{O}_9$$

for some $g_2 \sim 0$, and again the number of terms in degree 8 (outside g_2) has not increased. Repeating again with an analogous automorphism φ_3 gives

$$\varphi_3\varphi_2\varphi_1(f) = xy^2 - (\lambda_{51} - \lambda_{42} + \lambda_{33}) x^6 y^2 + g_3 + \mathcal{O}_9$$

with $g_3 \sim 0$. We are now in a position to apply 6.8 to leave only g_3 in degree 8.

The proof below confirms that this inductive idea works more generally.

Proof. If the middle sum in the expression for f_t is zero, we are done by 6.8 (with $g_t = 0$), so we may assume that the sum is nonzero.

Suppose that the middle sum contains a term $\mathsf{t}_1 = \lambda_b x^{b_1} y \cdots x^{b_r} y$ with $r > 1$. In this case, consider the unitriangular automorphism φ defined by $x \mapsto x$, $y \mapsto y - \lambda_b x^{b_1} y \cdots yx^{b_r - 1}$, where we have simply cancelled xy from the right-hand side of the target term t_1. As in 6.8, $\varphi(m) \equiv m \mod \mathfrak{n}^{t+1}$ whenever $\deg(m) \geq 4$, so any change in degree $\leq t$ comes from $\varphi(xy^2)$, and thus

$$\begin{aligned}\varphi(f) = xy^2 &+ f_4 + \cdots + f_{t-1} \\ &+ (f_t - \lambda_b xyx^{b_1} y \cdots yx^{b_r - 1} - \lambda_b x^{b_1+1} y \cdots yx^{b_r-1} y) + \mathcal{O}_{t+1}\end{aligned} \tag{6.F}$$

Writing $\mathsf{g}_1 = \mathsf{t}_1 - \lambda_b xyx^{b_1}y\cdots yx^{b_r-1} \sim 0$, then the degree t term of (6.F) equals

$$\mathsf{g}_1 + h\cdot y^2 + \sum \lambda_a x^{a_1}y\dots x^{a_r}y + \alpha x^t$$

where, under the summand, the target term t_1 has been replaced by a term of the form $\mathsf{t}_2 = -\lambda_b x^{b_1+1}y\cdots yx^{b_r-1}y$, so the sum has the same number of terms or fewer (depending on whether t_2 cancels with existing terms or not).

If $b_r = 1$, then the new term t_2 equals hy^2 for $h = -\lambda_b x^{b_1+1}y\cdots yx^{b_{r-1}}$, and we so may apply 6.8 to (6.F) to find ψ such that

$$\psi\varphi(f) = xy^2 + f_4 + \cdots + f_{t-1} + (f_t - \lambda_b xyx^{b_1}y\cdots yx^{b_r-1}) + \mathcal{O}_{t+1}$$

where the degree t term is equal to

$$\begin{aligned} f_t - \lambda_b xyx^{b_1}y\cdots yx^{b_r-1} &= \mathsf{g}_1 + f_t - \lambda_b x^{b_1}y\cdots x^{b_r}y \\ &= \mathsf{g}_1 + h\cdot y^2 + \sum_{a\neq b}\lambda_a x^{a_1}y\dots x^{a_r}y + \alpha x^t \end{aligned}$$

and the number of terms under the summand is now strictly reduced.

Otherwise, $b_r > 1$. Set $\varphi_1 = \varphi$, and repeating the original construction of a unitriangular automorphism by cancelling xy from the right, we can construct φ_2 such that

$$\begin{aligned} \varphi_2\varphi_1(f) = xy^2 &+ f_4 + \cdots + f_{t-1} \\ &+ \big(\mathsf{g}_2 + h\cdot y^2 + \sum \lambda_a x^{a_1}y\dots x^{a_r}y + \alpha x^t\big) + \mathcal{O}_{t+1} \end{aligned} \tag{6.G}$$

where $\mathsf{g}_2 \sim 0$ is the sum of g_1 and another binomial ~ 0, and in the sum we have replaced the term $-\lambda_b x^{b_1+1}y\cdots yx^{b_r-1}y$ by $\lambda_b x^{b_1+2}y\cdots yx^{b_r-2}y$. Repeating this, we find unitriangular automorphisms $\varphi_1,\dots,\varphi_{b_r-1}$ so that $\varphi_{b_r-1}\cdots\varphi_2\varphi_1(f)$ has the form of (6.G), and the sum has the same number of terms or fewer, but in which the target monomial we are focussing on has become $x^{b_1+b_r-1}yx^{b_2}\cdots yxy$. A further repetition with a unitriangular automorphism φ_{b_r} replaces that term by one that contains y^2, and once again we may apply 6.8 to find a unitriangular automorphism ψ that moves this term into higher degree. Thus after applying the single unitriangular automorphism $\psi\varphi_{b_r}\cdots\varphi_1$ to f, the number of terms in the summation when parsed in the form (6.E) has strictly reduced.

We repeat this process inductively, and it will terminate when there are no terms under the summation sign of the form $\lambda_a x^{a_1}y\dots x^{a_r}y$ with $r > 1$. Each step was achieved by a single unitriangular automorphism (itself built as a composition of unitriangular automorphisms), and composing each of these gives a single unitriangular automorphism ϑ such that

$$\vartheta(f) = xy^2 + f_4 + \cdots + f_{t-1} + (\mathsf{g} + h\cdot y^2 + \lambda x^{t-1}y + \alpha x^t) + \mathcal{O}_{t+1}$$

for some g with $\mathsf{g} \sim 0$.

To conclude, the unitriangular automorphism φ defined by $x \mapsto x - h$, $y \mapsto y - \frac{\lambda}{2}x^{a_1-1} = y - \frac{\lambda}{2}x^{t-2}$ has depth $t-3$, so again $\varphi(m) \equiv m \mod \mathfrak{n}^{t+1}$ whenever $\deg(m) \geq 4$ and thus

$$\varphi\vartheta(f) = xy^2 + f_4 + \cdots + f_{t-1} + (\mathsf{g} + \underbrace{\lambda x^{t-1}y - \tfrac{\lambda}{2}xyx^{t-2} - \tfrac{\lambda}{2}x^{t-2}y}_{\mathsf{h}} + \alpha x^t) + \mathcal{O}_{t+1}.$$

Set $g_t = \mathsf{g} + \mathsf{h}$, then since both $\mathsf{g} \sim 0$ and $\mathsf{h} \sim 0$, we are done. $\square$

From here, the strategy of §6.1 remains: first find a standard power series form of each potential, then simplify into polynomial normal form.

Proposition 6.11. *Suppose that $f = f_3 + \mathcal{O}_4$ where f_3^{ab} has two distinct linear factors. Then $f \cong xy^2 + q(x)$ for some power series $q(x) \in \mathbb{C}[\![x]\!]$ with $\mathrm{ord}(q) \geq 4$.*

Recall the Conventions 1.7, used in 6.3, on graded pieces of sequence elements: namely sequence elements $\mathsf{f}_n \in \mathbb{C}\langle\!\langle x,y\rangle\!\rangle$ are in Greek font, whilst $(\mathsf{f}_n)_t$ is the degree t piece of f_n.

Proof. We construct a sequence of power series $\mathsf{f}_1, \mathsf{f}_2, \dots$ and unitriangular automorphisms $\varphi^1, \varphi^2, \dots$ inductively, with each f_t having the form of the target power series $xy^2 + q(x)$ in low degree. Summary 3.7(3) will then construct $\mathsf{f} = \lim \mathsf{f}_i$ of the required form with $\mathrm{Jac}(f) \cong \mathrm{Jac}(\mathsf{f})$.

By 4.13, $f \cong g$ where $g_3 = xy^2$. After grouping together the terms containing y^2, then the terms that contain y but not y^2, and cyclically permuting, we may write

$$g_4 \sim \mathsf{h}_2 \cdot y^2 + \sum \lambda_a x^{a_1} y \ldots x^{a_r} y + \mu_4 x^4$$

for $\mathsf{h}_2 \in \mathbb{C}\langle x, y\rangle_2$, $r \geq 1$ and each $a_i \geq 1$ and $\mu_4 \in \mathbb{C}$ and where we use the abbreviated notation $\lambda_a := \lambda_{a_1 \cdots a_r} \in \mathbb{C}$. It is convenient to write the sum as $\sum \lambda_a x^{a_1} y \ldots x^{a_r} y$ by analogy with the general case, noting that here it is nothing more than $\lambda_{11} xyxy + \lambda_3 x^3 y$.

Hence we begin the induction by setting

$$\mathsf{f}_1 = xy^2 + (\mathsf{h}_2 \cdot y^2 + \sum \lambda x^{a_1} y \ldots x^{a_r} y + \mu_4 x^4) + g_{\geq 5}$$

and note that $\mathsf{f}_1 \sim g \cong f$. Thus f_1 is in the desired form in degrees ≤ 3 and has its degree 4 piece prepared in standard form for further analysis.

For the inductive step more generally, we may suppose that $\mathsf{f}_t \in \mathbb{C}\langle\langle x, y\rangle\rangle$ has been constructed of the form

$$\mathsf{f}_t = \big(xy^2 + \mathsf{q}_{t+2}(x)\big) + \big(\mathsf{h}_{t+1} \cdot y^2 + \sum \lambda_a x^{a_1} y \ldots x^{a_r} y + \mu_{t+3} x^{t+3}\big) + \mathcal{O}_{t+4}$$

with $\mathsf{q}_3 = 0$ and

(1) $(\mathsf{f}_t)_{\leq t+2} = xy^2 + \mathsf{q}_{t+2}(x)$, for some polynomial $\mathsf{q}_{t+2} \in \mathbb{C}[x]_{\geq 4}$ of degree $\leq t+2$, where the polynomials $\mathsf{q}_3, \ldots, \mathsf{q}_{t+2}$ satisfy $\mathsf{q}_{i+1} - \mathsf{q}_i = \mu_{i+1} x^{i+1}$ for $\mu_{i+1} \in \mathbb{C}$, and
(2) $(\mathsf{f}_t)_{t+3} = \mathsf{h}_{t+1} \cdot y^2 + \sum \lambda_a x^{a_1} y \ldots x^{a_r} y + \mu_{t+3} x^{t+3}$ for some homogeneous form h_{t+1} of degree $t+1$, each $a_i \geq 1$, $r \geq 1$ and $r + \sum a_i = t+3$, and $\mu_{t+3} \in \mathbb{C}$.

By 6.9 there exists a unitriangular φ^t of depth t such that

$$\varphi^t(\mathsf{f}_t) = xy^2 + q_{t+2}(x) + (k_{t+3} + \mu_{t+3} x^{t+3}) + \mathcal{O}_{t+4}.$$

where $k_{t+3} \sim 0$. In degree $t+4$, again grouping together the terms containing y^2, then the terms that contain y but not y^2, and cyclically permuting, we may write

$$\varphi^t(\mathsf{f}_t)_{t+4} \sim \mathsf{h}_{t+2} \cdot y^2 + \sum \lambda_a x^{a_1} y \ldots x^{a_r} y + \mu_{t+4} x^{t+4}.$$

Thus after setting $\mathsf{q}_{t+3}(x) = \mathsf{q}_{t+2}(x) + \mu_{t+3} x^{t+3}$, define

$$\mathsf{f}_{t+1} = xy^2 + \mathsf{q}_{t+3}(x) + \big(\mathsf{h}_{t+2} \cdot y^2 + \sum \lambda_a x^{a_1} y \ldots x^{a_r} y + \mu_{t+4} x^{t+4}\big) + \varphi^t(\mathsf{f}_t)_{\geq t+5}.$$

Note that $\varphi^t(\mathsf{f}_t) \sim \mathsf{f}_{t+1}$, and that $\varphi^t(\mathsf{f}_t) - \mathsf{f}_{t+1} \in \mathfrak{n}^{t+3} \subset \mathfrak{n}^{t+1}$.

Thus we have constructed a sequence of power series $\mathsf{f}_1, \mathsf{f}_2, \ldots$ and unitriangular automorphisms $\varphi^1, \varphi^2, \ldots$ to which 3.7(3) applies. Since at each stage $\mathsf{q}_s = \mathsf{q}_{s-1} + \mu_s x^s$, it is clear that $q := \lim \mathsf{q}_s = \sum_{s=4}^{\infty} \mu_s x^s$, and that $\mathsf{f} = \lim \mathsf{f}_i = xy^2 + q$, since the difference $(\mathsf{f}_i - (xy^2 + q))_{i \geq 1}$ converges to zero. Hence $\mathcal{J}\mathrm{ac}(f) \cong \mathcal{J}\mathrm{ac}(xy^2 + q)$, as required. □

The next step is to reduce the options for $q(x)$, using the following preliminary lemma.

Lemma 6.12. *Let $u \in \mathbb{C}[\![x]\!]$ be an even power series: that is, u is a power series in x^2.*

(1) *If u is a unit, then u^{-1} and $\sqrt[n]{u}$ are also even power series for any $n \geq 2$.*
(2) *Let $U \in \mathbb{C}[\![x]\!]$ be a unit and $n \in \mathbb{Z}$ a nonzero integer. Then there is a unit $t \in \mathbb{C}[\![x]\!]$ with $t^n = U(xt)$. Furthermore, if U is even then t is even.*

Proof. (1) Consider $v \in \mathbb{C}[\![z]\!]$ with $u(x) = v(x^2)$. If u is a unit, then v is a unit and v^{-1} and $\sqrt[n]{v} \in \mathbb{C}[\![z]\!]$ for all $n \geq 2$. Then $u^{-1}(x) = v^{-1}(x^2)$ and $\sqrt[n]{u}(x) = \sqrt[n]{v}(x^2)$.
(2) Write $U = a_0 + a_1 x + a_2 x^2 + \cdots$ with $a_0 \neq 0$. Consider the case $n > 0$. We show that we may solve inductively for the coefficients b_d of the expansion $t = b_0 + b_1 x + b_2 x^2 + \cdots$ in the equation $t^n = U(xt)$.

It is clear that the coefficient of x^d in t^n is a sum of $nb_0^{n-1} b_d$ with terms involving only coefficients b_i with $i < d$. On the other side of the equation, the coefficient of x^d in $U(xt)$ is a sum of terms involving a_i and b_j with $i \leq d$ and $j < d$. Putting these together, b_d does not appear in the coefficient of x^i for any $i < d$, and it appears linearly with nonzero coefficient for the first time in the coefficient of x^d, and so we may solve for it. Working inductively in increasing $d \geq 0$, and taking the limit, determines t as claimed. For $n < 0$, the same argument proves the existence of the unit t^{-1}, which is equivalent.

Suppose that U is even, and let b_{2n+1} be the smallest nonzero odd-degree coefficient of t. Then the odd-degree term with smallest degree in t^n is $nb_0^{n-1} b_{2n+1} x^{2n+1}$, while in

$U(xt)$ it is $2a_2b_0b_{2n+1}x^{2n+3}$ which appears in the summand $a_2(xt)^2$, a contradiction. So t must be even. $\square$

The point is now simple: y is under control, and so there is a relation $xy+yx$ in the Jacobi algebra. Then 6.7 yields the following key preparation result.

Proposition 6.13. *If $f = xy^2 + p(x)$ where $p(x) \in \mathbb{C}[\![x]\!]$ with $\operatorname{ord}(p) \geq 4$, then*

$$f \cong xy^2 + \alpha\, x^{2n} + \beta\, x^{2m+1}$$

for some $n, m \geq 2$, and some $\alpha, \beta \in \{0,1\}$. Furthermore, $\alpha = 1$ if and only if p has a nonzero even-degree term, in which case $2n$ is the least even degree appearing in p, and similarly the analogous criterion for $\beta = 1$ and least odd degree term in p.

Proof. First note that

$$\mathfrak{Jac}(f) = \mathbb{C}\langle\!\langle x,y\rangle\!\rangle/(\!(xy+yx, y^2+\delta_x p)\!).$$

We exhibit an automorphism of $\mathbb{C}\langle\!\langle x,y\rangle\!\rangle$ that takes the two generators of the Jacobian ideal to $(xy+yx)$(unit) and $\big(y^2+\beta x^{2m}+\alpha x^{2n-1}\big)$ (unit), respectively, where either $\alpha = 0$ or $2n \geq 4$ is the least even degree appearing in p, and either $\beta = 0$ or $2m+1 \geq 5$ is the least odd degree appearing in p. This proves all the claims.

Parsing $\delta_x p$ into even and odd terms, write the Jacobi algebra relations as

$$xy+yx \quad \text{and} \quad y^2 + ax^{2N}u + bx^{2M-1}v$$

where $u, v \in \mathbb{C}[\![x^2]\!]$ are each either a unit or zero, $N, M \geq 2$, and $a, b \in \mathbb{C}$ are any nonzero numbers that carry through the calculation undisturbed; we choose $a = 2N+1$ and $b = 2M$ at the end.

Suppose in the first place that $u \neq 0$, then fix a square root $s = \sqrt{u} \in \mathbb{C}[\![x^2]\!]$ and consider the unitriangular automorphism φ sending $x \mapsto x$, $y \mapsto ys$. By 6.7 this induces a topological isomorphism

$$\bar{\varphi}\colon \frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(xy+yx)\!)} \xrightarrow{\sim} \frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(xy+yx)\!)}.$$

In the codomain of this map, y commutes with x^2 and thus commutes with $s \in \mathbb{C}[\![x^2]\!]$. It follows that $\bar{\varphi}(y^2) = ysys = y^2s^2 = yu$ and thus

$$\bar{\varphi}(y^2+\delta_x p) = (y^2+ax^{2N})u + bx^{2M-1}v.$$

By 2.11(1)(3) after right multiplying by the unit u^{-1}, we obtain an isomorphism

$$\frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(xy+yx, y^2+\delta_x p)\!)} \xrightarrow{\sim} \frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(xy+yx, y^2+ax^{2N}+bx^{2M-1}\frac{v}{u})\!)}. \tag{6.H}$$

If $v=0$, then (6.H) asserts that $\mathfrak{Jac}(f) \cong \mathfrak{Jac}(xy^2+x^{2N+1})$, and so we are done. Hence we may assume that $v \neq 0$.

As u and v are both unit power series in x^2, so is $\frac{v}{u}$. By 6.12(2), since $2N-2M+1$ is nonzero, we may choose a unit $t \in \mathbb{C}[\![x^2]\!]$ such that $t^{2N} = t^{2M-1}v(xt)/u(xt)$. Consider the unitriangular automorphism ψ sending $x \mapsto xt$, $y \mapsto yt^N$. Again by 6.7 there is an induced topological isomorphism

$$\bar{\psi}\colon \frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(xy+yx)\!)} \xrightarrow{\sim} \frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(xy+yx)\!)}.$$

Clearly x commutes with $t, t^N \in \mathbb{C}[\![x^2]\!]$, and further in the codomain of $\bar{\psi}$ the element y commutes with x^2 and thus commutes with $t, t^N \in \mathbb{C}[\![x^2]\!]$. Thus

$$\begin{aligned}\bar{\psi}(y^2+ax^{2N}+bx^{2M-1}\tfrac{v}{u}) &= y^2t^{2N} + ax^{2N}t^{2N} + bx^{2M-1}t^{2M-1}u(xt)/v(xt)\\ &= (y^2+ax^{2N}+bx^{2M-1})t^{2N}.\end{aligned}$$

Again 2.11(1)(3) then induces an isomorphism

$$\frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(xy+yx, y^2+ax^{2N}+bx^{2M-1}\frac{v}{u})\!)} \xrightarrow{\sim} \frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(xy+yx, y^2+ax^{2N}+bx^{2M-1})\!)}. \tag{6.I}$$

Setting $a = 2N+1$ and $b = 2M$, the right hand side is $\mathfrak{Jac}(xy^2+x^{2N+1}+x^{2M})$. Hence composing (6.H) with (6.I) gives an isomorphism $\mathfrak{Jac}(f) \cong \mathfrak{Jac}(xy^2+x^{2N+1}+x^{2M})$.

The case $u = 0$ and $v \neq 0$ works in exactly the same way as the case $u \neq 0$ and $v = 0$ above, applying an automorphism with $s = \sqrt{v}$, while the case $u = v = 0$ is trivial. $\square$

The above is not quite yet in normal form, since some of the polynomial potentials in 6.13 have isomorphic Jacobi algebras. The next step is to discard cases where the odd term in $p(x)$ has significantly greater degree than the even term.

Lemma 6.14. *If $f = xy^2 + x^{2n} + \varepsilon x^{2m+1}$ where $\varepsilon \in \{0, 1\}$ and $m \geq n$, then $y^3 \in (\delta_x f, \delta_y f)$. In particular $y^3 \in (\!(\delta_x f, \delta_y f)\!)$.*

Proof. Set $I = (\delta_x f, \delta_y f) = (y^2 + 2nx^{2n-1} + \varepsilon(2m+1)x^{2m}, xy + yx)$, and below write $\equiv$ for an equality mod I. Since $y^2 \equiv -2nx^{2n-1} - \varepsilon(2m+1)x^{2m}$, multiplying on the left by y and on the right by y gives

$$\begin{aligned} -2nx^{2n-1}y - \varepsilon(2m+1)x^{2m}y \equiv y^3 &\equiv -2nyx^{2n-1} - \varepsilon(2m+1)yx^{2m} \\ &\equiv 2nx^{2n-1}y - \varepsilon(2m+1)x^{2m}y \end{aligned}$$

where the last line holds since $xy \equiv -yx$ using the second generator of I. Inspecting the right and lefthand sides, the $x^{2m}y$ terms cancel, and so $4nx^{2n-1}y \equiv 0$, thus $x^{2n-1}y \equiv 0$. Finally, since $m \geq n$, taking out the common factor we see that

$$y^3 \equiv (2n - \varepsilon(2m+1)x^{2m-2n-1})x^{2n-1}y \equiv 0.$$

Thus $y^3 \in I$. The final statement follows immediately. $\square$

Corollary 6.15. *If $f = xy^2 + x^{2n} + \varepsilon x^{2m+1}$ where $m \geq 2n - 1$, then $x^{4n-2} \in (\!(\delta_x f, \delta_y f)\!)$.*

Proof. Continue to write $\equiv$ for an equality mod $(\delta_x f, \delta_y f)$. Then

$$\begin{aligned} x^{4n-2} = (-x^{2n-1})^2 &\equiv \tfrac{1}{(2n)^2}(y^2 + \varepsilon(2m+1)x^{2m})^2 && (\text{since } \delta_x f \equiv 0) \\ &\equiv \tfrac{\varepsilon(2m+1)}{(2n)^2}(y^2x^{2m} + x^{2m}y^2 + \varepsilon(2m+1)x^{4m}) && (y^3 \equiv 0 \text{ by } 6.14) \\ &\equiv \tfrac{\varepsilon(2m+1)}{(2n)^2}(2x^{2m}y^2 + \varepsilon(2m+1)x^{4m}). && (xy \equiv -yx) \end{aligned}$$

Taking out the x^{2m} common factor from the front, we may write $x^{4n-2} \equiv x^{2m}g$ for some g with no constant term. Then, since $2m \geq 4n - 2$ by assumption, we see that $x^{4n-2} \equiv x^{4n-2}(x^{2m-(4n-2)}g)$, and so $x^{4n-2}(1 - x^{2m-(4n-2)}g) \equiv 0$.

Given this statement holds mod $(\delta_x f, \delta_y f)$, it also holds mod $(\!(\delta_x f, \delta_y f)\!)$, hence $x^{4n-2}(1 - x^{2m-(4n-2)}g) = 0$ in $\mathfrak{J}\mathrm{ac}(f)$. But there, $1 - x^{2m-(4n-2)}g$ is a unit, and hence $x^{4n-2} = 0$ in $\mathfrak{J}\mathrm{ac}(f)$, as required. $\square$

The above two results combine to remove the case when the odd-degree x term is sufficiently larger than the even-degree x term, as follows.

Corollary 6.16. *If $f = xy^2 + x^{2n} + x^{2m+1}$ where $m \geq 2n - 1$, then $f \cong xy^2 + x^{2n}$.*

Proof. By 6.15 we have $x^{4n-2} \in (\!(\delta_x f, \delta_y f)\!)$ and $x^{4n-2} \in (\!(y^2 + 2nx^{2n-1}, xy + yx)\!)$. Since $2m \geq 4n - 2$, it follows that x^{2m} belongs to *both* of the ideals above, and thus

$$\begin{aligned} (\!(\delta_x f, \delta_y f)\!) &:= (\!(y^2 + 2nx^{2n-1} + (2m+1)x^{2m}, xy + yx, x^{2m})\!) \\ &= (\!(y^2 + 2nx^{2n-1}, xy + yx, x^{2m})\!) \\ &= (\!(y^2 + 2nx^{2n-1}, xy + yx)\!). \end{aligned}$$

As this final ideal is obtained from $xy^2 + x^{2n}$ by differentiation, the result follows. $\square$

Summarising the above gives the following, which is the main result of this subsection.

Corollary 6.17. *Suppose that $f \in \mathbb{C}\langle\!\langle x, y\rangle\!\rangle_{\geq 3}$ where $(f_3)^{\mathrm{ab}}$ has two roots. Then either*

$$f \cong \begin{cases} xy^2 & \\ xy^2 + x^{2m+1} & m \geq 2 \\ xy^2 + x^{2n} & n \geq 2 \\ xy^2 + x^{2m+1} + x^{2n} & 2 \leq m \leq n-1 \\ xy^2 + x^{2n} + x^{2m+1} & 2 \leq n \leq m \leq 2(n-1) \end{cases}$$

All of the above are mutually non-isomorphic.

Proof. The fact that the stated list covers all cases follows from 6.13, using 6.16 to discount the case when the odd-degree x term is sufficiently larger than the even-degree x term. We now claim that the potentials listed give pairwise non-isomorphic Jacobi algebras.

The first two families both have infinite dimensional Jacobi algebras, whereas the bottom three are all finite dimensional. As such, the only possibilities for isomorphisms are between members in families one and two, or between members in families three, four and five. But $\dim_{\mathbb{C}} \mathfrak{Jac}(xy^2)^{\mathrm{ab}} = \infty$, whereas $\dim_{\mathbb{C}} \mathfrak{Jac}(xy^2 + x^{2m+1})^{\mathrm{ab}} = 2m+2$, and so all members of families one and two are mutually non-isomorphic.

For the final three families, all members of families three and four and mutually non-isomorphic, as can be seen by extending the method of [BW, 4.7], or by using [Ka, 5.10] directly. Further, all members of family five are also mutually non-isomorphic for dimension reasons, since for f in family five $\dim_{\mathbb{C}} \mathfrak{Jac}(f)^{\mathrm{ab}} = 2m+2$ and $\dim_{\mathbb{C}} \mathfrak{Jac}(f) = (2m+2) + 4(n-1)$ by either [vG, §5] or §4.3, and thus we can distinguish between all different m and n. The only remaining possibility is an isomorphism between a member of family five, and a member of family three or four. But by above the dimension of $\mathfrak{Jac}(f)$ for f in family five is even, and the dimension of $\mathfrak{Jac}(g)$ for g in families three and four is odd [Ka, 5.10], so there can be no such isomorphisms. $\square$

6.4. **Overview of Type D normal forms.** The previous subsections combine to give the following, which is the main result of this section.

Theorem 6.18. *Let $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq 2}$ with $\mathfrak{Crk}(f) = 2$ and $\mathfrak{Crk}_2(f) = 2$. Then either*

$$f \cong \begin{cases} z_1^2 + \cdots + z_{d-2}^2 + xy^2 & \\ z_1^2 + \cdots + z_{d-2}^2 + xy^2 + x^{2m+1} & \textit{with } m \geq 1 \\ z_1^2 + \cdots + z_{d-2}^2 + xy^2 + x^{2n} & \textit{with } n \geq 2 \\ z_1^2 + \cdots + z_{d-2}^2 + xy^2 + x^{2n} + x^{2m+1} & \textit{with } n \geq 2,\ n \leq m \leq 2n-2 \\ z_1^2 + \cdots + z_{d-2}^2 + xy^2 + x^{2m+1} + x^{2n} & \textit{with } m \geq 1,\ n \geq m+1. \end{cases}$$

The Jacobi algebras of these potentials are all mutually non-isomorphic, and furthermore the following statements hold.

(1) *Every f in the first two families satisfies* $\mathrm{Jdim}\, \mathfrak{Jac}(f) = 1$.
(2) *Every f in the last three families satisfies* $\mathrm{Jdim}\, \mathfrak{Jac}(f) = 0$.
   (a) *For any fixed $n \geq 2$, the algebras in families three and four combine to give $n-1$ non-isomorphic Jacobi algebras, all of which satisfy* $\dim_{\mathbb{C}} \mathfrak{Jac}(f)^{\mathrm{ab}} = 2n+1$ *and* $\dim_{\mathbb{C}} \mathfrak{Jac}(f) = (2n+1) + 4(n-1) = 6n-3$.
   (b) *In the fifth family,* $\dim_{\mathbb{C}} \mathfrak{Jac}(f)^{\mathrm{ab}} = 2m+2$ *and* $\dim_{\mathbb{C}} \mathfrak{Jac}(f) = (2m+2) + 4(n-1)$.

Proof. By the Splitting Lemma 4.5 the condition $\mathfrak{Crk}(f) = 2$ implies that $f \cong z_1^2 + \ldots + z_{d-2}^2 + g$ for some $g \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$. The condition $\mathfrak{Crk}_2(f) = 2$ is then equivalent to the first two cases in 4.13, namely those $g \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ with $g_3 \neq 0$ for which g_3^{ab} has either two or three distinct linear factors. The options for all such g thus follow from combining 6.6 and 6.17

The fact that $\mathrm{Jdim}\, \mathfrak{Jac}(xy^2 + x^3) = 1$ follows since $\mathfrak{Jac}(xy^2 + x^3) \cong \mathfrak{Jac}(x^3 + y^3)$ by linear change in coordinates, and $\mathrm{Jdim}\, \mathfrak{Jac}(x^3 + y^3) = 1$ by 6.5(1). The statements that $\mathrm{Jdim}\, \mathfrak{Jac}(xy^2 + x^{2m-1}) = 1$ for all $m \geq 2$ and $\mathrm{Jdim}\, \mathfrak{Jac}(xy^2) = 1$ can be shown by a very similar explicit method as in the proof of 6.5(1), or alternatively by using 8.5 below, once we know (in 8.12) that all such Jacobi algebras are contraction algebras. The stated vector space dimensions of the Jacobi algebras in all remaining cases have already been justified in the proofs of 6.6 and 6.17 respectively.

The fact that the above are all mutually non-isomorphic, and thus a list of normal forms, then follows. Indeed, by inspecting $\mathfrak{J}$-dimension, the only possible isomorphisms are between members of families one and two, or between members of families three, four and five. Given we have just added the normal forms of 6.6 to the normal forms of 6.17, the only remaining possible isomorphisms are between these two cases. But again, either the dimension of the abelianisation, or the dimension of the contraction algebra itself, distinguishes in all cases. $\square$

7. Central Elements and General Elephants

This section algebraically extracts ADE information from the normal forms in §1.2, using generic central elements and contracted preprojective algebras.

7.1. **Generic Central Sections.** For any $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq 2}$, set $\mathcal{Z} = \mathcal{Z}(\mathfrak{Jac}(f))$ to be the centre of $\mathfrak{Jac}(f)$, and write $\mathfrak{m}_{\mathcal{Z}}$ for the Jacobson radical of $\mathcal{Z}$. Recall that $\mathfrak{J}$ denotes the Jacobson radical of the local ring $\mathfrak{Jac}(f)$.

Lemma 7.1. *We have $\mathfrak{m}_{\mathcal{Z}} = \mathfrak{J} \cap \mathcal{Z}$ and $\mathcal{Z}/\mathfrak{m}_{\mathcal{Z}} \cong \mathbb{C}$. Thus $\mathcal{Z}$ is also a local ring.*

Proof. Set $J_f = \langle\langle \delta_1 f, \ldots, \delta_d f \rangle\rangle$, then it is clear that $\mathfrak{J} \cap \mathcal{Z} = \{g + J_f \in \mathcal{Z} \mid g \in \mathfrak{n}\}$. This set is clearly a two-sided ideal of $\mathcal{Z}$, and further $1 + \mathfrak{J} \cap \mathcal{Z}$ consists of units in $\mathcal{Z}$. These two properties imply that $\mathfrak{J} \cap \mathcal{Z}$ equals the Jacobson radical $\mathfrak{m}_{\mathcal{Z}}$ of $\mathcal{Z}$, see e.g. [L, 4.5]. The fact that $\mathcal{Z}/\mathfrak{m}_{\mathcal{Z}} \cong \mathbb{C}$ is clear. □

Generic elements of the centre $\mathcal{Z}$ will be used to intrinsically extract ADE information.

Definition 7.2. Given $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq 2}$, we say that $\mathfrak{Jac}(f)$ has Type X if for all finite dimensional vector spaces $V \subset \mathfrak{m}_{\mathcal{Z}}$ such that $V \twoheadrightarrow \mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$, there exists a Zariski open subset U of V such that $\mathfrak{Jac}(f)/(u) \cong e\Pi e$ for all $u \in U$, where Π is the preprojective algebra of Type X, and e is an idempotent marked $\circ$ in (1.C).

Equivalently, in the language of [R1, 2.5], $\mathfrak{Jac}(f)$ has Type X provided that a general hyperplane section u of $\mathcal{Z}$ satisfies $\mathfrak{Jac}(f)/(u) \cong e\Pi e$ where Π is the preprojective algebra of Type X, and e is an idempotent marked $\circ$ in (1.C). We also remark that there are two different Type E_8's in 7.2, corresponding to the two different choices of $\circ$ in E_8 in (1.C). This feature matches the two different E_8 cases in the length classification of flops [KM].

Much like the definition of cDV singularities, 7.2 is only designed to be useful in specific situations. Indeed, for general $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq 2}$, it is not clear whether the centre $\mathcal{Z}$ of $\mathfrak{Jac}(f)$ is noetherian, nor whether its maximal ideal $\mathfrak{m}_{\mathcal{Z}}$ is finitely generated as a $\mathcal{Z}$-module. Consequently, work is required to establish that $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$ is finite dimensional, which is needed for there to exist a finite dimensional vector space V surjecting onto it.

When $\mathfrak{Jac}(f)$ is finite dimensional, these difficulties disappear, since $\mathfrak{Jac}(f)$ and thus $\mathcal{Z}$, $\mathfrak{m}_{\mathcal{Z}}$ and $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$ are all finite dimensional vector spaces. Other cases are more tricky, but for our purposes the following suffices.

Lemma 7.3. *If f is a normal form from 6.18, then the following statements hold.*

(1) *If $u \in \mathfrak{m}_{\mathcal{Z}}$, then $u \equiv \lambda x^2 + h$ in $\mathfrak{Jac}(f)$ for some $h \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$.*
(2) *If further $f \in D_{\infty,\infty}$ or $D_{\infty,m}$, then $\mathcal{Z} \cong \mathbb{C}[\![x^2]\!]$.*

In particular, in all cases $\dim_{\mathbb{C}}(\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2) < \infty$.

Proof. (1) In all cases $\delta_y f = xy + yx$, and so certainly x^2 commutes with y in $\mathfrak{Jac}(f)$. Obviously x^2 commutes with x, thus since we are considering closed ideals, it follows that x^2 is central in $\mathfrak{Jac}(f)$. Similarly y^2 is central.

We next claim that there are no elements in $\mathfrak{m}_{\mathcal{Z}}$ that contain linear terms. Write $u \in \mathfrak{m}_{\mathcal{Z}}$ as $u = k + g$ for some $k = \lambda_1 x + \lambda_2 y$ and $g \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$. Now u remains central after factoring by $\mathfrak{J}^3$, so set $I = \langle\langle \delta_x f, \delta_y f, \mathfrak{n}^3 \rangle\rangle$, and observe that

$$0 + I = [x + I, u + I] = 2\lambda_2 xy + I$$

Since xy forms part of a basis of $\mathbb{C}\langle\langle x, y \rangle\rangle/I$, it follows that $\lambda_2 = 0$. Repeating using the commutator $[y + I, u + I]$ shows that $\lambda_1 = 0$.

Thus $u = g$ for some $g \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$. Using the relation $xy + yx$ to move x's to the left, and the other relation to move y^2 either to zero, to x^2, or into higher degree, write

$$u \equiv \lambda_1 x^2 + \lambda_2 xy + h$$

in $\mathfrak{Jac}(f)$, for some $h \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$. We next claim that $\lambda_2 = 0$. Since u is central, and x^2 is central, it follows that $v := \lambda_2 xy + h$ is also central in $\mathfrak{Jac}(f)$. In particular it is still central after factoring by $\mathfrak{J}^4$. Set $I = \langle\langle \delta_x f, \delta_y f, \mathfrak{n}^4 \rangle\rangle$, so that

$$0 + I = [x + I, v + I] = 2\lambda_2 x^2 y + I$$

and thus $2\lambda_2 x^2 y \in I$. But $x^2 y$ forms part of the basis of $\mathbb{C}\langle\langle x, y \rangle\rangle/I$, so $\lambda_2 = 0$.

(2) Either set $I = (xy + yx, y^2)$, or $(xy + yx, y^2 + x^{2m})$, so by assumption $\mathcal{J}\mathrm{ac}(f) \cong \mathbb{C}\langle\!\langle x, y\rangle\!\rangle/(\!(I)\!)$. Consider an arbitrary element $u \in \mathcal{Z}$. By using the first relation to move all the x's to the left, and the second relation to either move y^2 to zero or to higher powers of x, we may write $u \equiv p + qy$ in $\mathcal{J}\mathrm{ac}(f)$, where by (1) $p \in \mathbb{C}[\![x]\!]_{\geq 2}$ and $q \in \mathbb{C}[\![x]\!]_{\geq 2}$. Observe that in $\mathcal{J}\mathrm{ac}(f)$

$$0 \equiv [x, u] \equiv [x, p + qy] = 2xqy$$

and so $xqy \in (\!(I)\!)$. Thus $xq_{\leq t-3}y \in I + \mathfrak{n}^t$ for all $t \geq 3$. Now $\mathbb{C}\langle\!\langle x, y\rangle\!\rangle/(I + \mathfrak{n}^t)$ has basis $1, x, \ldots, x^{t-1}, y, xy, \ldots, x^{t-2}y$. Write $q_{\leq t-3} = \sum_{i=2}^{t-3} \lambda_i x^i$, then the second part of this basis on the equation $xq_{\leq t-3}y \in I + \mathfrak{n}^t$ shows that $\lambda_2 = \ldots = \lambda_{t-3} = 0$. This holds for all t, and so $q = 0$.

Thus the central element $u \equiv p$. Splitting into even and odd terms, write $u \equiv P(x^2) + xQ(x^2)$ in $\mathcal{J}\mathrm{ac}(f)$ for some $P, Q \in \mathbb{C}[\![x]\!]_{\geq 1}$. Then, in $\mathcal{J}\mathrm{ac}(f)$, since x^2 is central

$$0 \equiv [y, u] \equiv [y, P + xQ] \equiv -2(xQ)y.$$

Using the same argument as above, $Q = 0$, and so $u \equiv P(x^2)$, as claimed. This shows that $\mathcal{Z} \subseteq \mathbb{C}[\![x^2]\!]$, but since x^2 is central by (1), equality holds, proving (2).

For the very last statement, all the finite dimensional $\mathcal{J}\mathrm{ac}(f)$ satisfy $\dim_{\mathbb{C}}(\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2) < \infty$. Since the only other potentials in 6.18 are those in (2), where visibly $\dim_{\mathbb{C}}(\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2) = 1$, the final statement follows. □

It follows from 7.3 that $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$ is finite dimensional for any $f \in \mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle_{\geq 2}$ with $\mathcal{C}\mathrm{rk}(f) = 2$ and $\mathcal{C}\mathrm{rk}_2(f) = 2$. The is a rather remarkable use of normal forms: we have no method to prove such a result without using 6.18.

7.2. **ADE preliminaries.** The next problem is establishing that the centre $\mathcal{Z}$ is non-trivial. For Type A and D this turns out to be easy, but Type E requires the following preparation. Consider the elements

$$g_{6,n} := \begin{cases} x^2 + xyx + yx^2 & \text{if } n = 3t + 1 \text{ with } t \geq 1 \\ x^2 + xyx + yx^2 + (-1)^t 3^t (3t + 2) x^{2t+1} & \text{if } n = 3t + 2 \text{ with } t \geq 1 \\ x^2 + xyx + yx^2 + (-1)^{t+1} 3^t t(xyx^{2t-2} + yx^{2t-1}) & \text{if } n = 3t \text{ with } t \geq 2. \end{cases}$$

The following establishes, in the cases $E_{6,n}$, that the centre of $\mathcal{J}\mathrm{ac}(f)$ is non-trivial, and that $\mathfrak{m}_{\mathcal{Z}}$ is at least two dimensional as a vector space.

Lemma 7.4. *If $f \in E_{6,n}$, then x^2 is central in $\mathcal{J}\mathrm{ac}(f)$, as is $g_{6,n}$.*

Proof. The first statement follows from the relation $3x^2 + y^3 \equiv 0$, which implies that $yx^2 \equiv -\frac{1}{3}y^3 \equiv x^2 y$ and thus x^2 commutes with y. Since x^2 clearly commutes with x and we are considering closed ideals, it follows that x^2 is central in $\mathcal{J}\mathrm{ac}(f)$.

For the second statement, we establish the first case, with the proofs of all other cases being similar. For this, it suffices to show that $xyx + yx^2$ is also central in $\mathcal{J}\mathrm{ac}(f)$ when $n = 3t + 1$ with $t \geq 1$. We first claim that $xyxy - yxyx \in (\!(\delta_x f, \delta_y f)\!)$. This follows since $n - 1 = 3t$, then $y^{n-1} \equiv (-3x^2)^t$ is central, and thus the commutator

$$(\!(\delta_x f, \delta_y f)\!) \ni [x, \delta_y f] \equiv (xyxy - yxyx) + n(xy^{n-1} - y^{n-1}x) \equiv xyxy - yxyx.$$

Using this, again with the fact that x^2 is central, it follows that

$$[x, xyx + yx^2] \equiv (yx^3 + x^3 y) - (x^3 y + yx^3) = 0$$
$$[y, xyx + yx^2] \equiv (yxyx + y^2 x^2) - (xyxy + yx^2 y) \equiv 0.$$

Thus $xyx + yx^2$ commutes with both x and y, and so is central in $\mathcal{J}\mathrm{ac}(f)$. □

7.3. **Extracting ADE.** We are now in a position to extract ADE using general hyperplane sections of the centre. In what follows, in the case that $\mathcal{J}\mathrm{ac}(f)$ is finite dimensional, all ideals are automatically closed. In the cases when $\mathrm{Jdim}\,\mathcal{J}\mathrm{ac}(f) = 1$ this fact is also true for Type A by inspection, and for Type D by e.g. 8.4 and 8.13 below. As such, in the following technically we should temporarily write $\mathcal{J}\mathrm{ac}(f)/(\!(g)\!)$ when considering Type $D_{\infty,m}$ and $D_{\infty,\infty}$ until we have established 8.4 and 8.13. However, since 8.4 and 8.13 are logically independent of what follows, we refrain from doing so, and drop the double bracket to ease notation.

Theorem 7.5. *Consider the normal forms A_n, $D_{n,m}$, $D_{n,\infty}$, $E_{6,n}$, A_∞, $D_{\infty,m}$, $D_{\infty,\infty}$ and $E_{6,\infty}$ from §1.2. In each case, define an element s as follows*

Type	Normal form	Conditions	s
A	$z_1^2+\ldots+z_{d-2}^2+x^2+\varepsilon_1 y^n$	$n\in\mathbb{N}_{\geq 2}\cup\{\infty\}$	y
D	$z_1^2+\ldots+z_{d-2}^2+xy^2+\varepsilon_2 x^{2n}+\varepsilon_3 x^{2m-1}$	$m,n\in\mathbb{N}_{\geq 2}\cup\{\infty\}$	x^2
E	$z_1^2+\ldots+z_{d-2}^2+x^3+xy^3+\varepsilon_4 y^n$	$n\in\mathbb{N}_{\geq 4}$	$g_{6,n}$

where $g_{6,n}$ is defined in §7.2 above. Then the following statements hold.

(1) *The element s is central in $\mathfrak{J}\mathrm{ac}(f)$, and $\mathfrak{J}\mathrm{ac}(f)/(s)\cong e\Pi e$, where Π is the preprojective algebra of Type A_1, D_4, or E_6, and e is the idempotent marked $\circ$.*

(2) *Normal forms of Type A and D give rise to Jacobi algebras which have Type A and D respectively, in the sense of 7.2.*

Proof. For (1), Type A is clear, since $\mathfrak{J}\mathrm{ac}(f)\cong\mathbb{C}[\![y]\!]/y^{n-1}$ or $\mathbb{C}[\![y]\!]$ depending on whether ε_1 is 1 or 0. In both cases y is central, and the quotient is $\mathfrak{J}\mathrm{ac}(f)/y\cong\mathbb{C}\cong e\Pi e$ where Π is the preprojective algebra of Type A_1.
Type D is similar. The fact x^2 is central was justified in 7.3(1). But then since (x^2) is a closed ideal of $\mathfrak{J}\mathrm{ac}(f)$, setting $\lambda_2=(2n)\varepsilon_2$ and $\lambda_3=(2m-1)\varepsilon_3$ it follows that

$$\begin{aligned}\mathfrak{J}\mathrm{ac}(f)/(x^2)&\cong\frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(x^2,xy+yx,y^2+\lambda_2x^{2n-1}+\lambda_3x^{2m-2})\!)}&&\text{(by 2.11)}\\&\cong\frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(x^2,xy+yx,y^2)\!)}&&\text{(since }2n-1\geq 2\text{ and }2m-2\geq 2)\\&\cong e\Pi e&&\text{(by [M2], see e.g. [CB, p53])}\end{aligned}$$

where Π is the preprojective algebra of Type D_4, and e is the central vertex.

Type E is more involved. All proofs turn out to be similar, so here we illustrate the technique by considering the case $f\in E_{6,n}$ with $n=3t+1$ and $t\geq 2$. Certainly $g_{6,n}$ is central by 7.4. After rescaling the x and y appropriately,

$$\mathfrak{J}\mathrm{ac}(f)\cong\frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(-x^2+y^3,xy^2+yxy+y^2x+y^{3t})\!)}=:\Gamma$$

and we work on the right-hand side. Under this identification, the element $g_{6,n}$ becomes $\lambda x^2+\mu(xyx+yx^2)$ for some non-zero scalars λ and μ. We now claim that for any non-zero scalars λ and μ the factor

$$A:=\frac{\Gamma}{(\lambda x^2+\mu(xyx+yx^2))}=\frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(-x^2+y^3,xy^2+yxy+y^2x+y^{3t},\lambda x^2+\mu(xyx+yx^2))\!)}$$

is isomorphic to the model algebra $B:=\mathbb{C}\langle\!\langle x,y\rangle\!\rangle/(\!(x^2,y^3,(x+y)^3)\!)\cong e\Pi e$ where Π is the preprojective algebra of Type E_6 and e is the central vertex [CB, p53]. The result will then follow, since for particular λ,μ, there is an isomorphism $A\cong\mathfrak{J}\mathrm{ac}(f)/(g_{6,n})$.

To establish the claim, note first that $x^3\equiv 0$ in A for any $t\geq 2$, as follows. The additional relation gives $-\lambda x^3\equiv\mu(x^2yx+xyx^2)$, which equals $\mu(yx^3+x^3y)$ since x^2 is central. Repeating, we may push x^3 into higher and higher degrees, and so x^3 belongs to the closed ideal defining A, as claimed. Given $x^3\equiv 0$ in A, it follows from the first relation that $y^6\equiv x^4\equiv 0$ in A, and since $t\geq 2$ also that $y^{3t}\equiv 0$ in A. Consequently

$$A\cong\frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(-x^2+y^3,xy^2+yxy+y^2x,\lambda x^2+\mu(xyx+yx^2),y^{3t})\!)}\tag{7.A}$$

$$=\frac{\mathbb{C}\langle\!\langle x,y\rangle\!\rangle}{(\!(-x^2+y^3,xy^2+yxy+y^2x,\lambda x^2+\mu(xyx+yx^2))\!)}.\tag{7.B}$$

where the last equality holds since y^6 belongs to the closed ideal in (7.A), and $t\geq 2$. This latter presentation has no dependence on t.

Now, composing the automorphism $\phi\colon\mathbb{C}\langle\!\langle x,y\rangle\!\rangle\to\mathbb{C}\langle\!\langle x,y\rangle\!\rangle$ defined by

$$x\mapsto x-(xy+yx)+yxy,\qquad y\mapsto(x+2y)-y^2-yx$$

with $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle \twoheadrightarrow B$ gives a surjective homomorphism $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle \twoheadrightarrow B$. It is elementary to check that the three relations of A in (7.B) map to zero, and hence since ϕ is continuous, it extends to the closure of ideals and thus induces a surjection $\phi\colon A \twoheadrightarrow B$.

Using the same method, it is also elementary to check that the automorphism $\mathbb{C}\langle\langle x,y \rangle\rangle \to \mathbb{C}\langle\langle x,y \rangle\rangle$ given by

$$x \mapsto -\tfrac{2}{3}x(3-2x) + (y-2)xy, \qquad y \mapsto x(1+\tfrac{25}{48}x) + y(1+\tfrac{1}{4}y+x)$$

descends to a surjective map $B \twoheadrightarrow A$. Thus $\dim A \leq \dim B = 12$, and in particular $\dim A$ is also finite. The previous surjection $A \twoheadrightarrow B$ then implies that $\dim A = \dim B$, so that $\phi\colon A \to B$, being a surjective map between algebras of the same dimension, is necessarily an isomorphism.

(2) For potentials of Type A, clearly $\mathfrak{Jac}(f)$ is either $\mathbb{C}[\![y]\!]/(y^{n-1})$ or $\mathbb{C}[\![y]\!]$, both of which are commutative, so $\mathcal{Z} = \mathfrak{Jac}(f)$ and further $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$ is spanned by the image of y. Given any finite dimensional vector space $V \subset \mathfrak{m}_{\mathcal{Z}}$ such that $\pi\colon V \twoheadrightarrow \mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$, set $\mathcal{U}_1 = \{\lambda \in \mathbb{A}^1 \mid \lambda \neq 0\}$, $\mathcal{U} = \pi^{-1}(\mathcal{U}_1)$, and let $u \in \mathcal{U}$. Then for all $u \in \mathcal{U}$, write $u = \lambda y + p$ in $\mathfrak{Jac}(f)$ for some $\lambda \neq 0$ and some $p \in \mathbb{C}[\![y]\!]_{\geq 2}$. In particular u equals y multiplied by a unit, and so $\mathfrak{Jac}(f)/(u) \cong \mathfrak{Jac}(f)/(y) \cong \mathbb{C} \cong e\Pi e$, where Π is the preprojective algebra of Type A.

Lastly, consider Type D. By 7.3 all potentials f in 6.18 satisfy $\dim_{\mathbb{C}}(\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2) < \infty$. Hence we can again consider a finite dimensional vector space $V \subset \mathfrak{m}_{\mathcal{Z}}$, such that $\pi\colon V \twoheadrightarrow \mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$. Since $x^2 \in \mathfrak{m}_{\mathcal{Z}}$ by (1), and $\mathfrak{m}_{\mathcal{Z}}$ contains no linear terms (as justified in 7.3(1)), x^2 is non-zero in $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$. Thus set $b_1 = x^2 + \mathfrak{m}_{\mathcal{Z}}^2$, and extend to a basis $b_1, \ldots, b_t$. Set $\mathcal{U}_1 = \{\sum \lambda_i b_i \mid \lambda_1 \neq 0\}$, and $\mathcal{U} = \pi^{-1}(\mathcal{U}_1)$.

Let $u \in \mathcal{U}$, then by 7.3(1) $u \equiv \lambda x^2 + h$ in $\mathfrak{Jac}(f)$, for some $h \in \mathbb{C}\langle\langle x,y \rangle\rangle_{\geq 3}$. The assumption $u \in \mathcal{U}$ ensures that $\lambda \neq 0$. By the relation $xy \equiv -yx$, we may pull all the x's in h to the left, and since h has order at least three, afterwards each term either starts with x^2, or ends with y^2. Thus in $\mathfrak{Jac}(f)$

$$h \equiv x^2 r + xy^2 p + y^2 q$$

for some $r \in \mathbb{C}\langle\langle x,y \rangle\rangle_{\geq 1}$, $q \in \mathbb{C}[\![y]\!]_{\geq 1}$ and $p \in \mathbb{C}[\![y]\!]$. Consequently $u \equiv x^2(\lambda + r) + xy^2 p + y^2 q$, and further using the relation $\delta_x f$, it follows that

$$u \equiv x^2 \big(\lambda + r - (\lambda_2 x^{2n-2} + \lambda_3 x^{2m-3})p - (\lambda_2 x^{2n-3} + \lambda_3 x^{2m-4})q\big)$$

where again $\lambda_2 = (2n)\varepsilon_2$ and $\lambda_3 = (2m-1)\varepsilon_3$. The term in brackets is a unit: since $n, m \geq 2$ its only degree zero term is λ, which by assumption is non-zero. It follows that $\mathfrak{Jac}(f)/(u) \cong \mathfrak{Jac}(f)/(x^2)$, and so the result follows by (1). □

Remark 7.6. The need for taking a generic, or at least well chosen, central element in 7.5 is essential. Indeed, by 7.4, for $f \in E_{6,n}$ the element x^2 is central in $\mathfrak{Jac}(f)$. However

$$\mathfrak{Jac}(f)/(x^2) \cong \frac{\mathbb{C}\langle\langle x,y \rangle\rangle}{\langle\langle x^2, y^3, (x+y)^3 - xyx \rangle\rangle} \ncong \frac{\mathbb{C}\langle\langle x,y \rangle\rangle}{\langle\langle x^2, y^3, (x+y)^3 \rangle\rangle} = e\Pi e \qquad (7.C)$$

even although both sides have dimension twelve. Write $\mathcal{U}$ for the open set given by the non-vanishing of the co-efficient of x^2, then (7.C) together with 7.5(1) assert that the isomorphism class of $\mathfrak{Jac}(f)/(u)$ is not constant along $u \in \mathcal{U}$. Consequently, for Type E a smaller generic open set is required.

Remark 7.7. In the proof of 7.5(1) above, the inverse of $\phi\colon A \to B$ is not the constructed map $B \twoheadrightarrow A$, rather ϕ^{-1} is induced by the much more non-obvious automorphism

$$\begin{aligned} x &\mapsto x + \tfrac{1}{2}(xy+yx) - \tfrac{1}{8}yxy \\ y &\mapsto \tfrac{1}{2}(-x+y) + \tfrac{1}{8}(x^2 - 3xy - yx + y^2) + \\ &\qquad\qquad \tfrac{1}{64}(5yx^2 + 12yxy + 16y^2x) + \tfrac{3}{64}y^2xy + \tfrac{7}{128}y^2x^2. \end{aligned}$$

8. Geometric Corollaries

The previous results have geometric consequences. Section 8.1 classifies contraction algebras, up to isomorphism, from all Type A and D flopping contractions. This has immediate application to GV invariants. Then §8.2 constructs the first, and conjecturally only, infinite family of Type D divisor-to-curve contractions. Using this, and known results from flops, we then prove that the Realisation Conjecture (1.10) is true, except possibly for some exceptional cases, establishing 1.11 in the introduction. The last subsection classifies contraction algebras that can arise from Type A and Type D_4 divisor-to-curve contractions.

8.1. **Classification of contraction algebras for A and D flops.** In this section we classify contraction algebras that can arise from Type A and D flops, in both cases without referring to any classification of such flops (noting in Type D a classification does not exist). We will show that the only possible options are those finite dimensional Jacobi algebras in 5.1 and 6.18 respectively. We use the notation of §1.5 freely; in particular, $\mathcal{R}$ is commutative noetherian and in applications $\operatorname{Spec}\mathcal{R}$ is the base of a simple 3-fold flop.

Remark 8.1. In both 8.2 and 8.9 below we classify the contraction algebras within a given type, but in fact more is true. By [HT, 4.6] if $\mathrm{B}_{\mathrm{con}}$ is the contraction algebra of a Type X flopping contraction which is isomorphic to a contraction algebra $\mathrm{A}_{\mathrm{con}}$ of a Type Y flopping contraction, then $X = Y$. Hence, the algebras in 8.2 below are the contraction algebras of all possible Type A flops, and only of Type A flops, and the algebras in 8.9 are the contraction algebras of all possible Type D flops, and only of Type D flops.

The classification in Type A is elementary.

Proposition 8.2. *If $\mathrm{A}_{\mathrm{con}}$ is a contraction algebra of a Type A flopping contraction, then $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(x^2 + y^n)$ for some $n \geq 2$. Furthermore, any other contraction algebra of any other type cannot be isomorphic to such a Jacobi algebra.*

Proof. Consider $\mathrm{A}_{\mathrm{con}}$ from an arbitrary Type A flop $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$. By a now standard argument of Van den Bergh [V2, A.1], any indecomposable CM $\mathcal{R}$-module necessarily has rank one. Further, since $\mathcal{R}$ is normal the endomorphism ring of any rank one CM module is isomorphic to $\mathcal{R}$. Hence $\mathrm{A}_{\mathrm{con}}$, being defined to be a factor of $\operatorname{End}_{\mathcal{R}}(N)$ for some indecomposable CM $\mathcal{R}$-module N, is thus a factor of $\mathcal{R}$, and hence is commutative.

Since $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(f)$ for some f, combining 5.1 and 5.4 we see that $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(x^2+y^n)$ or $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(x^2)$. The last case is impossible, since $\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}} < \infty$ given that $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$ is a flop [DW3]. □

Remark 8.3. A more direct proof of 8.2 uses Reid's Pagoda classification of Type A flops [R2] and then applies [DW1, 3.10] to conclude that $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(x^2 + y^n)$ for some n. The proof of 8.2 did not use Reid's classification, partly to show that it is not necessary, but mainly because in Type D below such a classification is not available.

Type D is much more involved, and requires multiple preliminary results. In the following, by an $\mathcal{R}$-algebra Γ we simply mean that there exists a homomorphism $\mathcal{R} \to Z(\Gamma)$, where $Z(\Gamma)$ is the centre. From 3.2, let $\mathfrak{J}(\Gamma)$ denote the Jacobson radical of Γ.

Lemma 8.4. *Let Γ be an $\mathcal{R}$-algebra, where $(\mathcal{R}, \mathfrak{m})$ is commutative local noetherian, and suppose that Γ is finitely generated as an $\mathcal{R}$-module.*

(1) *The $\mathfrak{J}(\Gamma)$-adic topology coincides with the $\mathfrak{m}$-adic topology on Γ.*
(2) *Every ideal of Γ is closed with respect to the $\mathfrak{J}(\Gamma)$-adic and the $\mathfrak{m}$-adic topologies.*
(3) *If $(\mathcal{R}, \mathfrak{m})$ is complete local, then Γ is complete with respect to both the $\mathfrak{J}(\Gamma)$-adic and the $\mathfrak{m}$-adic topologies.*

Proof. (1) Since $(\mathcal{R}, \mathfrak{m})$ is local, and Γ is finitely generated as an $\mathcal{R}$-module, it follows immediately from e.g. [L, 20.6] that there exists $n \geq 1$ such that

$$\mathfrak{J}(\Gamma)^n \subseteq \mathfrak{m}\Gamma \subseteq \mathfrak{J}(\Gamma). \tag{8.A}$$

From this, it is clear that the $\mathfrak{J}(\Gamma)$-adic and the $\mathfrak{m}$-adic topologies coincide.
(2) We show that for any finitely generated $\mathcal{R}$-module M, with submodule N, then N is closed in M. Given this, applying (1) to $M = \Gamma$ and $N = I$ proves the result. But since M is finitely generated, and $\mathcal{R}$ is noetherian, Krull's intersection theorem [M1, 8.1(1)] immediately shows that M/N is separated, and hence N is closed in M.
(3) Again, this is well known. Since Γ is a finitely generated $\mathcal{R}$-module, and $(\mathcal{R}, \mathfrak{m})$ is complete local and noetherian, it follows from e.g. [L, 21.34(1)] that Γ is $\mathfrak{m}$-adically complete, and hence the result follows by (1). □

Corollary 8.5. *If* $\mathrm{A}_{\mathrm{con}}$ *is a contraction algebra associated to a crepant* $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ *as above, then* $\operatorname{Jdim} \mathrm{A}_{\mathrm{con}} = 0, 1$. *Furthermore, the following statements hold.*

(1) $\operatorname{Jdim} \mathrm{A}_{\mathrm{con}} = 0$ *if and only if* $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ *is a flop.*
(2) $\operatorname{Jdim} \mathrm{A}_{\mathrm{con}} = 1$ *if and only if* $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ *is a divisorial contraction to a curve.*

Proof. $\mathrm{A}_{\mathrm{con}}$ is module finite over $\mathcal{R}$, being a factor of an NCCR [DW1].

Now if M is any finitely generated $\mathcal{R}$-module (e.g. $M = \mathrm{A}_{\mathrm{con}}$), since $(\mathcal{R}, \mathfrak{m})$ is local $\dim_{\mathcal{R}}(M)$ can be defined using the $\mathfrak{m}$-adic topology, as the growth rate of the function $\operatorname{length}(M/\mathfrak{m}^i M)$, see e.g. [E1, §12.1]. Taking suitable powers of the inclusions in (8.A), it is elementary to see that the two sets

$$\mathcal{S}_1 = \{r \in \mathbb{R} \mid \text{for some } c \in \mathbb{R},\ \dim_{\mathbb{C}}(\mathrm{A}_{\mathrm{con}}/\mathfrak{J}^n) \leq cn^r \text{ for every } n \in \mathbb{N}\}$$
$$\mathcal{S}_2 = \{r \in \mathbb{R} \mid \text{for some } c \in \mathbb{R},\ \dim_{\mathbb{C}}(\mathrm{A}_{\mathrm{con}}/\mathfrak{m}^n \mathrm{A}_{\mathrm{con}}) \leq cn^r \text{ for every } n \in \mathbb{N}\}$$

are equal, so $\operatorname{Jdim} \mathrm{A}_{\mathrm{con}} = \inf \mathcal{S}_1 = \inf \mathcal{S}_2 = \dim_{\mathcal{R}}(\mathrm{A}_{\mathrm{con}})$.

The main result of [DW2] shows that $\operatorname{Supp}_{\mathcal{R}}(\mathrm{A}_{\mathrm{con}})$ equals the contracted locus in $\operatorname{Spec} \mathcal{R}$. Hence $\dim_{\mathcal{R}}(\mathrm{A}_{\mathrm{con}})$ is either 0 for flops, or 1 for divisor-to-curves respectively. It follows that $\operatorname{Jdim}(\mathrm{A}_{\mathrm{con}})$ is either 0 or 1, respectively. □

Theorem 8.6. *If* $f \in \mathbb{C}\langle\langle x, y\rangle\rangle_{\geq 3}$ *and* $\mathcal{J}\mathrm{ac}(f)$ *is complete in its* $\mathfrak{J}$*-adic topology, then every central element* $g + (\!(\delta_x f, \delta_y f)\!) \in \mathcal{J}\mathrm{ac}(f)$ *which is not a unit satisfies* $g \in \mathfrak{n}^2$.

Proof. Set $R = \mathcal{J}\mathrm{ac}(f)$, and $\mathfrak{J} = \mathfrak{J}(R)$. Note that $I = (\!(\delta f)\!) \subseteq \mathfrak{n}^2$, hence by 3.2(1)

$$\mathfrak{J}^2 = (\mathfrak{n}^2 + I)/I = \mathfrak{n}^2/I. \tag{8.B}$$

Consider the central element $g' = g + I$ in R. Since g' is not a unit in R, certainly $g' \in \mathfrak{J}$. Further g cannot be a unit in $\mathbb{C}\langle\langle x, y\rangle\rangle$, or it would descend to a unit, so $g \in \mathfrak{n}$.

We now suppose that $g \notin \mathfrak{n}^2$, and aim for a contradiction. Since $I \subseteq \mathfrak{n}^2$ we see that $g' = g + I \notin \mathfrak{J}^2$, and hence $0 \neq g' + \mathfrak{J}^2 \in \mathfrak{J}/\mathfrak{J}^2$. But by (8.B) we have

$$\frac{\mathfrak{J}}{\mathfrak{J}^2} = \frac{\mathfrak{n}/I}{\mathfrak{n}^2/I} \cong \frac{\mathfrak{n}}{\mathfrak{n}^2}$$

and so $\dim_{\mathbb{C}} \mathfrak{J}/\mathfrak{J}^2 = 2$. Pick $0 \neq h + \mathfrak{J}^2$ to complete $g' + \mathfrak{J}^2$ to a basis of $\mathfrak{J}/\mathfrak{J}^2$.

But now since by assumption R is complete, and local, we may use [BIRS, 3.1] to present R. Consider the two-loop quiver Q, and map the trivial path to the identity of R, one of the loops ℓ_1 to h and the other loop ℓ_2 to g'. Then by [BIRS, 3.1] the completeness of R extends this to a surjective homomorphism

$$\varphi \colon \mathbb{C}\langle\langle \ell_1, \ell_2\rangle\rangle \twoheadrightarrow R$$

and the kernel is a closed ideal. Since the kernel contains the relation $\ell_1\ell_2 - \ell_2\ell_1$, given g' is central and so commutes with h, it follows that $(\!(\ell_1\ell_2 - \ell_2\ell_1)\!) \subseteq \operatorname{Ker} \varphi$. In particular φ induces a surjection $\mathbb{C}[\![\ell_1, \ell_2]\!] \twoheadrightarrow R$, and so R is commutative. Given this would contradict 5.4, we conclude that $g \in \mathfrak{n}^2$. □

We will also require the following fact.

Proposition 8.7. *Suppose that* $\mathrm{A}_{\mathrm{con}}$ *is the contraction algebra associated to a* D_4 *contraction, then there is a central element* $g \in \mathrm{A}_{\mathrm{con}}$ *such that*

$$\mathrm{A}_{\mathrm{con}}/(g) \cong \frac{\mathbb{C}\langle x, y\rangle}{(x^2, xy + yx, y^2)}.$$

Proof. Consider the 3-fold contraction $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, and for generic $g \in \mathcal{R}$ consider the pullback diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} \mathcal{R}/g \arrow[r] & \operatorname{Spec} \mathcal{R} \end{tikzcd}$$

By assumption, $\mathcal{R}/g$ is a D_4 Kleinian singularity. Now let $\mathrm{A} = \operatorname{End}_{\mathcal{R}}(N)$ be the NCCR associated to $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, and view $g \in \mathcal{R} = Z(\mathrm{A}) \subset \mathrm{A}$. Since g is generic, we can find such a g which is not contained in any associated prime ideal of $\operatorname{Ext}^1_{\mathcal{R}}(N, N)$, from which

[IW2, 5.19] (the assumptions there are Type A, but the method is general) shows that there is an isomorphism

$$\mathrm{A}/g \cong \mathrm{End}_{\mathcal{R}/g}(N/gN).$$

From this isomorphism [DW1, (3.C)] establishes that $\mathrm{A}_{\mathrm{con}}/g$ is isomorphic to the contraction algebra associated to the surfaces contraction $\mathcal{Y} \to \operatorname{Spec} \mathcal{R}/g$. The fact that the surfaces contraction algebra for this particular D_4 contraction is $\mathbb{C}\langle x, y\rangle/(x^2, xy + yx, y^2)$ can be deduced from [DW1, 8.7]; see also [M2]. □

We obtain the following remarkable consequence.

Corollary 8.8. *No $\mathfrak{J}\mathrm{ac}(f)$ with $f \in \mathbb{C}\langle\!\langle x, y\rangle\!\rangle_{\geq 3}$ such that either $f_3 = 0$ or $f_3 \cong x^3$, can arise as a contraction algebra of a D_4 flop, or a D_4 divisor-to-curve contraction.*

Proof. Given such an f, suppose that $\mathfrak{J}\mathrm{ac}(f) \cong \mathrm{A}_{\mathrm{con}}$ for a contraction algebra of a D_4 flop, or a D_4 divisor-to-curve contraction $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$. Since $\mathrm{A}_{\mathrm{con}}$ is module finite over $\mathcal{R}$, being a factor of a NCCR [DW1], 8.4 shows that $\mathrm{A}_{\mathrm{con}}$ and hence $\mathfrak{J}\mathrm{ac}(f)$ is complete with respect to its radical-adic topology, and further every ideal is closed.

Further, by 8.7 we can find a central g such that $\dim_{\mathbb{C}} \mathfrak{J}\mathrm{ac}(f)/(g) = 4$, and since $\mathfrak{J}\mathrm{ac}(f) \cong \mathrm{A}_{\mathrm{con}}$ is complete, we can use 8.6 to write $g = g' + (\!(\delta_x f, \delta_y f)\!)$ where $g' \in \mathfrak{n}^2$. But since all ideals in $\mathfrak{J}\mathrm{ac}(f)$ are closed, it follows that

$$\frac{\mathfrak{J}\mathrm{ac}(f)}{(g)} = \frac{\mathfrak{J}\mathrm{ac}(f)}{(\!(g)\!)} \overset{2.11}{\cong} \frac{\mathbb{C}\langle\!\langle x, y\rangle\!\rangle}{(\!(\delta_x f, \delta_y f, g')\!)}$$

has dimension four. We claim that this is impossible, by exhibiting a factor with higher dimension. Reusing the notation in 5.4, write $\mathcal{M}_3$ for the set of all noncommutative monomials of degree 3, and then we will factor by $(\!(\mathcal{M}_3)\!)$. In the two cases $f_3 = x^3$ and $f_3 = 0$, the factors are, respectively

$$\frac{\mathbb{C}\langle\!\langle x, y\rangle\!\rangle}{(\!(x^2, g', \mathcal{M}_3)\!)} \quad \text{and} \quad \frac{\mathbb{C}\langle\!\langle x, y\rangle\!\rangle}{(\!(g', \mathcal{M}_3)\!)}.$$

The right hand algebra surjects onto the left hand algebra, so it suffices to prove that $\dim_{\mathbb{C}} \mathbb{C}\langle\!\langle x, y\rangle\!\rangle/(\!(x^2, g', \mathcal{M}_3)\!) > 4$. But since $g' \in \mathfrak{n}^2$ by 8.6, inside the ideal we can replace g' by $\lambda_1 xy + \lambda_2 yx + \lambda_3 y^2$, which gives at most one linear relation between xy, yx and y^2. From this, the statement is clear. □

The above gives rise to the following, which is the main result in this subsection.

Corollary 8.9. *If $\mathrm{A}_{\mathrm{con}}$ is a contraction algebra of a Type D flopping contraction, then $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(x^2y + x^{2n})$ for some $n \geq 2$, or $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(xy^2 + x^{2n} + x^{2m-1})$ for some $m, n \geq 2$ with $m \leq 2n - 1$.*

Proof. Consider $\mathrm{A}_{\mathrm{con}}$ from an arbitrary Type D flop. By [KM] necessarily the elephant is D_4, so $\mathrm{A}_{\mathrm{con}}$ is not commutative by 8.7, since $\mathrm{A}_{\mathrm{con}}$ has a factor which is not commutative. As $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(f)$ for some $f \in \mathbb{C}\langle\!\langle x, y\rangle\!\rangle_{\geq 2}$, appealing to 5.4 then gives $f_2 = 0$.

From this, 8.8 asserts that $f_3 \neq 0$, and $f_3 \ncong \ell^3$. Hence by 6.18 $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(f)$ for some f in the list stated there. Only the bottom three families are possible, since $\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}} < \infty$ given the contraction is a flop [DW3]. □

The above classification of contraction algebras does not (yet) prove the classification of flops, as this is reliant on the Donovan–Wemyss conjecture [DW1] being true. Regardless, it still gives the following remarkable corollary. There are significant gaps in the possible GV invariants that can arise, establishing 1.6 in the introduction.

Corollary 8.10. *Consider $(a, b) \in \mathbb{N}^2$. Then (a, b) are the GV invariants for a Type D flopping contraction if and only if either*

(1) *$(a, b) = (2m + 3, m)$ for some $m \geq 1$, or*
(2) *$(a, b) = (2n, b)$ for some $n \geq 2$, with $b \geq n - 1$.*

Further, when $a = 2m + 3$ there are precisely $m + 1$ distinct contraction algebras realising (a, b), up to isomorphism, whilst for any given $(2n, b)$ the contraction algebra is unique.

Proof. Suppose $\mathrm{A}_{\mathrm{con}}$ is the contraction algebra of a Type D flop. By 8.9, it is isomorphic to $\mathfrak{J}\mathrm{ac}(f)$ for some f in the bottom three families in 6.18. By 6.18(2) and Toda's dimension formula [T, 1.1], we can read off the GV invariants. Indeed, by [T] the pair (n_1, n_2) where $n_1 = \dim_{\mathbb{C}} \mathfrak{J}\mathrm{ac}(f)^{\mathrm{ab}}$ and $n_2 = \frac{1}{4}(\dim_{\mathbb{C}} \mathfrak{J}\mathrm{ac}(f) - \dim_{\mathbb{C}} \mathfrak{J}\mathrm{ac}(f)^{\mathrm{ab}})$ are precisely the GV invariants for length two flops. Only those pairs in Figure 1 appear. The last statement regarding isomorphism classes is contained in 6.18. ☐

8.2. **Constructing divisor to curve contractions.** In the list of potentials in 6.18, the first appears as the contraction algebra of a divisor-to-curve contraction in [DW4, 2.18]. The second family, with $m = 1$, is isomorphic to $x^3 + y^3$, and so appears as a contraction algebra in [DW4, 2.25]. All the other three families are contraction algebras of D_4 flops by [vG, Ka], and the above subsection.

Motivated by Conjecture 1.10, this subsection will fill the last remaining gap, and show that the whole of the second family in 6.18, with arbitrary m, are realised as the contraction algebra of a divisor-to-curve contraction.

Remark 8.11. In the proof below we will first construct the contraction algebraically, before passing to the formal fibre to realise the contraction algebra. This algebraic construction is advantageous, since it conceptually distinguishes between the cases: in $\operatorname{Spec} R_\infty$ below, which locally realises $D_{\infty,\infty}$, the origin is cD_4 whilst all other points on the singular locus are cA_2. In contrast, in $\operatorname{Spec} R_m$ below, which locally realises $D_{\infty,m}$, the origin is cD_4 whilst all other points on the singular locus are cA_1. Compare the pictures in [DW4, 2.18] and [DW4, 2.25], and also [W].

Proposition 8.12. *Consider the element of* $\mathbb{C}[\![X,Y,Z,T]\!]$ *defined by*

$$F_m := \begin{cases} Y(X^m+Y)^2 + XZ^2 - T^2 & \text{if } m \geq 1 \\ Y^3 + XZ^2 - T^2 & \text{if } m = \infty \end{cases}$$

and set $\mathcal{R}_m = \mathbb{C}[\![X,Y,Z,T]\!]/F_m$. *Then the following statements hold.*

1. $\operatorname{Sing}(\mathcal{R}_m)^{\mathrm{red}} = (X^m + Y, Z, T)$ *if* $m \geq 1$, *and* (Y, Z, T) *if* $m = \infty$.
2. *In either case, blowing up this locus gives rise to a crepant Type D divisorial contraction to a curve* $\mathcal{X}_m \to \operatorname{Spec} \mathcal{R}_m$ *where* $\mathcal{X}_m$ *is smooth.*
3. *The contraction algebra of* $\mathcal{X}_m \to \operatorname{Spec} \mathcal{R}_m$ *is isomorphic to* $\mathfrak{J}\mathrm{ac}(xy^2 + x^{2m+1})$ *when* $m \geq 1$, *respectively* $\mathfrak{J}\mathrm{ac}(xy^2)$ *when* $m = \infty$.
4. $\mathcal{R}_m \cong \mathcal{R}_n$ *if and only if* $m = n$, *and so the* F_m *are all distinct up to isomorphism and so form an infinite family.*

Proof. (1) is immediate.
(2) Working first on the case of finite $m \geq 1$, consider the affine algebra

$$R_m = \frac{\mathbb{C}[r,s,u,v]}{u^2 - r(r-s^m)^2 - sv^2}$$

whose completion at the origin is $\mathcal{R}_m$, in coordinates $(r, s, u, v) = (-Y, X, T, Z)$. The blowup along $(u, v, r-s^m)$ is covered by two affine patches: the first is $U = \operatorname{Spec} \mathbb{C}[s, y_0, y_1]$, with $y_0 = u/(r - s^m)$ and $y_1 = v/(r - s^m)$, the second chart is a smooth hypersurface, and the map from U to the base is given by

$$(s, y_0, y_1) \mapsto (y_0^2 - sy_1^2,\ s,\ y_0(y_0^2 - sy_1^2 - s^m),\ y_1(y_0^2 - sy_1^2 - s^m)).$$

The exceptional locus in U is the divisor $y_0^2 - sy_1^2 - s^m = 0$. Pulling the canonical basis of differentials on $\operatorname{Spec} R_m$ back to U gives

$$f_2^*\left(\frac{dr \wedge ds \wedge dv}{u}\right) = \frac{d(y_0^2 - sy_1^2) \wedge ds \wedge d(y_1(y_0^2 - sy_1^2 - s^m))}{y_0(y_0^2 - sy_1^2 - s^m)} = 2ds \wedge dy_0 \wedge dy_1$$

which is a regular differential on U, and in particular has no zero or pole along the exceptional divisor. Thus the map is crepant, as claimed. The case $m = \infty$ is similar, but easier, as both open charts are affine 3-space.
(3) The easiest way to establish the claim is to recognise F_m as a pullback from the

universal D_4 flop and apply restriction theorems for contraction algebras. Consider the six-dimensional universal D_4 flop, given in [K, (1.1)] as

$$\mathcal{R} = \frac{\mathbb{C}[r,s,t,u,v,w,z]}{u^2 - rw^2 + 2zvw - sv^2 + (rs - z^2)t^2}$$

and its universal family $\mathcal{Y} \longrightarrow \operatorname{Spec}\mathcal{R}$, which is an isomorphism away from the singular locus in $\operatorname{Spec}\mathcal{R}$. As observed by Van Garderen [vG2, §2.2.3], slicing by the sequence $g_1 = z$, $g_2 = r - w - s^m$, and $g_3 = t$ yields a commutative diagram

$$\begin{tikzcd}
Y = Y_3 \arrow[r] \arrow[d, "f"'] & Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\
\operatorname{Spec}\mathcal{R}_3 \arrow[r] & \operatorname{Spec}\mathcal{R}_2 \arrow[r] & \operatorname{Spec}\mathcal{R}_1 \arrow[r] & \operatorname{Spec}\mathcal{R}
\end{tikzcd}$$

where $\mathcal{R}_1 = \mathcal{R}/g_1$, $\mathcal{R}_2 = \mathcal{R}_1/g_2$ and $\mathcal{R}_3 = \mathcal{R}_2/g_3$. The result is the affine algebra

$$\mathcal{R}_3 = \frac{\mathbb{C}[r,s,u,v]}{u^2 - r(r - s^m)^2 - sv^2}$$

whose completion at the origin is $\mathcal{R}_m$. The pullback $f\colon Y \to \operatorname{Spec}\mathcal{R}_3$ is visibly an isomorphism away from $\operatorname{Sing}(\mathcal{R}_3)^{\mathrm{red}} = (u, v, r - s^m)$, so in particular is birational.

Van Garderen observes that $\mathcal{R}_3$ is an integral domain [vG2, 2.12] and Y is smooth [vG2, 2.13], and that each g_i is a *slice*, in the terminology of [vG2, 2.9], which implies that f is projective and surjective with $\mathbf{R}f_*\mathcal{O} = \mathcal{O}$ [vG2, 2.10].

Furthermore, the tilting bundle yielding a derived equivalence between $\mathcal{Y}$ and $\Lambda \in \operatorname{CM}\mathcal{R}$ restricts to give a derived equivalence between Y and $\Lambda \otimes \mathcal{R}_3$ [vG2, (2.11)]. Since g_1, g_2, g_3 is a regular sequence, $\Lambda \otimes \mathcal{R}_3 \in \operatorname{CM}\mathcal{R}_3$ and so in particular f is crepant [IW1, 4.14]. Since visibly both the blowup in (2) and f are crepant resolutions of the same variety, and both containing no flopping curves, they must be isomorphic (as varieties over the base $\operatorname{Spec} R_m$) since minimal models are unique up to flop. Thus the contraction algebra associated to (2) is isomorphic to the contraction algebra of the formal fibre of f. But by [vG2, 2.8] this is the claimed Jacobi algebra, namely $\mathcal{J}\mathrm{ac}(xy^2 + x^{2m+1})$ when $m \geq 1$, respectively $\mathcal{J}\mathrm{ac}(xy^2)$ when $m = \infty$.
(4) If $\mathcal{R}_m \cong \mathcal{R}_n$ are isomorphic, then the contraction algebras of $\mathcal{X}_m \to \operatorname{Spec}\mathcal{R}_m$ and of $\mathcal{X}_n \to \operatorname{Spec}\mathcal{R}_n$ must be isomorphic. But then their abelianisations must also be isomorphic, and so in particular must have the same dimension. But the abelianisations have dimension $2(m+1)$ and $2(n+1)$ respectively, and hence $m = n$. ☐

The start of this subsection, combined with 8.12, then gives the following.

Corollary 8.13. *All the potentials in 6.18 are geometric.*

In turn, this establishes 1.11 in the introduction.

Corollary 8.14. *Conjecture 1.10 is true, except for the one remaining unresolved case when $f \cong x^3 + \mathcal{O}_4$, where some further analysis is required.*

Proof. Every $\mathrm{A}_{\mathrm{con}} \cong \mathcal{J}\mathrm{ac}(f)$ for some $f \in \mathbb{C}\langle\!\langle x, y\rangle\!\rangle_{\geq 2}$. If $f_2 \neq 0$ the result is 5.2, so we can assume $f_2 = 0$. We need $f_3 \neq 0$ so that $\operatorname{Jdim}\mathcal{J}\mathrm{ac}(f) \leq 1$, and 4.13 splits into three cases. The first two cases are covered by 6.18, and 8.13 asserts that these are all geometric. The only remaining, unresolved, case from 4.13 is when $f_3 \cong x^3$. ☐

8.3. **Classification for A and D divisor-to-curve contractions.** This section is the divisor-to-curve analogue of §8.1, and justifies our philosophy that divisor-to-curve contractions should be 'limits' of flops.

Proposition 8.15. *If $\mathrm{A}_{\mathrm{con}}$ is a contraction algebra of a Type A divisor-to-curve contraction, then $\mathrm{A}_{\mathrm{con}} \cong \mathcal{J}\mathrm{ac}(x^2)$.*

Proof. For the same homological reason as in 8.15, in Type A the contraction algebra $\mathrm{A}_{\mathrm{con}}$ is necessarily commutative. Since the contraction is divisor-to-curve, necessarily $\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}} = \infty$ [DW3]. Combining 5.4 and 5.1, we see that $\mathrm{A}_{\mathrm{con}} \cong \mathcal{J}\mathrm{ac}(x^2)$, since x^2 is the only infinite dimensional example in 5.1. ☐

The following is the analogue of 8.9. However, it is slightly weaker, due to two key geometric facts having only been developed in the flops setting: (1) Katz–Morrison [KM] asserts that Type D is generically Type D_4 for flops, but this is open in the divisor-to-curve setting, and (2) Hua–Toda [HT] asserts that the isomorphism class of a contraction algebra determines the type, but again only for flops.

However, we can say the following, without using any geometric classifications.

Proposition 8.16. *If* $\mathrm{A}_{\mathrm{con}}$ *is a contraction algebra of a Type* D_4 *divisor-to-curve contraction, then* $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(x^2y)$*, or* $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(xy^2 + x^{2m+1})$ *for some* $m \geq 2$.

Proof. The proof is very similar to 8.9. The algebra $\mathrm{A}_{\mathrm{con}}$ is not commutative by 8.7, since $\mathrm{A}_{\mathrm{con}}$ has a factor which is not commutative. As $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(f)$ for some $f \in \mathbb{C}\langle\langle x, y\rangle\rangle_{\geq 2}$, appealing to 5.4 then gives $f_2 = 0$.

From this, 8.8 asserts that $f_3 \neq 0$, and $f_3 \ncong \ell^3$. Hence by 6.18 $\mathrm{A}_{\mathrm{con}} \cong \mathfrak{J}\mathrm{ac}(f)$ for some f in the list stated there. Only the top two families are possible, since $\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}} = \infty$ given the contraction is divisor-to-curve [DW3]. □

Appendix A. $\mathfrak{J}$-dimension restrictions

The papers [ISm, IS] introduce several new ideas that substantially strengthen the Golod–Shafarevich estimates [GS] for the growth of algebraic Jacobi rings, and prove that almost all have cubic or higher $\mathfrak{J}$-dimension. In this appendix we extend the main results of [IS] into the setting of formal noncommutative Jacobi algebras of 3.1, in a manner which should be viewed as the analogue of Vinberg's [V3] extension of Golod–Shafarevich into the setting of topological rings. This extension is not entirely trivial, as we need to convince the reader (and ourselves!) that the growth in the $\mathfrak{J}$-dimension of $\mathfrak{J}\mathrm{ac}(f)$ can be approximated using algebraic Jacobi algebras, where results of [IS] can then be applied.

A.1. **Algebraic Notation.** Throughout this appendix we let $d \geq 2$ and consider $\mathbb{C}\langle \mathsf{x}\rangle = \mathbb{C}\langle x_1, \ldots, x_d\rangle$. An element $F \in \mathbb{C}\langle \mathsf{x}\rangle$ is called a *superpotential* if it is cyclically symmetric, in the sense of 3.3. For $m \geq k \geq 3$ write

$$\mathsf{SP}_{k,m} = \{F \in \mathbb{C}\langle \mathsf{x}\rangle \mid F \text{ is a superpotential with } F_j = 0 \text{ for } j < k \text{ and } j > m\},$$

where F_j is the homogeneous component of F of degree j, as in §2.1. In the special case $m = k$, write $\mathsf{SP}_k := \mathsf{SP}_{k,k}$, which consists of all homogeneous superpotentials of degree k, together with zero. Throughout, we will write elements of $\mathbb{C}\langle \mathsf{x}\rangle$ and $\mathbb{C}\langle\langle \mathsf{x}\rangle\rangle$ by small letters f and g, and superpotentials by capital letters F, G.

With the (left) strike-off derivatives $\eth_i$ defined as in (3.A), the *algebraic Jacobi algebra* associated to a superpotential F is the algebra

$$\mathrm{A}\mathfrak{J}\mathrm{ac}(F) := \frac{\mathbb{C}\langle x_1, \ldots, x_d\rangle}{(\eth_1 F, \ldots, \eth_d F)} = \frac{\mathbb{C}\langle \mathsf{x}\rangle}{I_F}$$

where $I_F = (\eth_1 F, \ldots \eth_d F)$ is the two-sided ideal generated by $\eth_1 F, \ldots, \eth_d F$. We write $\mathfrak{m} = (x_1, \ldots, x_d) \subset \mathbb{C}\langle \mathsf{x}\rangle$, a maximal two-sided ideal, and denote its image in $\mathrm{A}\mathfrak{J}\mathrm{ac}(F)$ by $\mathfrak{R} = \mathfrak{m}/I_F$, the powers of which are $\mathfrak{R}^i = (\mathfrak{m}^i + I_F)/I_F$.

The use of strike-off derivatives $\eth_i$ on superpotentials, as we do here to align with the statements and results of [ISm, IS], or cyclic derivatives δ_i on any potential, as in 3.1, give equivalent theories but with minor differences in detail, which we address in §A.5.

A.2. **Exact Potentials and Hilbert series.** The differentiation package has two useful tools. The first is the following Euler relation.

Lemma A.1. [ISm, 3.5] *If* F *is a superpotential, then* $\sum_{i=0}^{d} [x_i, \eth_i F] = 0$.

The second is a sequence of right $\mathrm{A}\mathfrak{J}\mathrm{ac}(F)$-modules

$$0 \to \mathrm{A}\mathfrak{J}\mathrm{ac}(F) \xrightarrow{\mathsf{d}_3} \mathrm{A}\mathfrak{J}\mathrm{ac}(F)^{\oplus d} \xrightarrow{\mathsf{d}_2} \mathrm{A}\mathfrak{J}\mathrm{ac}(F)^{\oplus d} \xrightarrow{\mathsf{d}_1} \mathrm{A}\mathfrak{J}\mathrm{ac}(F) \xrightarrow{\mathsf{d}_0} \mathbb{C} \to 0 \tag{A.A}$$

defined in e.g. [ISm, 3.4]. The precise form of the d_i will not concern us, as below we will only require the following two facts.

(1) [ISm, 3.6] For any superpotential F, the sequence (A.A) is a complex, which is exact at the three right-most non-zero terms.

(2) If further F is homogeneous, say $0 \neq F \in \mathsf{SP}_k$, then the morphisms in the complex (A.A) satisfy $\deg(\mathsf{d}_3) = 1$, $\deg(\mathsf{d}_2) = k-2$, $\deg(\mathsf{d}_1) = 1$, and $\deg(\mathsf{d}_0) = 0$. Note that $\mathsf{d}_0(f) = f_0$ is the natural augmentation map.

Definition A.2. An element $F \in \mathsf{SP}_{k,m}$ is called *exact* if (A.A) is exact.

If G is homogeneous, then the ideal $(\partial_1 G, \dots \partial_d G)$ is a homogeneous ideal and so the graded decomposition of $\mathbb{C}\langle \mathsf{x} \rangle$ induces a decomposition

$$\mathrm{A}\mathfrak{J}\mathrm{ac}(G) = \bigoplus_{n \geq 0} \mathrm{A}\mathfrak{J}\mathrm{ac}(G)_i. \tag{A.B}$$

For $G \in \mathsf{SP}_k$, the boundary maps d_i are homogeneous, and so furthermore the sequence (A.A) also decomposes into graded pieces, or *homogeneous slices*, each of which is a complex of finite-dimensional vector spaces, exact at the codomains of the restrictions of d_0, d_1 and d_2.

Definition A.3. For $G \in \mathsf{SP}_k$, (A.B) determines the *Hilbert series* of $\mathrm{A}\mathfrak{J}\mathrm{ac}(G)$

$$\mathcal{H}_G = \sum_{i \geq 0} \dim_{\mathbb{C}} \big(\mathrm{A}\mathfrak{J}\mathrm{ac}(G)_i\big) t^i \in \mathbb{C}[\![t]\!].$$

In the following, and throughout, recall that SP_k and $\mathsf{SP}_{k,m}$ are only defined when $k \geq 3$.

Lemma A.4. *Let $G \in \mathsf{SP}_k$ be exact, and write $a_i = \dim_{\mathbb{C}} \mathrm{A}\mathfrak{J}\mathrm{ac}(G)_i$. Then the following statements hold.*

(1) *The elements $\partial_1 G, \dots, \partial_d G$ are linearly independent over $\mathbb{C}$.*
(2) *For all $j \geq k$ there is an equality*

$$a_j - d a_{j-1} + d a_{j-k+1} - a_{j-k} = 0. \tag{A.C}$$

(3) *Setting $a_i = 0$ for $i < 0$, the equation (A.C) holds for all $j > 0$.*

Proof. To ease notation, temporarily write $\Lambda = \mathrm{A}\mathfrak{J}\mathrm{ac}(G)$, so that $a_i = \dim \Lambda_i$.
(1) Since $k-1 \geq 2$, and in (A.A) $\mathbb{C}$ lies in degree zero, taking the appropriate homogeneous slice of (A.A) yields an exact sequence

$$0 \to 0 \xrightarrow{\mathsf{d}_3} \Lambda_0^{\oplus d} \xrightarrow{\mathsf{d}_2} \Lambda_{k-2}^{\oplus d} \xrightarrow{\mathsf{d}_1} \Lambda_{k-1} \xrightarrow{\mathsf{d}_0} 0 \to 0.$$

It follows that $a_{k-1} = d a_{k-2} - d a_0$. But clearly $a_0 = 1$, and $a_{k-2} = d^{k-2}$ since there are no relations until degree $k-1$. Hence $a_{k-1} = d^k - d$, and consequently the d homogeneous relations $\partial_1 G, \dots, \partial_d G$ of degree $k-1$ must be linearly independent.
(2) If $j \geq k (\geq 3)$, in a very similar way to the above, (A.A) yields an exact sequence

$$0 \to \Lambda_{j-k} \xrightarrow{\mathsf{d}_3} \Lambda_{j-k+1}^{\oplus d} \xrightarrow{\mathsf{d}_2} \Lambda_{j-1}^{\oplus d} \xrightarrow{\mathsf{d}_1} \Lambda_j \xrightarrow{\mathsf{d}_0} 0 \to 0 \tag{A.D}$$

from which the claim follows.
(3) Admitting trivial summands in negative degree, $\Lambda_i = \{0\}$ for $i < 0$, to the direct sum decomposition of $\mathrm{A}\mathfrak{J}\mathrm{ac}(G)$, the sequence (A.D) is defined and exact for all $0 < j < k$. □

It follows that (A.A) computes the Hilbert series in the exact homogeneous case.

Corollary A.5. *If $G \in \mathsf{SP}_k$ is exact, then its Hilbert series is $\mathcal{H}_G = (1 - dt + dt^{k-1} - t^k)^{-1}$.*

Proof. The only homogeneous slice (A.D) of (A.A) not considered in A.4 is when $j = 0$, but that is simply the identification $\mathrm{A}\mathfrak{J}\mathrm{ac}(G)_0 = \mathbb{C}$ of the augmentation map d_0. Therefore multiplying (A.C) by t^j and summing over all $j \geq 0$ gives

$$\mathcal{H}_G - dt\mathcal{H}_G + dt^{k-1}\mathcal{H}_G - t^k \mathcal{H}_G - 1 = 0.$$ □

The linear independence in A.4(1) extends to one-sided independence over $\mathbb{C}\langle \mathsf{x} \rangle$.

Corollary A.6. *Let $G \in \mathsf{SP}_k$ be exact. If $\sum_{i=1}^d (\partial_i G) u_i = 0$ for $u_1, \dots, u_d \in \mathbb{C}\langle \mathsf{x} \rangle$, then $u_i = 0$ for all i.*

Proof. Since $\mathbb{C}\langle \mathsf{x}\rangle$ is graded, we can assume that the u_i are also homogeneous, say of degree t. From here is a simple induction, with the $t = 0$ case being A.4(1). For each i, clearly we may write $u_i = \sum_{j=1}^{d} w_j x_j$ for some w_j. Hence

$$0 = \sum_{i=1}^{d} (\partial_i G) u_i = \sum_{i=1}^{d} (\partial_i G) \sum_{j=1}^{d} w_j x_j = \sum_{j=1}^{d} \left(\sum_{i=1}^{d} (\partial_i G) w_i \right) x_j.$$

Given this holds in the free algebra, it follows that $\sum_{i=1}^{d} (\partial_i G) w_i = 0$, and so by induction the result follows. $\square$

It will be convenient to consider the following subset of homogeneous superpotentials:

$$\mathsf{ESP}_k := \{G \in \mathsf{SP}_k \mid G \text{ is an exact potential, and } \mathrm{A}\mathfrak{J}\mathrm{ac}(G) \xrightarrow{x_1 \cdot} \mathrm{A}\mathfrak{J}\mathrm{ac}(G) \text{ is injective}\}.$$

Lemma A.7. *If $G \in \mathsf{ESP}_k$ then there is a set $N \subset \mathbb{C}\langle \mathsf{x}\rangle$ of monomials, satisfying $x_1 N \subset N$, which descends to a basis of the $\mathbb{C}$-vector space $\mathrm{A}\mathfrak{J}\mathrm{ac}(G)$.*

Proof. Setting $M_0 = \{1\}$, inductively by degree in (A.B) we can construct sets of monomials M_j of $\mathbb{C}\langle \mathsf{x}\rangle$ of degree j such that the following two conditions are satisfied.

(1) There is an inclusion $x_1 M_j \subseteq M_{j+1}$.
(2) M_j descends to give a basis of $\mathrm{A}\mathfrak{J}\mathrm{ac}(G)_j$.

It is clear that the necessarily disjoint union $N = \bigcup_{j \geq 0} M_j$ descends to a basis of $\mathrm{A}\mathfrak{J}\mathrm{ac}(G)$, since it does so in each degree. $\square$

Recall $d \geq 2$ is the number of variables in $\mathbb{C}\langle \mathsf{x}\rangle$.

Lemma A.8. *If $k \geq 3$ with $(d, k) \neq (2, 3)$, then $\mathsf{ESP}_k \neq \emptyset$.*

Proof. This is [IS]. Set

$$g = \begin{cases} \sum_{\sigma \in \mathfrak{S}_{d-1}} x_d^{k-d+1} x_{\sigma(1)} \dots x_{\sigma(d-1)} & \text{if } k \geq d \\ x_d x_{d-1} \dots x_{d-k+1} + \sum_{j \in \mathcal{S}} x_j x_d \mathsf{m}_j & \text{if } k < d \end{cases}$$

where $\mathfrak{S}_{d-1}$ is the symmetric group, and $\mathcal{S} = \{j \mid 1 \leq j \leq d-1,\ j \neq d-k+1\}$, and the m_j are explicit monomials explained in [IS, 3.6]. It is a reasonably elementary calculation to show that $G = \mathrm{sym}(g) \in \mathsf{ESP}_k$; see [IS, 3.5, 3.6]. $\square$

It turns out, although we do not need this, that if $d = 2$ then $\mathsf{ESP}_3 = \emptyset$. This is why the argument below fails in the $(d, k) = (2, 3)$ case.

The basis in A.7 adapted to x_1 gives tighter control in Vinberg's argument [V3]. With notation M_j as in the proof of A.7, consider the following subset of $\mathbb{C}\langle \mathsf{x}\rangle$, defined by

$$M_j^+ = \{m \in M_j \mid m \neq x_1 n \text{ for any monomial } n\}.$$

Set $A_j = \mathrm{Span}_{\mathbb{C}}(M_j)$ and $A_j^+ = \mathrm{Span}_{\mathbb{C}}(M_j^+)$, both of which are linear subspaces of $\mathbb{C}\langle \mathsf{x}\rangle$, and note that since $M_j^+ \subseteq M_j \backslash x_1 M_{j-1}$, there is an inequality $|M_j^+| \leq a_j - a_{j-1}$.

Lemma A.9. *Let $G \in \mathsf{ESP}_k$ and set $I = (\partial_1 G, \dots \partial_d G)$. Then for all $w \geq 0$,*

$$I_{k+w} = V I_{k+w-1} \oplus R' A_{w+1} \oplus (\partial_1 G) A_{w+1}^+.$$

Proof. Recall $V = \mathrm{Span}_{\mathbb{C}}\{x_1, \dots, x_d\}$ and $R = \mathrm{Span}_{\mathbb{C}}\{\partial_1 G, \dots, \partial_d G\}$. The Vinberg trick adapted to x_1 gives

$$\begin{aligned} I_{k+w} &= V I_{k+w-1} + R V^{w+1} \\ &= V I_{k+w-1} + R A_{w+1} && \text{(since } M_{w+1} \text{ descends to a basis)} \\ &= V I_{k+w-1} + R' A_{w+1} + (\partial_1 G) A_{w+1}^+ && \text{(applying } \eta) \end{aligned}$$

where $R' = \mathrm{Span}_{\mathbb{C}}\{\partial_2 G, \dots, \partial_d G\}$. Estimating the dimension of each of the three individual summands, then summing, gives

$$\begin{aligned} \dim I_{k+w} &\leq n(n^{k+w-1} - a_{k+w-1}) + (n-1) a_{w+1} + (a_{w+1} - a_w) \\ &= n^{k+w} - n a_{k+w-1} + n a_{w+1} - a_w. \end{aligned}$$

But $\dim I_{k+w} = n^{k+w} - n a_{k+w-1} + n a_{w+1} - a_w$ by A.4(2), and this equality of dimensions implies the sum of three terms is direct. $\square$

A.3. **Modules of Syzygies.** For $F \in \mathsf{SP}_{k,m}$ define Ω_F, the module of syzygies of the presentation, to be the kernel

$$0 \to \Omega_F \to (\mathbb{C}\langle \mathsf{x}\rangle \otimes_{\mathbb{C}} \mathbb{C}\langle \mathsf{x}\rangle^{\mathrm{op}})^{\oplus d} \xrightarrow{\varphi_F} (\partial_1 F, \dots \partial_d F) \to 0 \qquad \text{(A.E)}$$

where the surjective map $\varphi = \varphi_F$ is defined on simple tensors by

$$(u_1 \otimes v_1, \dots, u_d \otimes v_d) \mapsto \sum_{i=1}^{d} u_i(\partial_i F)v_i$$

and extended linearly. Since φ is a $\mathbb{C}\langle \mathsf{x}\rangle$-bimodule homomorphism, the kernel Ω_F is automatically a bimodule. For each $1 \le i, j \le d$ and any $r \in \mathbb{C}\langle \mathsf{x}\rangle$, there is a *trivial syzygy*

$$\tau^r_{i,j} := (0, \dots, 0, 1 \otimes r(\partial_j F), 0, \dots, 0) - (0, \dots, 0, (\partial_i F)r \otimes 1, 0, \dots, 0)$$

where the first term has non-trivial entry in the ith component, and the second has non-trivial entry in the jth. In addition, the *Euler syzygy* is defined as

$$\eta := (1 \otimes x_1 - x_1 \otimes 1, \dots, 1 \otimes x_d - x_d \otimes 1).$$

It is clear that $\eta \in \Omega_F$, and $\tau^r_{i,j} \in \Omega_F$ for all $r \in \mathbb{C}\langle \mathsf{x}\rangle$ and all $1 \le i, j \le d$.

Proposition A.10. *If $G \in \mathsf{ESP}_k$ then Ω_G is generated as a $\mathbb{C}\langle \mathsf{x}\rangle$-bimodule by the set $\{\tau^r_{i,j}, \eta \mid r \in \mathbb{C}\langle \mathsf{x}\rangle, 1 \le i, j \le d\}$.*

Proof. Since G is homogeneous, the sequence (A.E) is graded. If Ω_G is not generated as claimed, then there is a homogeneous element $0 \ne \alpha = (\alpha_1, \dots, \alpha_d) \in \Omega_G$ of minimal degree w (that is, $\deg \alpha_i = w$ for every nonzero component α_i) that is not in the span of trivial syzygies and η. Considering those summands of each α_i that have a variable x_i leading the left-hand tensor factor, we may write

$$\alpha = (1 \otimes u_1, \dots, 1 \otimes u_d) + x_1\xi_1 + \dots + x_d\xi_d$$

where each u_i is homogenous of degree w and ξ_i is homogeneous of degree $w-1$. If $w = 0$, then $\varphi_G(\alpha) = \sum(\partial_i G)u_i = 0$ where $u_i \in \mathbb{C}$ are constants, which contradicts A.4(1) unless $\alpha = 0$, so we may suppose that $w \ge 1$.

We first simplify the u_i. With notation as above A.9, write $u_i = u_i^+ + u_i^- \in A_w \oplus I_w = \mathbb{C}\langle \mathsf{x}\rangle_w$ and note that $u_i^- \in I_w$ is a sum of terms of the form $a(\partial_j F)b$. We may then use the trivial syzygy $\tau^a_{i,j}$ to write each such term as

$$\begin{aligned}(0, \dots, 0, 1 \otimes a(\partial_j G)b, 0, \dots, 0) &= (0, \dots, 0, 1 \otimes a\partial_j G, 0, \dots, 0)b && \text{in the } i\text{th component}\\ &= (0, \dots, 0, (\partial_i G)a \otimes 1, 0, \dots, 0)b && \text{in the } j\text{th component.}\end{aligned}$$

Since $\deg G = k \ge 3$, the final expression has a variable on the left of each term, and so absorbing all such terms into $x_1\xi_1 + \dots + x_d\xi_d$ we may suppose that every $u_i \in A_w$.

Furthermore, we may assume $u_1 \in A_w^+$ by the Euler syzygy η: indeed we may remove any summand $1 \otimes x_1 v$ from the first component by

$$(1 \otimes x_1 v, 0, \dots, 0) = -(0, 1 \otimes x_2 v, \dots, 1 \otimes x_d v) + (x_1 \otimes v, \dots, x_d \otimes v)$$

and the second summand may again be absorbed into $x_1\xi_1 + \dots + x_d\xi_d$. It then follows that $\varphi(\alpha) \in I_{k+w-1}$ has the form

$$\sum_{i=1}^{d}(\partial_i G)u_i + \sum a_i(\partial_i G)b_i \in \big((\partial_1 G)A_w^+ \oplus R'A_w\big) \oplus VI_{k+w-2} \overset{\text{A.9}}{=} I_{k+w-1}$$

were $u_1 \in A_w^+$ and $u_j \in A_w$ for $j = 2, \dots, d$. Since $\varphi(\alpha) = 0$ and the sum is direct, we must also have $\sum(\partial_i G)u_i = 0$. By A.6 $u_1 = \dots = u_d = 0$, and so

$$0 = \varphi(\alpha) = \varphi\big(\sum_{i=1}^{d} x_i\xi_i\big) = \sum_{i=1}^{d} x_i\varphi(\xi_i).$$

Since $\mathbb{C}\langle \mathsf{x}\rangle$ is free, each $\varphi(\xi_i) = 0$, and so each ξ_i is a syzygy of smaller degree. By minimality of w, each ξ_i is therefore generated by trivial syzygies and the Euler syzygy. But therefore so is α, which is a contradiction. $\square$

Recall that y_i denotes the ith graded piece of y.

Proposition A.11. *Suppose that $F \in \mathsf{SP}_{k,m}$ such that $F_k \in \mathsf{ESP}_k$. If $0 \neq y \in (\partial_1 F, \dots, \partial_d F)$ with $y_i = 0$ for all $0 \leq i < t$, then we can write*

$$y = \sum_{\operatorname{ord}(u_i v_i) \geq t-k+1} u_i (\partial_{j(i)} F) v_i$$

with $u_i, v_i \in \mathbb{C}\langle \mathsf{x} \rangle$ monomials and $j(i) \in \{1, \dots, d\}$ for each i.

Proof. Since y is in the ideal, there is an expression

$$y = \sum_h u_h (\partial_{j(h)} F) v_h$$

with $u_h, v_h \in \mathbb{C}\langle \mathsf{x} \rangle$ monomials. Set $N = \min\{\deg(u_h v_h) \mid u_h v_h \neq 0\}$, so that in particular $N+k-1 \leq \operatorname{ord}(y)$. If equality holds, then the proof is complete, hence we can suppose that $N+k-1 < \operatorname{ord}(y)$. Proceeding inductively, we construct a new expression $y = \sum u(\partial_j F)v$ in which each $\deg(uv) > N$.

We first consider the derivatives of the lowest degree term F_k. For each $i = 1, \dots, d$ set $\rho_i = \partial_i F_k$. Since $F_k \in \mathsf{ESP}_k$, in particular F_k is exact, by A.4(1) each ρ_i is non-zero and homogeneous of degree $k-1$. The cancellation $\sum_{\deg(u_h v_h) = N} u_h \rho_{j(h)} v_h = 0$ in $\mathbb{C}\langle \mathsf{x} \rangle$ of the lowest degree terms expresses a syzygy from Ω_{F_k}, which by A.10 may be written as

$$\sum_{\deg(u_h v_h) = N} u_h \otimes v_h = \sum a^r_{i,j} \tau^r_{i,j} b^r_{i,j} + \sum_s c_s \eta \, d_s$$

for homogeneous elements $r, a^r_{i,j}, b^r_{i,j}, c_s, d_s \in \mathbb{C}\langle \mathsf{x} \rangle$ satisfying $\deg(a^r_{i,j}\, r\, b^r_{i,j}) = N - k + 1$ and $\deg(c_s d_s) = N - k$; that is, since Ω_G is graded, we may express a homogeneous element in terms of homogeneous generators as a sum of terms in a single degree.

We now consider the derivatives of the whole of F. To avoid confusion, we denote the trivial syzygies in Ω_F by $\mathcal{T}^r_{i,j}$. Note that $\tau^r_{i,j}$ is the lowest order piece of $\mathcal{T}^r_{i,j}$, and the Euler syzygy η is the same element for F and F_k. Thus

$$y = y - 0 = \sum_h u_h (\partial_{j(h)} F) v_h - \big(\sum a^r_{i,j} \varphi(\mathcal{T}^r_{i,j}) b^r_{i,j} + \sum_s c_s \varphi(\eta) b_s \big)$$

where the lowest order piece of each $a^r_{i,j} \varphi(\mathcal{T}^r_{i,j}) b^r_{i,j}$ is exactly $a^r_{i,j} \rho^r_{i,j} b^r_{i,j}$, of degree N, and each $c_s \varphi(\eta) b_s$ is homogeneous, also of degree N. Thus, in this new expression for y, those terms with $\deg u_h v_h = N$ in the first summand are cancelled identically by the lowest order piece of the second summand. $\square$

Recall that $\mathfrak{R}^i = (\mathfrak{m}^i + I_F)/I_F$.

Corollary A.12. *If $F \in \mathsf{SP}_{k,m}$ with $F_k \in \mathsf{ESP}_k$, then left multiplication*

$$x_1 \colon \mathrm{A}\mathfrak{J}\mathrm{ac}(F)/\mathfrak{R}^j \to \mathrm{A}\mathfrak{J}\mathrm{ac}(F)/\mathfrak{R}^{j+1}$$

is injective for all $j \geq 1$.

Proof. Setting $I = (\partial_1 F, \dots, \partial_d F)$, the claim is that left multiplication by x_1

$$\mathbb{C}\langle \mathsf{x} \rangle/(I + \mathfrak{m}^j) \to \mathbb{C}\langle \mathsf{x} \rangle/(I + \mathfrak{m}^{j+1}) \tag{A.F}$$

is injective. Suppose that $u \in \mathbb{C}\langle \mathsf{x} \rangle$ satisfies $x_1 u \in I + \mathfrak{m}^{j+1}$ but $u \notin I + \mathfrak{m}^j$. Set $\ell = \operatorname{ord}(u)$, so that $u = u_\ell + \mathcal{O}_{\ell+1}$. Since $u \notin I + \mathfrak{m}^j$, necessarily $u \notin \mathfrak{m}^j$, so $\ell < j$. Hence we can choose $u \in \mathbb{C}\langle \mathsf{x} \rangle$ of maximal possible order for which $x_1 u \in I + \mathfrak{m}^{j+1}$, but $u \notin I + \mathfrak{m}^j$.

Now if $x_1 u \in \mathfrak{m}^{j+1}$, then $u \in \mathfrak{m}^j$, which contradicts $u \notin I + \mathfrak{m}^j$. Hence $x_1 u \in I + \mathfrak{m}^{j+1}$ implies there exists $0 \neq y \in I$ such that

$$x_1 u - y \in \mathfrak{m}^{j+1}. \tag{A.G}$$

Writing $y = \sum y_t$ and taking the degree t pieces of (A.G), we see that $y_t = 0$ for all $t \leq \ell$, and $x_1 u_\ell - y_{\ell+1} = 0$.

Setting $t = \ell + 1$, we may write $y = \sum u_i (\partial_{j(i)} F) v_i$ as in A.11, where the u_i and v_i are monomials that satisfy $\operatorname{ord}(u_i v_i) \geq \ell - k + 2$. Taking leading terms gives

$$\sum_{\operatorname{ord}(u_i v_i) = \ell - k + 2} u_i (\partial_{j(i)} F_k) v_i = x_1 u_\ell$$

in $\mathbb{C}\langle\mathsf{x}\rangle$. But $F_k \in \mathsf{ESP}_k$, and so x_1 is injective on $\mathrm{A}\mathcal{J}\mathrm{ac}(F_k)$. Thus $u_\ell \in (\eth_1 F_k, \ldots, \eth_d F_k)_\ell$ and so we can find monomials a_i, b_i satisfying $\mathrm{ord}(a_i b_i) = \ell - k + 1$, such that

$$u_\ell = \sum_i a_i(\eth_{j(i)} F_k) b_i. \tag{A.H}$$

Replacing F_k by F in this expression, $v := u - \sum_i a_i(\eth_{j(i)}F)b_i$ is an element of $\mathbb{C}\langle\mathsf{x}\rangle$, with $\mathrm{ord}(v) \geq \ell + 1$ by (A.H). Since $u \equiv v \bmod I$, clearly $x_1 v \in I + \mathfrak{m}^{j+1}$ but $v \notin I + \mathfrak{m}^j$. Thus v contradicts the maximal choice of u, and so (A.F) is injective, as claimed. □

A.4. **Very general elements.** Fixing, once and for all, a basis $f_1, f_2, \ldots, f_r$ of SP_k, we treat SP_k as an irreducible algebraic family of superpotentials, and identify it with $\mathbb{C}^r = \operatorname{Spec} \mathbb{C}[\mathsf{t}]$, where $\mathsf{t} = t_1, \ldots, t_r$ are parameter variables and any element of SP_k is the specialisation $\mathsf{t} = a$ at a point $a \in \mathbb{C}^r$ of the 'generic' superpotential $G = \sum t_i f_i$. In particular, SP_k inherits the Zariski topology from $\mathbb{C}^r$. Under this identification, it is natural to abbreviate $G_a \in \mathsf{SP}_k$ by $a \in \mathsf{SP}_k$. (This is a low brow but nevertheless precise approach to the *variety of algebras* $\mathcal{W}_k = \{\mathrm{A}\mathcal{J}\mathrm{ac}(G) \mid G \in \mathsf{SP}_k\}$ used in [ISm, IS].)

Lemma A.13 (cf. [ISm, 3.9])**.** *Fix (d, k) and consider SP_k with its Zariski topology as above. For any $i \geq 0$,*

1. *there is a non-empty Zariski open subset $U_i \subset \mathsf{SP}_k$ on which $\dim \mathrm{A}\mathcal{J}\mathrm{ac}(G_a)_i$ is constant and takes the minimum value of any $G \in \mathsf{SP}_k$.*
2. *there is a largest Zariski open subset $V_i \subset U_i \cap U_{i+1}$ on which the rank of the restriction $\mathsf{d}_3 \colon \mathrm{A}\mathcal{J}\mathrm{ac}(G_a)_i \to \mathrm{A}\mathcal{J}\mathrm{ac}(G_a)_{i+1}$ for any $a \in V_i$ is the maximum possible for a linear map between spaces of these dimensions (i.e. is injective).*
3. *there is a largest Zariski open subset $W_i \subset V_i \cap U_{i+k-2}$ on which the rank of the restriction $\mathsf{d}_2 \colon \mathrm{A}\mathcal{J}\mathrm{ac}(G_a)_{i+1} \to \mathrm{A}\mathcal{J}\mathrm{ac}(G_a)_{i+k-1}$ for any $a \in W_i$ is the maximum possible for a linear map whose kernel contains the image of d_3.*

Thus if W_i not empty, then for $a \in W_i$, the homogeneous slice (A.D) *of $\mathrm{A}\mathcal{J}\mathrm{ac}(G_a)$ with domain of d_3 in degree i is an exact sequence of finite-dimensional vector spaces.*

Proof. (1) This is [IS, 2.2], proved as in [IS, 2.1]. The point is this: if the minimum is achieved at a_0, then choose a subset of the set Φ_i of all monomials of degree i for which the (square) coefficient matrix of a basis of the kernel of $\mathbb{C}\langle\mathsf{x}\rangle_i \to \mathrm{A}\mathcal{J}\mathrm{ac}(G_{a_0})_i$ with respect to Φ_i is invertible; the entries of this matrix are algebraic in a, so it remains invertible as a varies in a Zariski open subset $a_0 \in U_i \subset \mathsf{SP}_k$, proving that the kernel cannot get smaller so must have constant dimension, and thus the minimum is also achieved for all $a \in U_i$.
(2) Within $U_i \cap U_{i+1}$, the map $\mathsf{d}_3 \colon \mathrm{A}\mathcal{J}\mathrm{ac}(G_a)_i \to \mathrm{A}\mathcal{J}\mathrm{ac}(G_a)_{i+1}$ is determined by a matrix of fixed size whose entries are functions in a. Maximising its rank is an open condition on a, since it occurs on the complement of a locus of vanishing minors. (It is possible that all relevant minors vanish identically, in which case the conclusion is simply that V_i is empty.)
(3) Similarly, maximising the rank of d_2 is an open condition prescribed by minors that are functions in a; the condition that the kernel contains the image of d_3 is already imposed by the entries of the matrix, since $a \in V_i$ and (A.A) is a complex. □

For formal power series $\varphi(t)$ and $\psi(t)$, write $\varphi \geq \psi$ to mean that the coefficients of $\varphi - \psi$ are all non-negative. If $\mathcal{P}$ is a family of power series, then $\psi \in \mathcal{P}$ is called the *minimum* if $\varphi \geq \psi$ for all $\varphi \in \mathcal{P}$, noting that the minimum does not necessarily exist.

Proposition A.14 (cf. [ISm, 3.8])**.** *If $(d, k) \neq (2, 3)$, then there exists a countable intersection $\mathcal{U}$ of non-empty Zariski opens of SP_k such that for all $G \in \mathcal{U}$, the following statements hold.*

1. *$G \in \mathsf{ESP}_k$*
2. *$\mathcal{H}_G$ is the minimum in $\{\mathcal{H}_H \mid H \in \mathsf{SP}_k\}$.*

In particular, the minimum Hilbert series in $\{\mathcal{H}_F \mid F \in \mathsf{SP}_k\}$ is $(1 - dt + dt^{k-1} - t^k)^{-1}$.

Proof. Let $\mathcal{U}_1$ be the countable intersection over all $i \geq 0$ of the Zariski open subsets W_i of A.13. By A.8, this intersection is not empty.

Similarly, there is another such countable intersection $\mathcal{U}_2 \subset \mathsf{SP}_k$ on which the left-multiplication map $x_1 \colon \mathrm{A}\mathcal{J}\mathrm{ac}(G) \to \mathrm{A}\mathcal{J}\mathrm{ac}(G)$ is injective, as injectivity maximises rank in each degree. Since x_1 is injective on the free algebra (at $\mathsf{t}=0$), or again applying A.8, this intersection is not empty.

By A.13(1), minimising any coefficient of the Hilbert series as G varies over SP_k is an open condition. Therefore there is another countable intersection $\mathcal{U}_3 \subset \mathsf{SP}_k$ of non-empty Zariski open subsets on which every coefficient is minimised. In particular, any potential in this non-empty intersection necessarily achieves the minimum Hilbert series. (Compare with [IS, 2.2], where this fact is attributed to Ufnarovskij [U].)

Thus $\mathcal{U}=\mathcal{U}_1\cap\mathcal{U}_2\cap\mathcal{U}_3$ is a non-empty countable intersection on which conditions (1) and (2) are satisfied. The final line follows from (A.5). $\square$

For a given $F\in\mathsf{SP}_{k,m}$, recall the notation $\mathfrak{m}$ and I_F from §A.1. The *Poincaré series* of F is defined to be

$$\mathcal{P}_F=\sum_{i\geq 0}\dim_{\mathbb{C}}\left(\tfrac{\mathbb{C}\langle \mathsf{x}\rangle}{\mathfrak{m}^{i+1}+I_F}\right)t^i \tag{A.I}$$

This measures the growth of the quotients of $\mathrm{A}\mathcal{J}\mathrm{ac}(F)$ by the ideals $(\mathfrak{m}^{i+1}+I_F)/I_F$. The general situation is more delicate than the homogeneous case of A.14, but still a minimum is achieved by a very general element.

Corollary A.15. *If $(d,k)\neq(2,3)$, then for any $m\geq k$ there exists $F\in\mathsf{SP}_{k,m}$ such that the following statements hold.*

(1) *$\mathcal{P}_F$ is the minimum in $\{\mathcal{P}_H\mid H\in\mathsf{SP}_{k,m}\}$.*
(2) *$F_{\mathrm{ord}(F)}\in\mathcal{U}$, where $\mathcal{U}\subset\mathsf{SP}_k$ is defined in A.14.*

Proof. As in A.14 (cf. [IS, 2.1] or [ISm, 3.8]), minimising each coefficient of the Poincaré series is an open condition in a family, and so the minimum is realised on a countable intersection of non-empty Zariski open subsets $\mathcal{V}\subset\mathsf{SP}_{k,m}$.

Consider the map $\mathsf{SP}_{k,m}\to\mathsf{SP}_k$ given by $F\mapsto F_k$. Intersecting the preimage of $\mathcal{U}\subset\mathsf{SP}_k$ with $\mathcal{V}$ determines a countable intersection of open subsets of $\mathsf{SP}_{k,m}$ on which the claims hold. Since $\mathbb{C}$ is uncountable, this set does contain a closed point. $\square$

A.5. **Power Series.** Given $f\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$, its Poincaré series is defined to be

$$\hat{\mathcal{P}}_f=\sum(\dim_{\mathbb{C}}\mathcal{J}\mathrm{ac}(f)/\mathfrak{J}^i)t^i.$$

When $F\in\mathsf{SP}_{k,m}$, we can view F as either a polynomial superpotential and form $\mathcal{P}_F$ in (A.I), or we can view F as an element of $\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ and form $\hat{\mathcal{P}}_F$. Since $\mathcal{J}\mathrm{ac}(f)$ is defined with respect to the cyclic derivatives δ, and $\mathrm{A}\mathcal{J}\mathrm{ac}(f)$ is defined with respect to strike-off derivatives ∂, it is not quite true that $\hat{\mathcal{P}}_F=\mathcal{P}_F$.

Lemma A.16. *Given $G\in\mathsf{SP}_{k,m}$, then $\mathcal{P}_G=\hat{\mathcal{P}}_{\mathbb{G}}$, where $\mathbb{G}=\sum_{i=k}^m\frac{1}{i}G_i$. Thus, if $G\in\mathsf{G}_k$, then $\hat{\mathcal{P}}_{\mathbb{G}}=\mathcal{P}_G=\frac{1}{1-t}\mathcal{H}_G$ where $\mathbb{G}=\frac{1}{k}G$.*

To remedy this, for $f\in\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle$ and for any $i\geq 0$, set $\mathsf{f}_i=f_{\leq i}$ and $\mathsf{F}_i=\mathrm{sym}(\mathsf{f}_i)$. Then, continuing the notation of 2.2, and recalling that $\mathfrak{R}^i=(\mathfrak{m}^i+I_F)/I_F$,

$$\begin{aligned}\mathcal{J}\mathrm{ac}(f)/\mathfrak{J}^i&\cong\frac{\mathbb{C}\langle\!\langle \mathsf{x}\rangle\!\rangle}{\langle\!\langle\mathfrak{n}^i,\delta_1 f,\dots,\delta_d f\rangle\!\rangle}\\&\cong\frac{\mathbb{C}\langle \mathsf{x}\rangle}{(\mathfrak{m}^i,\delta_1\mathsf{f}_i,\dots,\delta_d\mathsf{f}_i)}\\&\cong\frac{\mathbb{C}\langle \mathsf{x}\rangle}{\big(\mathfrak{m}^i,\partial_1(\mathrm{sym}\,\mathsf{f}_i),\dots,\partial_d(\mathrm{sym}\,\mathsf{f}_i)\big)}\cong\mathrm{A}\mathcal{J}\mathrm{ac}(\mathsf{F}_i)/\mathfrak{R}^i\end{aligned}\tag{A.J}$$

This gives a term-by-term algebraicisation of the Poincaré series, by

$$\hat{\mathcal{P}}_f=\sum_{i\geq 0}(\dim_{\mathbb{C}}\mathrm{A}\mathcal{J}\mathrm{ac}(\mathsf{F}_i)/\mathfrak{R}^i)t^i$$

A.6. **Main Results.** The main result, A.18, requires the following elementary lemma.

Lemma A.17. *For $d \in \mathbb{R}$ and $k \geq 2$ consider the formal power series*

$$\frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)} = \sum_{i\geq 0} b_i t^i.$$

Setting $b_j = 0$ for $j < 0$, the following statements hold.

(1) *There is an equality $b_j = b_{j-1}d - b_{j-k+1}d + b_{j-k} + 1$ for all $j \geq 0$.*
(2) *$b_0 = 1$, and further $b_j = 1 + d + \ldots + d^j$ for all $1 \leq j \leq k-2$.*

Proof. (1) Treating the series as a sum over $i \in \mathbb{Z}$, with $b_{<0} = 0$, and multiplying up shows at once that $b_0 = 1$ and that for any $j \in \mathbb{Z} \setminus \{0\}$ the coefficient of t^j is

$$b_j - (d+1)b_{j-1} + db_{j-2} + db_{j-k+1} - (d+1)b_{j-k} + b_{j-k-1} = 0.$$

When $j = 0$, the claimed equality in (1) holds. For $j \geq 1$, splitting off a single b_{j-1} summand from the equation above we see by induction

$$\begin{aligned} b_j &= db_{j-1} - db_{j-2} - db_{j-k+1} + (d+1)b_{j-k} - b_{j-k-1} \\ &\qquad + (db_{j-2} - db_{j-k} + b_{j-k-1} + 1) \\ &= db_{j-1} - db_{j-k+1} + b_{j-k} + 1 \end{aligned}$$

as claimed.
(2) For $1 \leq j \leq k-2$, the equality in (1) reads

$$b_j = b_{j-1}d + 1$$

and so the result holds at once since $b_0 = 1$. □

The following is the main result of this appendix, and it asserts that, in almost all cases, the $\mathfrak{J}$-dimension of $\mathcal{J}\mathrm{ac}(f)$ is ≥ 3. In particular, in almost all cases the Jacobi algebra is infinite dimensional, as a vector space. Recall from §2.1 that $\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq k}$ consists of all those $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ for which $f_j = 0$ for all $j < k$, and note that $0 \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq k}$.

Theorem A.18. *Suppose that $d = 2$ and $k \geq 4$, or $d \geq 3$ and $k \geq 3$. If $f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle$ has order k, then $\mathrm{Jdim}\, \mathcal{J}\mathrm{ac}(f) \geq 3$.*

In fact the proof shows that the coefficients of the Poincaré series of $\mathcal{J}\mathrm{ac}(f)$ are no smaller than those of

$$\frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)}.$$

When $d = 2$ and $k = 4$ this lower bound is $1/\big((1-t)^3(1-t^2)\big)$, and when $d = k = 3$ it is $1/(1-t)^4$, both of which have polynomial growth of degree 3. For all other d, k in the scope of the theorem, the growth is exponential and $\mathrm{Jdim}\, \mathcal{J}\mathrm{ac}(f) = \infty$.

Proof. Associated to the fixed $k = \mathrm{ord}(f)$ and d is the power series

$$\frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)} = \sum_{i\geq 0} b_i t^i.$$

The b_is are positive integers, that depend only on k and d. Similarly, associated to k and d are the positive integers $a_0, a_1, \ldots$ defined to be

$$\begin{aligned} a_j &:= \min\{\dim_{\mathbb{C}} \big(\tfrac{\mathcal{J}\mathrm{ac}(f)}{\mathfrak{J}^{j+1}}\big) \mid f \in \mathbb{C}\langle\langle \mathsf{x} \rangle\rangle_{\geq k}\} \\ &= \min\{\dim_{\mathbb{C}} \big(\tfrac{\mathcal{J}\mathrm{ac}(f)}{\mathfrak{J}^{j+1}}\big) \mid f \in \mathbb{C}\langle \mathsf{x} \rangle_{\geq k}\} && \text{(truncate terms mod } \mathfrak{J}^{j+1}) \\ &= \min\{\dim_{\mathbb{C}} \big(\tfrac{\mathcal{J}\mathrm{ac}(f)}{\mathfrak{J}^{j+1}}\big) \mid f \in \mathbb{C}\langle \mathsf{x} \rangle_{\geq k},\, f_t = 0 \text{ for } t > j+1\} \\ &= \begin{cases} \dim_{\mathbb{C}} \big(\frac{\mathbb{C}\langle\langle \mathsf{x} \rangle\rangle}{\mathfrak{n}^{j+1}}\big) & \text{if } j \leq k-2 \\ \min\{\dim_{\mathbb{C}} \big(\frac{\mathrm{A}\mathcal{J}\mathrm{ac}(F)}{\mathfrak{R}^{j+1}}\big) \mid F \in \mathsf{SP}_{k,j+1}\} & \text{if } j \geq k-1. \end{cases} && \text{(by (A.J))} \end{aligned}$$

Certainly $\hat{\mathcal{P}}_f \geq \sum_{i\geq 0} a_i t^i$, by the minimality of the a_i. We claim that $a_i = b_i$ for all $i \geq 0$, since then

$$\hat{\mathcal{P}}_f \geq \sum_{i\geq 0} a_i t^i = \sum_{i\geq 0} b_i t^i = \frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)},$$

which has the prescribed growth as in the statement of the result.

So, from here on we discard the original f, and instead prove that $a_i = b_i$ for all $i \geq 0$. This is a statement which depends only on k and d.

Since by assumption $(d,k) \neq (2,3)$, by A.8 $\mathsf{ESP}_k \neq \emptyset$ and so choose $G \in \mathsf{ESP}_k$. Since G is exact, $\mathcal{H}_G = (1 - dt + dt^{k-1} - t^k)$, and so by A.16 for $\mathbb{G} = \frac{1}{k}G$ we have

$$\hat{\mathcal{P}}_{\mathbb{G}} = \frac{1}{1-t} \cdot \mathcal{H}_G = \frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)} = \sum_{i \geq 0} b_i t^i.$$

Since $\mathbb{G}$ exists, it follows immediately by minimality of the a_is that $a_i \leq b_i$ for all $i \geq 0$.

Now clearly $a_0 = b_0 = 1$, and further for all $1 \leq j \leq k-2$, we have $a_j = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle}{\mathfrak{n}^{j+1}}\right)$, which equals b_j by A.17(2). Further, since d relations of degree k can cut down the dimension of $\mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle / \mathfrak{J}^{k+1}$ by at most d, it follows that $a_{k-1} \geq 1 + d + \ldots + d^k - d$. This equals b_{k-1} by A.17(2), and so $a_{k-1} \geq b_{k-1}$, which in turn forces $a_{k-1} = b_{k-1}$.

Thus, by induction we can suppose that $a_j = b_j$ for all $0 \leq j \leq s$, for some $s \geq k-1$. The proof will be completed once we can show that $a_{s+1} = b_{s+1}$.

Now by A.15 applied to $m = s+2$, there exists $F \in \mathsf{SP}_{k,s+2}$ for which $\mathcal{P}_F$ is the minimum in $\{\mathcal{P}_H \mid H \in \mathsf{SP}_{k,s+2}\}$, and further $F_k \in \mathcal{U}$ where $\mathcal{U}$ is from A.14. By the first of these facts, since for all $j \leq s+1$ by truncation

$$a_j = \min\{\dim_{\mathbb{C}} \left(\tfrac{\mathrm{A}\mathfrak{J}\mathrm{ac}(\mathsf{H})}{\mathfrak{R}^{j+1}}\right) \mid \mathsf{H} \in \mathsf{SP}_{k,j+1}\} = \min\{\dim_{\mathbb{C}} \left(\tfrac{\mathrm{A}\mathfrak{J}\mathrm{ac}(\mathsf{H})}{\mathfrak{R}^{j+1}}\right) \mid \mathsf{H} \in \mathsf{SP}_{k,s+1}\}$$

it follows that

$$\mathcal{P}_F = \sum_{j=0}^{s+1} a_j t^i + \mathcal{O}_{s+2}. \tag{A.K}$$

On the other hand, since $F_k \in \mathcal{U}$, by definition $F_k \in \mathsf{ESP}_k$. Thus, by A.12 for all $j \geq 1$ the left multiplication by x_1

$$\mathrm{A}\mathfrak{J}\mathrm{ac}(F)/\mathfrak{R}^j \xrightarrow{x_1 \cdot} \mathrm{A}\mathfrak{J}\mathrm{ac}(F)/\mathfrak{R}^{j+1}$$

is injective. Set $I = (\eth_1 F, \ldots, \eth_d F)$, then the above asserts that there is an injection

$$\frac{\mathbb{C}\langle \mathsf{x} \rangle}{I + \mathfrak{m}^j} \xrightarrow{x_1 \cdot} \frac{\mathbb{C}\langle \mathsf{x} \rangle}{I + \mathfrak{m}^{j+1}}$$

for all $j \geq 1$. This allows us to pick inductively, starting with $M_0 = \{1\}$, sets of monomials M_j of $\mathbb{C}\langle \mathsf{x} \rangle$ of degree j, such that the following two conditions are satisfied.

(1) There is an inclusion $x_1 M_j \subseteq M_{j+1}$.
(2) The necessarily disjoint union $N_j = M_0 \cup \ldots \cup M_j$ projects down to give a basis of $\mathbb{C}\langle \mathsf{x} \rangle / (I + \mathfrak{m}^{j+1})$.

To fix notation, set $B_j = \mathrm{Span}_{\mathbb{C}}(N_j) \subset \mathbb{C}\langle \mathsf{x} \rangle$, and define b_j via the equality

$$\dim_{\mathbb{C}} B_j = \dim_{\mathbb{C}} \left(\tfrac{\mathbb{C}\langle \mathsf{x} \rangle}{I + \mathfrak{m}^{j+1}}\right) = \mathsf{b}_j.$$

Note that the second equality implies that $\mathcal{P}_F = \sum_{i \geq 0} \mathsf{b}_j t^j$.

Now, write $V = \mathrm{Span}_{\mathbb{C}}\{x_1, \ldots, x_d\} \subseteq \mathbb{C}\langle \mathsf{x} \rangle$, $R = \mathrm{Span}_{\mathbb{C}}\{\eth_1 F, \ldots, \eth_d F\} \subseteq \mathbb{C}\langle \mathsf{x} \rangle$, and for $j \geq 0$ consider the quotient map $\pi_j \colon \mathbb{C}\langle \mathsf{x} \rangle \to \mathbb{C}\langle \mathsf{x} \rangle / \mathfrak{m}^{j+1}$. By the definition of b_j, for every $j \geq 0$,

$$\dim_{\mathbb{C}} \pi_j(I) = 1 + d + \ldots + d^j - \mathsf{b}_j. \tag{A.L}$$

Now we apply the adapted Vinberg argument a second time: either an element of the two-sided ideal I starts with an x_i or it does not, so we can write

$$I = VI + R\,\mathbb{C}\langle \mathsf{x} \rangle.$$

Applying π_{j+1} then gives an equality

$$\pi_{j+1}(I) = \pi_{j+1}(VI) + \pi_{j+1}(R\,\mathbb{C}\langle \mathsf{x} \rangle).$$

Since B_j descends to span $\mathbb{C}\langle \mathsf{x} \rangle / (I + \mathfrak{m}^{j+1})$, every element of $\mathbb{C}\langle \mathsf{x} \rangle$ may be written as an element in B_j, plus an element in I, plus an element in $\mathfrak{m}^{j+1}$. Projecting down this sum via π_{j+1}, and noting that RI gets absorbed into $\pi_{j+1}(I)$, and elements of R have degree $\geq k$, it follows that mod $\mathfrak{m}^{j+1}$ there is an equality

$$\pi_{j+1}(I) = \pi_{j+1}(VI) + \pi_{j+1}(RB_{j+2-k}).$$

Write $R' = \mathrm{Span}_{\mathbb{C}}\{\partial_2 F, \ldots, \partial_d F\} \subseteq \mathbb{C}\langle\mathsf{x}\rangle$, then using the Euler relation $\sum_{i=0}^{d}[x_i, \partial_i F] = 0$ of A.1 we may get rid of any appearance of $(\partial_1 F)x_1$ at the cost of terms in the other summands. It follows that

$$\pi_{j+1}(I) = \pi_{j+1}(VI) + \pi_{j+1}(R'B_{j+2-k}) + \pi_{j+1}((\partial_1 F)B^+_{j+2-k}),$$

where $B^+_{j+2-k} = \mathrm{Span}_{\mathbb{C}}\{n \in N_{j+2-k} \mid n \neq x_1 m \text{ for any } m\}$.

The proof is completed by estimating the dimension of each of the three individual summands. Applying (A.L) for the first summand,

$$\dim_{\mathbb{C}} \pi_{j+1}(I) \leq d(1 + d + \ldots + d^j - \mathsf{b}_j) + (d-1)\mathsf{b}_{j+2-k} + (\mathsf{b}_{j+2-k} - \mathsf{b}_{j+1-k})$$

since $\dim_{\mathbb{C}} B^+_{j+2-k} \leq \mathsf{b}_{j+2-k} - \mathsf{b}_{j+1-k}$ holds by construction of the N_j in (1). Plugging (A.L) for π_{j+1} into the above displayed equation, and then cancelling, it follows that

$$1 - \mathsf{b}_{j+1} \leq -d\mathsf{b}_j + d\mathsf{b}_{j+2-k} - \mathsf{b}_{j+1-k}.$$

which after re-arranging gives

$$\mathsf{b}_{j+1} \geq \mathsf{b}_j d - \mathsf{b}_{j+2-k} d + \mathsf{b}_{j+1-k} + 1. \tag{A.M}$$

Since $\mathcal{P}_F = \sum_{i\geq 0} \mathsf{b}_j t^j$, by (A.K) we see that $\mathsf{b}_j = a_j$ for $0 \leq j \leq s+1$, and hence

$$\begin{aligned} a_{s+1} &\geq a_s d - a_{s+2-k} d + a_{s+1-k} + 1 && \text{((A.M) for } j = s) \\ &= b_s d - b_{s+2-k} d + b_{s+1-k} + 1 && (a_j = b_j \text{ for } j \leq s \text{ by induction}) \\ &= b_{s+1} && \text{(A.17(1) for } j = s+1) \end{aligned}$$

Since we already know $a_{s+1} \leq b_{s+1}$ by minimality, the above forces $a_{s+1} = b_{s+1}$. Hence by induction $a_j = b_j$ for all $j \geq 0$, and the result follows. □

References

[AM] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.

[AB] K. Ardakov and K. A. Brown, *Ring-theoretic properties of Iwasawa algebras: a survey*, Doc. Math. 2006, Extra Vol., 7–33.

[A1] V. I. Arnold, *Local normal forms of functions*, Invent. Math. **35** (1976), 87–109.

[A2] J. August, *The tilting theory of contraction algebras*, Adv. Math. **374** (2020), 107372, 56 pp.

[BCP] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., **24** (1997), 235–265. http://www.magma.usyd.edu

[BW] G. Brown and M. Wemyss, *Gopakumar–Vafa invariants do not determine flops*, Comm. Math. Phys. **361** (2018), no. 1, 143–154.

[BW2] G. Brown and M. Wemyss, *Noncommutative local normal forms: exceptional cases*, in preparation.

[BIRS] A. B. Buan, O. Iyama, I. Reiten, and D. Smith, *Mutation of cluster-tilting objects and potentials*, Amer. J. Math. **133** (2011), no. 4, 835–887.

[BGS] R.-O. Buchweitz, G.-M. Greuel, and F.-0. Schreyer, *Cohen-Macaulay modules on hypersurface singularities. II.* Invent. Math. **88** (1987), no. 1, 165–182.

[CB] W. Crawley-Boevey, *Noncommutative algebra 1*, Bielefeld, Summer Semester 2019 pdf.

[CBH] W. Crawley-Boevey and M. P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no. 3, 605–635.

[D] B. Davison, *Refined invariants of finite-dimensional Jacobi algebras.* arXiv:1903.00659.

[DGPS] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, SINGULAR 4-1-0 — *A computer algebra system for polynomial computations*, http://www.singular.uni-kl.de (2017).

[DWZ] H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations. I. Mutations.* Selecta Math. (N.S.) **14** (2008), no. 1, 59–119.

[DW1] W. Donovan and M. Wemyss, *Noncommutative deformations and flops*, Duke Math. J. **165** (2016), no. 8, 1397–1474.

[DW2] W. Donovan and M. Wemyss, *Contractions and deformations*, Amer. J. Math. **141** (2019), no. 3, 563–592.

[DW3] W. Donovan and M. Wemyss, *Twists and braids for general 3-fold flops*, J. Eur. Math. Soc. (JEMS), **21** (2019), no. 6, 1641–1701.

[DW4] W. Donovan and M. Wemyss, *Noncommutative enhancements of contractions*, Adv. Math. **344** (2019), 99–136.

[Du] T. Ducat, *Divisorial extractions from singular curves in smooth 3-folds.* Internat. J. Math. **27** (2016), no. 1, 1650005, 23 pp.

[E1] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, **150**. Springer-Verlag, New York, 1995.

[E2] M. Ershov, *Golod–Shafarevich groups: a survey*, Internat. J. Algebra Comput., **22** (2012), no. 5, 1230001, 68 pp.

[GS] E. S. Golod and I. R. Šafarevič, *On the class field tower*, Izv. Akad. Nauk SSSR Ser. Mat., **28**, 261–272, 1964.

[HT] Z. Hua and Y. Toda, *Contraction algebra and invariants of singularities*, Int. Math. Res. Not. IMRN 2018, no. 10, 3173–3198.

[HZ] Z. Hua, and G. Zhou, *Quasi-homogeneity of potentials*, J. Noncommut. Geom. **15** (2021), no. 2, 399–422.

[IW1] O. Iyama and M. Wemyss, *Singular Derived Categories of $\mathbb{Q}$-factorial terminalizations and Maximal Modification Algebras*, Adv. Math. **261** (2014), 85–121.

[IW2] O. Iyama and M. Wemyss, *Reduction of triangulated categories and Maximal Modification Algebras for cA_n singularities*, J. Reine Angew. Math. **738** (2018), 149–202.

[I] N. Iyudu, *Classification of contraction algebras and pre-Lie algebras associated to braces and trusses*, arXiv:2008.06033.

[IS] N Iyudu and S. Shkarin, *Potential algebras with few generators*, arXiv:1806.06829 v1

[ISm] N Iyudu and A. Smoktunowicz, *Golod-Shafarevich-type theorems and potential algebras*, Int. Math. Res. Not. IMRN 2019, no. 15, 4822–4844.

[KM] S. Katz and D. R. Morrison, *Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups*, J. Algebraic Geom. **1** (1992) 449–530.

[K] J. Karmazyn, *The length classification of threefold flops via noncommutative algebras*, Adv. Math. **343** (2019), 393–447.

[Ka] Y. Kawamata, *Non-commutative deformations of perverse coherent sheaves and rational curves*, arXiv:2006.09547.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

[KL] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, revised edition. Graduate Studies in Mathematics, **22**. American Mathematical Society, Providence, RI, 2000. x+212 pp. ISBN: 0-8218-0859-1.

[L] T. Y. Lam, *A first course in noncommutative rings*, Second edition. Graduate Texts in Mathematics, **131**. Springer-Verlag, New York, 2001. xx+385 pp.

[M1] H. Matsumura, *Commutative ring theory*, Second edition. Cambridge Studies in Advanced Mathematics, **8**. Cambridge University Press, Cambridge, 1989. xiv+320 pp.

[M2] A. Mellit, *Kleinian singularities and algebras generated by elements that have given spectra and satisfy a scalar sum relation*, Algebra Discrete Math. 2004, no. 3, 89–110.

[MN] D. Mond, and J. J. Nuño-Ballesteros *Singularities of mappings. The local behaviour of smooth and complex analytic mappings*, Grundlehren der Mathematischen Wissenschaften **357**, Springer, xv, 567 p. (2020).

[M3] S. Mori, *On 3-dimensional terminal singularities*, Nagoya Math. J. **98** (1985), 43–66.

[R1] M. Reid, *Canonical 3-folds*, Journées de Géometrie Algébrique d'Angers, Juillet 1979, pp. 273–310, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[R2] M. Reid, *Minimal models of canonical 3-folds*, Algebraic varieties and analytic varieties (Tokyo, 1981), 131–180, Adv. Stud. Pure Math., 1, North-Holland, Amsterdam, 1983.

[T] Y. Toda, *Non-commutative width and Gopakumar–Vafa invariants*, Manuscripta Math. **148** (2015), no. 3-4, 521–533.

[T2] N. Tziolas, *Terminal 3-fold divisorial contractions of a surface to a curve. I*, Compositio Math. **139** (2003), no. 3, 239–261.

[T3] N. Tziolas, *Three-fold divisorial extremal neighborhoods over cE7 and cE6 compound DuVal singularities*, Internat. J. Math. **21** (2010), no. 1, 1–23.

[U] V. A. Ufnarovskij, *Combinatorial and asymptotic methods in algebra*, In *Algebra VI*, volume **57** of Encyclopaedia Math. Sci., pages 1–196. Springer, Berlin, 1995.

[V1] M. Van den Bergh, *Calabi-Yau algebras and superpotentials*, Selecta Math. (N.S.) **21** (2015), no. 2, 555–603.

[V2] M. Van den Bergh, *Three-dimensional flops and noncommutative rings*, Duke Math. J. **122** (2004), no. 3, 423–455.

[vG] O. van Garderen, *Donaldson–Thomas invariants of length 2 flops*, arXiv:2008.02591.

[vG2] O. van Garderen, *Donaldson–Thomas invariants of threefold flops*, PhD thesis, University of Glasgow, 2021.

[V3] E. B. Vinberg, *On the theorem concerning the infinite-dimensionality of an associative algebra*, Izv. Akad. Nauk SSSR Ser. Mat., **29**. 209–214, 1965.

[Wa] S. Warner, *Topological rings*, North-Holland Mathematics Studies, **178**. North-Holland Publishing Co., Amsterdam, 1993. x+498 pp.

[W] P. M. H. Wilson, *The Kähler cone on Calabi-Yau threefolds*, Invent. Math. **107** (1992), no. 3, 561–583. *Erratum: 'The Kähler cone on Calabi-Yau threefolds' [Invent. Math. 107 (1992), no. 3, 561–583]*, Invent. Math. **114** (1993), no. 1, 231–233.

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