

LOCAL NORMAL FORMS OF NONCOMMUTATIVE FUNCTIONS

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ABSTRACT. This article describes local normal forms of functions in noncommuting variables, up to equivalence generated by isomorphism of noncommutative Jacobi algebras, extending singularity theory in the style of Arnold's commutative local normal forms into the noncommutative realm. This generalisation unveils many new phenomena, including an ADE classification when the Jacobi ring has dimension zero and, by suitably taking limits, a further ADE classification in dimension one. These are natural generalisations of the simple singularities and those with infinite multiplicity in Arnold's classification. We obtain normal forms away from some exceptional Type E cases. Remarkably these normal forms have no moduli, and the key new feature is that the noncommutative world affords larger families.

This theory has a range of immediate consequences to the birational geometry of 3-folds. The normal forms of dimension zero *are* the analytic classification of smooth 3-fold flops, and one outcome of NC singularity theory is the first complete list of all Type D flopping germs, with variants covering Type E . The normal forms of dimension one have further ramifications to divisorial contractions to a curve. In addition, the general techniques also give strong evidence towards new contractibility criteria for rational curves.

1. INTRODUCTION

Any smooth point on a d -dimensional variety is, up to complete local change in coordinates, the power series ring in d variables. Complete locally, all smooth points look the same. A similar phenomenon applies to more interesting singular points: as observed by Arnold and others in various landmark papers [A2, M3, BGS], if we pass to the completion, there are many fewer singular points than might naively be expected, and they often have surprising connections to other parts of mathematics. The purpose of this paper is to extend both the theory and the key classifications into the noncommutative context. In the process, we uncover surprising algebraic facts with geometric corollaries.

1.1. Noncommutative Singularity Theory. For $d \geq 1$ consider the noncommutative formal power series ring $\mathbb{C}\langle\mathbf{x}\rangle = \mathbb{C}\langle x_1, \dots, x_d \rangle$, which is the complete local version of the free algebra. From the perspective of this paper, the algebra $\mathbb{C}\langle\mathbf{x}\rangle$ replaces the commutative power series ring $\mathbb{C}[x_1, \dots, x_d]$ from classical singularity theory.

For any $f \in \mathbb{C}\langle\mathbf{x}\rangle$, it is possible to cyclically differentiate f with respect to a variable x_i to obtain an element $\delta_i f$. The collection of such elements generate a closed two-sided ideal $(\delta_1 f, \dots, \delta_d f)$, the details of which are recalled in §2.2. The resulting quotient

$$\mathcal{J}\text{ac}(f) = \frac{\mathbb{C}\langle x_1, \dots, x_d \rangle}{(\delta_1 f, \dots, \delta_d f)}$$

is called the *Jacobi algebra* of f , and the element f is called the *potential*.

We will regard f and g as being equivalent if their Jacobi algebras are isomorphic, remarking that in the noncommutative setting, given the hidden dependence on cyclic equivalence, naive versions of the Tjurina algebra do not exist (see 4.2). With the ring $\mathbb{C}\langle\mathbf{x}\rangle$ fixed and the equivalence relation established, the overarching aim of singularity theory remains: to classify all equivalence classes of potentials satisfying numerical criteria, and to develop powerful theory in the situation where classification is not possible.

Whenever $d > 1$ the algebra $\mathbb{C}\langle\mathbf{x}\rangle$ is not noetherian, and the exponential explosion in its growth means that factoring by only d elements often results in Jacobi algebras with pathological properties. As in the classical case, pathologies turn out not to matter: the complexity of *some* singularities prevents neither the development of a general theory, nor various classification results for those which satisfy reasonable numerical conditions.

Writing \mathfrak{J} for the Jacobson radical of $\mathcal{J}ac(f)$, the first and natural restriction to impose on f is to numerically constrain the growth of successive quotients of the chain of ideals

$$\mathcal{J}ac(f) \supseteq \mathfrak{J} \supseteq \mathfrak{J}^2 \supseteq \dots$$

This numerical growth, defined in 3.4, is called the \mathfrak{J} -dimension, and will be written $\text{Jdim } \mathcal{J}ac(f)$. As explained in 3.6, since $\mathcal{J}ac(f)$ is a factor of a complete ring, there is no reasonable Gelfand–Kirillov dimension, and so the \mathfrak{J} -dimension replaces it.

Alongside the development of a more general theory, the motivating problem is to extend Arnold-style classification of germs into the above noncommutative setting.

Problem 1.1. For any finite $n \geq 0$, produce a set of potentials \mathcal{S}_n that realise every Jacobi algebra of \mathfrak{J} -dimension n , up to isomorphism.

We furthermore insist that the elements of \mathcal{S}_n should be *normal forms*, namely that if $f, g \in \mathcal{S}_n$ with $f \neq g$, then the resulting Jacobi algebras are not isomorphic. Building on foundational algebraic results of Iyudu–Shkarin [IS2], in Appendix A we show that, for small \mathfrak{J} -dimension n , 1.1 essentially reduces to a problem in $d \leq 2$ variables.

Below we will focus mainly on the situation $n \leq 1$, which is already highly non-trivial. Exactly as in Arnold, such precise numerical restrictions are often only motivated afterwards, by their answer and by the incredibly rich families that they describe. The restriction $n \leq 1$ is also, happily, the condition needed for the applications to birational geometry. We do however remark that it is not even clear that the set \mathcal{S}_0 is countable, never mind \mathcal{S}_1 , and there is certainly no *prima facie* reason why ADE should enter.

1.2. Noncommutative ADE Normal Forms. We now introduce the ADE families that will turn out to solve 1.1 when $n \leq 1$. The main results regarding what precisely these families classify are stated later, in §1.3.

As remarked by Arnold, it is often not possible to rigorously define a series until *after* it has been classified. In the subsections that follow we will use various different phenomena to explain the ADE names of the families, but it is only after classification that one can make the moves needed to extract this ADE information. As such, the definition of the families below follows the usual pattern of classical singularity theory: their definition comes first, and their justification comes afterwards.

Below, we view the families with $n = 0$ as the noncommutative version of simple singularities à la Arnold, whereas we view the $n = 1$ families as the ‘limit’ of the $n = 0$ case, and thus the noncommutative version of the tame singularities A_∞ and D_∞ of [BGS].

With the above caveats, for any $d \geq 2$ consider the following families, which below will fully classify the $n = 0$ case in 1.1. The big \mathcal{O} notation is explained in §1.8.

Type	Name	Normal form	Conditions
A	A_n	$z_1^2 + \dots + z_{d-2}^2 + x^2 + y^n$	$n \geq 2$
D	$D_{n,m}$	$z_1^2 + \dots + z_{d-2}^2 + xy^2 + x^{2n} + x^{2m-1}$	$n, m \geq 2, m \leq 2n - 1$
	$D_{n,\infty}$	$z_1^2 + \dots + z_{d-2}^2 + xy^2 + x^{2n}$	$n \geq 2$
E	$E_{6,n}$	$z_1^2 + \dots + z_{d-2}^2 + x^3 + xy^3 + y^n$	$n \geq 4$
		$z_1^2 + \dots + z_{d-2}^2 + x^3 + \mathcal{O}_4$	(various cases)

TABLE 1. \mathfrak{J} -dimension 0 normal forms

It is possible to write Type D in the unified manner $z_1^2 + \dots + z_{d-2}^2 + xy^2 + x^{2n} + \varepsilon x^{2m-1}$ where ε is either 0 or 1, but often it will be preferable to regard them as two distinct families, both of Type D . In addition to the fact that Type D is larger than in the classical case, what is perhaps much more remarkable is that in Type E there are infinitely many cases: the family $E_{6,n}$ stated, together with various other examples all of the form $x^3 + \mathcal{O}_4$, whose expressions are more complicated, and will be optimised elsewhere [BW4].

Taking the limit $n \rightarrow \infty$ of the above forms gives the following, where again all are optimised, except the very last line.

Type	Name	Normal form	Conditions
A	A_∞	$z_1^2 + \dots + z_{d-2}^2 + x^2$	
D	$D_{\infty,m}$	$z_1^2 + \dots + z_{d-2}^2 + xy^2 + x^{2m-1}$	$m \geq 2$
	$D_{\infty,\infty}$	$z_1^2 + \dots + z_{d-2}^2 + xy^2$	
E	$E_{6,\infty}$	$z_1^2 + \dots + z_{d-2}^2 + x^3 + xy^3$	
		$z_1^2 + \dots + z_{d-2}^2 + x^3 + \mathcal{O}_4$	

TABLE 2. \mathfrak{J} -dimension 1 normal forms

The classical case admits precisely two examples, namely the singularities A_∞ and D_∞ of [BGS]. The above noncommutative families are thus again larger: Type D splits into two and there are infinitely many examples within $D_{\infty,m}$, and Type E is non-empty.

With the benefit of hindsight, there are two reasons why to expect that the $n = 1$ case should be the limits of $n = 0$. First, as Arnold remarks, his simple A_n and D_n families give rise, under limits, to the germs x^2 and xy^2 , and the noncommutative families above generalise this passage from the isolated to the non-isolated. Second, in terms of the birational geometry of §1.7 below, contraction algebras should make sense of the feeling that divisor-to-curve contractions are limits of infinite families of flops.

In this paper, we prove that every $\mathcal{J}\text{ac}(f)$ with $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$ is isomorphic to a normal form in Type A or D above, or has the general form stated for E . We remark that the precise Type E normal forms stated, namely $E_{6,n}$ and $E_{6,\infty}$, are indeed genuine examples with \mathfrak{J} -dimension zero and one respectively. However, we refrain from describing the general case here, as we will treat all the exceptional Type E cases together, in a more technical companion paper [BW4].

We now outline our results in more detail, before describing their applications.

1.3. Main Noncommutative Singularity Theory Results. Since constants differentiate to zero, and elements with linear terms differentiate to units, we can and do assume that f has only quadratic and higher terms, which we write as $f \in \mathbb{C}\langle\langle x \rangle\rangle_{\geq 2}$ or equivalently as an explicit sum of its homogeneous pieces

$$f = f_2 + f_3 + f_4 + \dots$$

Just as in the classical theory, a Splitting Lemma 4.5 identifies a coordinate system which separates variables of the non-degenerate quadratic part from variables of a higher order potential, so that without loss of generality

$$f = x_1^2 + \dots + x_r^2 + f_{\geq 3}(x_{r+1}, \dots, x_d)$$

and thus we may turn attention to the potential $f_{\geq 3}$ in, typically, fewer variables. The number $d - r$ is called the *corank*, and as in the classical case there is a more intrinsic way of characterising it (4.3), namely as

$$\text{Crk}(f) = d - \dim_{\mathbb{C}} \left(\frac{\mathfrak{n}^2 + I}{\mathfrak{n}^2} \right) \quad (1.A)$$

where $\mathfrak{n} = (x_1, \dots, x_d)$ and $I = (\delta_1 f, \dots, \delta_d f)$. By the above and A.13 it turns out, in a manner pleasantly reminiscent of classical simple singularities, that the case when $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$ reduces to that of two variables. We rename the variables $z_1, \dots, z_{d-2}, x, y$ to emphasise this fact.

The following, a consequence of the Splitting Lemma together with a degree three preparation result, then characterises commutative Jacobi algebras in two variables. These are precisely our Type A families in §1.2. Below we adopt the convenient abuse of notation $f \cong g$ to mean $\mathcal{J}\text{ac}(f) \cong \mathcal{J}\text{ac}(g)$.

Proposition 1.2 (5.1, 5.4). *If $f \in \mathbb{C}\langle\langle x \rangle\rangle_{\geq 2}$, then the following hold.*

(1) $\text{Crk}(f) \leq 1$ if and only if

$$f \cong \begin{cases} z_1^2 + \dots + z_{d-2}^2 + x^2 \\ z_1^2 + \dots + z_{d-2}^2 + x^2 + y^n \end{cases} \text{ for some } n \geq 2.$$

Each member of the bottom family has finite dimensional Jacobi algebra, whereas in the top case the algebra is infinite dimensional, with $\text{Jdim } \mathcal{J}\text{ac}(f) = 1$.

(2) If $d = 2$, i.e. $f \in \mathbb{C}\langle\langle x, y \rangle\rangle$, then $\mathcal{J}\text{ac}(f)$ is commutative if and only if $\text{Crk}(f) \leq 1$.

Thus most Jacobi algebras are strictly not commutative and so new noncommutative invariants are needed to classify them. The equation (1.A) does admit an obvious generalisation, namely the *higher coranks* defined in §4.1, where for $f \in \mathbb{C}\langle\langle x \rangle\rangle_{\geq 3}$ the second corank is

$$\text{Crk}_2(f) = d^2 - \dim_{\mathbb{C}} \left(\frac{\mathbf{n}^3 + I}{\mathbf{n}^3} \right). \quad (1.B)$$

In classifying all f with $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$, A.13 together with 1.2 then reduces us to the case where $\text{Crk}(f) = 2$ and $\text{Crk}_2(f) = 2, 3$. The lowest case $\text{Crk}_2(f) = 2$ turns out to be given by the Type D families in the tables of §1.2.

Theorem 1.3 (6.18). *Suppose that $f \in \mathbb{C}\langle\langle x \rangle\rangle_{\geq 2}$ with $\text{Crk}(f) = 2$ and $\text{Crk}_2(f) = 2$.*

(1) *Then either*

$$f \cong \begin{cases} z_1^2 + \dots + z_{d-2}^2 + xy^2 & D_{\infty, \infty} \\ z_1^2 + \dots + z_{d-2}^2 + xy^2 + x^{2m+1} & \text{with } m \geq 1 \quad D_{\infty, m} \\ z_1^2 + \dots + z_{d-2}^2 + xy^2 + x^{2n} & \text{with } n \geq 2 \quad D_{n, \infty} \\ z_1^2 + \dots + z_{d-2}^2 + xy^2 + x^{2n} + x^{2m+1} & \text{with } 2n - 2 \geq m \geq n \geq 2 \quad D_{n, m} \\ z_1^2 + \dots + z_{d-2}^2 + xy^2 + x^{2m+1} + x^{2n} & \text{with } n > m \geq 1 \quad D_{n, m} \end{cases}$$

These f all have mutually non-isomorphic Jacobi algebras.

(2) *Furthermore, those labelled $D_{\infty, *}$ satisfy $\text{Jdim } \mathcal{J}\text{ac}(f) = 1$, whilst those labelled $D_{n, *}$ satisfy $\text{Jdim } \mathcal{J}\text{ac}(f) = 0$.*

It is remarkable that all normal forms are polynomial, and even more remarkable that all coefficients are integers. Indeed, all coefficients equal 1, and there are no moduli.

The last remaining case for which $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$ holds is when $\text{Crk}(f) = 2$ and $\text{Crk}_2(f) = 3$. After a suitable change in coordinates, all such f have the form

$$f \cong z_1^2 + \dots + z_{d-2}^2 + x^3 + f_{\geq 4}(x, y).$$

with some extra conditions on $f_{\geq 4}(x, y)$ that ensures $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$. We refer to these potentials as Type E . The families described in both Types $E_{6, n}$ and $E_{6, \infty}$ in §1.2 are genuine examples. However there are many others; see [BW4]. Their classification depends, in a rather more subtle manner, on naturally-defined higher coranks (see 4.9). For example, the potential $x^3 + xy^3$ of Type $E_{6, \infty}$ has second corank equal to 3, with all higher coranks equal to 4, while in contrast the potentials f_n of Type $E_{6, n}$ for $n \geq 5$ trim those coranks to

$$\text{Crk}_2(f_n), \text{Crk}_3(f_n), \dots, \text{Crk}_{n+6}(f_n) = 3, 4, 4, \dots, 4, 4, 3, 3, 2, 1, 1.$$

In particular $\mathcal{J}\text{ac}(f_n)$ has dimension $4(n+3)$. Controlling normal forms in such situations is both theoretically and computationally more difficult.

1.4. Intrinsically Extracting ADE. It turns out that there are two, completely distinct, ways to extract ADE behaviour from the families defined above, and thus explain the ADE naming conventions. In this section we explain the purely algebraic method; the birational geometry method is explained in §1.5 below.

The first method is the most surprising. Consider the six algebras defined explicitly by

$$\frac{\mathbb{C}\langle x, y, z \rangle}{x + y + z, \quad x, y, z} \quad \frac{\mathbb{C}\langle x, y, z \rangle}{x + y + z, \quad x^2, y^2, z^2} \quad \frac{\mathbb{C}\langle x, y, z \rangle}{x + y + z, \quad x^2, y^3, z^3} \quad \frac{\mathbb{C}\langle x, y, z \rangle}{x + y + z, \quad x^2, y^3, z^4} \quad \frac{\mathbb{C}\langle x, y, z \rangle}{x^2 + y + z, \quad \text{(see 7.1)}} \quad \frac{\mathbb{C}\langle x, y, z \rangle}{x + y + z, \quad x^2, y^3, z^5} \quad (1.C)$$

These have dimension 1, 4, 12, 24, 40, and 60 respectively. All the presentations in (1.C) are evidently related to ADE triples, where the fifth is slightly more subtle. A presentation-free description of all six algebras, which is conceptually more compelling, uniformly describes each in terms of the preprojective algebra of ADE Dynkin diagrams (see §7.1).

The following result allows us to associate ADE information directly to the normal forms in §1.2, by asserting that *all* such Jacobi algebras generically slice to one of the six algebras in (1.C) above. This is particularly striking, since nothing in the definition

of the families has involved any mention of only six algebras, nor any mention of the preprojective algebra, and aside from our naming conventions, any mention of ADE. It is not even clear that if $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$, then $\mathcal{J}\text{ac}(f)$ admits a non-unit central element.

In order to consider all cases $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$ together, below we adopt the convention that each ε_i can be either 0 or 1.

Theorem 1.4 (7.7). *Consider the normal forms $A_n, D_{n,m}, D_{n,\infty}, E_{6,n}, A_\infty, D_{\infty,m}, D_{\infty,\infty}$ and $E_{6,\infty}$ from §1.2. In each case, define an element s as follows*

Type	Normal form	Conditions	s
A	$z_1^2 + \dots + z_{d-2}^2 + x^2 + \varepsilon_1 y^n$	$n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$	y
D	$z_1^2 + \dots + z_{d-2}^2 + xy^2 + \varepsilon_2 x^{2n} + \varepsilon_3 x^{2m-1}$	$m, n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$	x^2
E	$z_1^2 + \dots + z_{d-2}^2 + x^3 + xy^3 + \varepsilon_4 y^n$	$n \in \mathbb{N}_{\geq 4}$	$g_{6,n}$

where $g_{6,n}$ is defined in §7. Then the following statements hold.

- (1) The element s is central in $\mathcal{J}\text{ac}(f)$, and furthermore $\mathcal{J}\text{ac}(f)/(s)$ is isomorphic to one of the six algebras in (1.C).
- (2) More specifically, $\mathcal{J}\text{ac}(f)/(s)$ is isomorphic to the first algebra in (1.C) when f is in the family A_* , the second algebra in (1.C) when f is in the family $D_{*,*}$, and the third algebra in (1.C) when f is in the family $E_{6,*}$.
- (3) For any generic central element g , the quotient $\mathcal{J}\text{ac}(f)/(g)$ is isomorphic to the first algebra in (1.C) when f is in the family A_* , and the second algebra in (1.C) when f is in the family $D_{*,*}$.

Most of the content in the theorem lies within the third part since generic elements, defined in 7.4, provide an *intrinsic* method of extracting the ADE information. The choice of central element $g_{6,n}$, which is rather involved, works for Type $E_{6,*}$, and there is also strong evidence that generic elements there also quotient to give the correct algebra in the sequence (1.C). Establishing this is computationally much harder, and will be addressed elsewhere [BW4]. We remark that all other examples we know within Type E , but which are not explicitly stated above, also factor to one of the six algebras in (1.C).

In the geometric context of §1.7 below, the generic central element g of 1.4 should be thought of as the noncommutative version of Reid's general elephant [R2]. Remarkably, the above theorem neither implies, nor is implied by, Reid's version. In both cases, taking a generic central element is essential to avoid pathological behaviour (see 7.8).

1.5. The Classification of Flops. The noncommutative singularity results in §1.3 have immediate applications in birational geometry. The slogan is simple: whilst Arnold's commutative normal forms classify Du Val singularities, noncommutative normal forms classify compound Du Val (cDV) singularities.

The key and most remarkable special case is that the normal forms in Table 1 classify smooth 3-fold flops [DW1, A3, JKM]. We first very briefly recall the geometric setting, where general background is left to e.g. [KMM], before outlining the new results.

Given any crepant projective birational morphism $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$, where \mathcal{R} is complete local cDV singularity, there is an associated *contraction algebra* A_{con} formed by considering noncommutative deformations of the curves above the unique closed point [DW1, DW3]. This is the finest known curve invariant associated to the contraction. When the contraction is furthermore *simple*, namely the reduced fibre above the origin is \mathbb{P}^1 , and further \mathcal{X} is smooth, then it is well known [DW1, V1] that $A_{\text{con}} \cong \mathcal{J}\text{ac}(f)$ for some $f \in \mathbb{C}\langle\langle x, y \rangle\rangle$ (see e.g. [BW1, 3.1(2)]).

Since cDV singularities are normal, necessarily $\text{Jdim } A_{\text{con}} \leq 1$, and there is a natural geometric dichotomy. Indeed, as explained in 8.5, if A_{con} is a contraction algebra associated to a crepant $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ as above, then

- (1) $\text{Jdim } A_{\text{con}} = 0$ if and only if $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ is a flop, and
- (2) $\text{Jdim } A_{\text{con}} = 1$ if and only if $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ is a divisorial contraction to a curve.

The only other fact we will require is that every $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ has an associated ADE type, since by Reid's general elephant theorem [R2] a generic $g \in \mathfrak{m}$ slices to give an ADE surface singularity \mathcal{R}/g . We will say \mathcal{R} has Type D if the generic slice is Type D , etc.

With this in mind, the results in §1.3 have immediate consequences. After first using the normal forms to classify contraction algebras, the following then gives the analytic classification of length two flops and beyond. It also gives the second, geometric, method to extract ADE information from the normal forms in §1.2.

Theorem 1.5 (8.2, 8.9, 8.17, 8.10). *With notation as above, the following hold:*

- (1) *The only contraction algebras for Type A and D flops are, up to isomorphism, the Jacobi algebras of Type A and D potentials in Table 1.*
- (2) *All Jacobi algebras in 1.2 and 1.3 are contraction algebras.*
- (3) (a) *Type A flops are classified by Type A normal forms in Table 1.*
 (b) *Type D flops are classified by Type D normal forms in Table 1.*

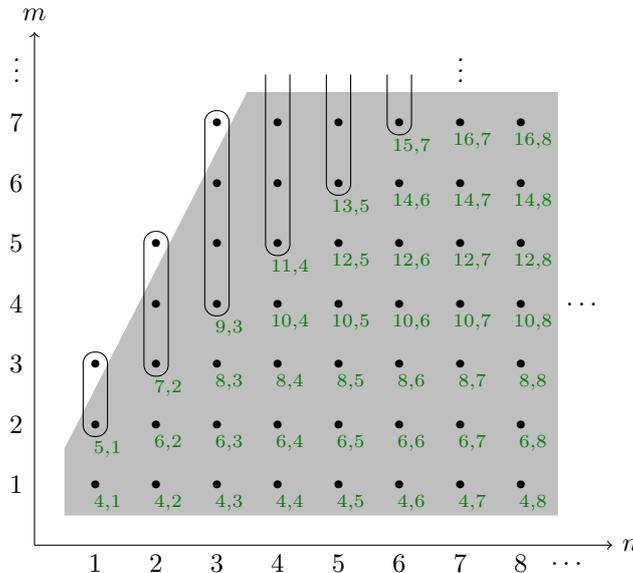
Furthermore, Type E flops are classified by Type E normal forms.

In the process of establishing 1.5 we use the examples of flops given in our previous work [BW1], together with their generalisations [vG, Ka]. Whilst the above classifies flops using noncommutative data, the following geometric description is perhaps more desirable.

Theorem 1.6 (8.11). *There is a one-to-one correspondence between lattice points in the following diagram and the base singularities $0 \in \text{Spec } \mathcal{R}$ of Type D flops, given by*

$$(n, m) \mapsto \text{Spec} \left(\frac{\mathbb{C}[[u, v, x, y]]}{u^2 + v^2y - x(y^{2n+1} + (x + \varepsilon y^m)^2)} \right)$$

where $\varepsilon = 1$ if the lattice point is contained within the shaded region, and $\varepsilon = 0$ otherwise.



In particular, Type D flops do not admit moduli. Furthermore, the following hold.

- (1) *The quasi-homogeneous Type D flops are precisely those outside the shaded region, and these are the standard Laufer family.*
- (2) *The GV invariants n_1, n_2 of the flopping contraction associated to a point (n, m) are written in green. The ovals group together flops with the same GV invariants.*

It is possible to instead index the GV invariants to the classifying potentials, which we do in Figure 3 on page 38. Either way, the important point is that not all pairs of GV invariants n_1, n_2 can be realised.

Corollary 1.7 (8.13). *There are no simple flopping contractions with GV invariants $5, n$ with $n \geq 2$. Similarly for $2m + 1, n$ with $m \geq 2$ and $n \neq m - 1$.*

More generally, the $E_{6,n}$ normal forms in Table 1 predict the first ever infinite family of E_6 flops, and indeed this family turns out to exist. Furthermore, the various other

type E normal forms not stated precisely in Table 1 both predict and classify E_7 and E_8 flops. Details will appear elsewhere [BW4], with the point being that noncommutative singularity theory predicts that there are in fact very few higher length flops, and that their GV invariants are extremely constrained.

1.6. Other cDV Applications. Our results also have applications to 3-fold divisorial contractions to a curve. Whilst there is an extensive literature [T2, T3, Du] on extremal (K -negative) divisorial contractions in the presence of terminal singularities, the K -trivial case considered here is much less studied, aside from the notable [W].

In contrast to the previous section §1.5, the analogue of the Donovan–Wemyss [DW1] conjecture for divisorial contractions to a curve remains open. A positive solution to this conjecture would immediately imply that the normal forms in Table 2 give the full classification of 3-fold divisorial contractions to a curve, exactly as in §1.5 where the normal forms in Table 1 classified flops. For now, whilst the generalised conjecture remains open, our noncommutative normal forms still have many consequences, and give unexpected predictions.

First of all, our control of normal forms allows us to constrain possible deformations of the fibres, by classifying the contraction algebras that can arise.

Proposition 1.8 (8.21, 8.22). *The only contraction algebras for Type A and D_4 smooth divisor-to-curve contractions are, up to isomorphism, the Jacobi algebras of the Type A and D potentials in Table 2.*

The normal forms in Table 2, together with 1.8 and the generalised Donovan–Wemyss conjecture, then predict the first and only infinite family of Type D divisorial contractions to a curve. None of these have been seen before; the following realises the whole family.

Proposition 1.9 (8.16). *Consider the element of $\mathbb{C}\llbracket X, Y, Z, T \rrbracket$ defined by*

$$F_m := \begin{cases} Y(X^m + Y)^2 + XZ^2 - T^2 & \text{if } m \geq 1 \\ Y^3 + XZ^2 - T^2 & \text{if } m = \infty \end{cases}$$

and set $\mathcal{R}_m = \mathbb{C}\llbracket X, Y, Z, T \rrbracket / F_m$. Then the following statements hold.

- (1) $\text{Sing}(\mathcal{R}_m)^{\text{red}} = (X^m + Y, Z, T)$ if $m \geq 1$, and (Y, Z, T) if $m = \infty$.
- (2) In either case, blowing up this locus gives rise to a crepant Type D divisorial contraction to a curve $\mathcal{X}_m \rightarrow \text{Spec } \mathcal{R}_m$ where \mathcal{X}_m is smooth.
- (3) The contraction algebra of $\mathcal{X}_m \rightarrow \text{Spec } \mathcal{R}_m$ is isomorphic to $\mathcal{J}\text{ac}(xy^2 + x^{2m+1})$ when $m \geq 1$, respectively $\mathcal{J}\text{ac}(xy^2)$ when $m = \infty$.

Thus in all cases the noncommutative forms $D_{\infty, *}$ are geometrically realised by F_* . The case $m = \infty$ appeared in [DW4, 2.18], the infinite family is new.

More generally, the rather lonely $E_{6, \infty}$ normal form in Table 2 predicts a divisorial contraction to a curve of Type E_6 . This also turns out to exist, and details will again appear elsewhere [BW4]. In fact, all the evidence now strongly suggests that $E_{6, \infty}$ is the final potential satisfying $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$, which gives the striking geometric prediction that divisorial contractions to a curve of Type E_7 and E_8 do not exist.

1.7. Contractibility and Realisation. It was conjectured in [Ka2, BW2] that in addition to being the classifying structure of contractible curves, noncommutative deformation theory (implicit in the above) also detects which curves can be contracted. Specifically, the conjecture asserts that a collection of crepant rational curves contracts to a point suitably locally, without contracting a divisor, if and only if its associated noncommutative deformation algebra is finite dimensional. This should be viewed as a wide-ranging generalisation of celebrated work of Artin [A1] (for surfaces) and Jiménez [J].

The key test case is when the curve is irreducible. One consequence of Appendix A is that the conjecture is very reasonable: the only open case is now that of $(-3, 1)$ curves.

Theorem 1.10 (A.14). *Let $C \subset \mathcal{X}$ be an irreducible rational curve in a smooth CY 3-fold, with NC deformation algebra Λ_{def} , such that $\mathcal{N}_{C|\mathcal{X}} \neq (-3, 1)$. Then $C \subset \mathcal{X}$ contracts to a point suitably locally, without contracting a divisor, if and only if $\dim_{\mathbb{C}} \Lambda_{\text{def}} < \infty$.*

Another consequence of this paper is a prediction regarding realising Jacobi algebras from geometry. We will call $f \in \mathbb{C}\langle\mathbf{x}\rangle$ *geometric* if it arises from geometry, that is, $\mathcal{J}ac(f)$ isomorphic to the contraction algebra of some $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ described in §1.5. Based partly on the results in this paper, and partly on extensive computer algebra searches using the software [BCP, DGPS], we conjecture the following.

Conjecture 1.11 (The Realisation Conjecture). Every $f \in \mathbb{C}\langle\mathbf{x}\rangle$ whose Jacobi algebra satisfies $\text{Jdim } \mathcal{J}ac(f) \leq 1$ is geometric.

The conjecture being true would imply that every finite dimensional $\mathcal{J}ac(f)$ is symmetric [A3, 2.6], that is $\text{Hom}_{\mathbb{C}}(\mathcal{J}ac(f), \mathbb{C}) \cong \mathcal{J}ac(f)$ as bimodules, a property which itself is far from clear. In 2014 our original expectation was that contraction algebras are a strict subset of Jacobi algebras and the task was to recognise them, but since then all computer searches and all papers (e.g. [D]) which have tried to disprove the conjecture have inadvertently ended up giving more evidence for it. This paper is no different.

Corollary 1.12 (8.18). *Conjecture 1.11 is true, except possibly for the one remaining unresolved case when $f \cong x^3 + \mathcal{O}_4$, where some further analysis is required.*

In the remaining cases, it does now seem likely that all potentials $f \cong x^3 + \mathcal{O}_4$ for which $\text{Jdim } \mathcal{J}ac(f) \leq 1$ are isomorphic to contraction algebras of cE_n singularities.

1.8. Notation and Conventions. Throughout we work over the complex numbers \mathbb{C} , which is necessary for various statements to hold, although any algebraically closed field of characteristic zero would suffice. In addition, we adopt the following notation.

- (1) Throughout $d \geq 1$ is fixed to be the number of variables. Set $\mathbf{x} = x_1, \dots, x_d$, and $\mathbb{C}\langle\mathbf{x}\rangle = \mathbb{C}\langle x_1, \dots, x_d \rangle$.
- (2) Vector space dimension will be written $\dim_{\mathbb{C}} V$.
- (3) $\mathbb{C}\langle\mathbf{x}\rangle_i$ or $\mathbb{C}\langle\mathbf{x}\rangle_i$ will denote the vector subspace of $\mathbb{C}\langle\mathbf{x}\rangle$ consisting of homogeneous degree i polynomials. For a formal power series $g \in \mathbb{C}\langle\mathbf{x}\rangle$ we denote the graded (necessarily polynomial) piece of degree i of g by $g_i \in \mathbb{C}\langle\mathbf{x}\rangle_i$.
- (4) Write $g_{<d} = \sum_{i<d} g_i$ and $g_{>d} = \sum_{i>d} g_i$, with natural self-documenting variations such as $g_{\geq d}$. Thus, for example, $g = g_3 + g_4 + g_{\geq 5}$ is a power series with no terms in degrees 0, 1 and 2, and no further conditions.
- (5) Given $g, h \in \mathbb{C}\langle\mathbf{x}\rangle$, write $g = h + \mathcal{O}_d$ as a shorthand for $g_{<d} = h_{<d}$.
- (6) The previous conventions on degree introduce one typographical difficulty, namely the compatibility with sequences. We will frequently work with sequences $(f_n)_{n \geq 1}$ of power series $f_n \in \mathbb{C}\langle\mathbf{x}\rangle$, and analogously we write $(f_n)_d$, $(f_n)_{<d} \in \mathbb{C}\langle\mathbf{x}\rangle$ and $(f_n)_{>d} \in \mathbb{C}\langle\mathbf{x}\rangle$ for its pieces in the indicated degrees. To scrupulously avoid confusion, we will systematically use Greek font f_n to denote the n th power series in a sequence, and not the n th degree graded piece of a single power series.
- (7) The notation § $x.y$ refers to Subsection $x.y$, $(n.m)$ refers to displayed equation $(n.m)$, and $n.m$ refers to statement $n.m$, where the type of statement – Definition, Theorem, and so on – is usually left unspecified.

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2. FORMAL AUTOMORPHISMS

This section reviews properties of the noncommutative formal power series $\mathbb{C}\langle\mathbf{x}\rangle$, and also constructions of various automorphisms of $\mathbb{C}\langle\mathbf{x}\rangle$, mainly following [DWZ, §2]. From the viewpoint of noncommutative singularity theory, it is the construction in §2.6 leading to 3.7(3) that will be used heavily in later sections.

2.1. Polynomial and Power Series Notation. As in the introduction, write $\mathbb{C}\langle\mathbf{x}\rangle$ for formal noncommutative power series in d variables, and further write $\mathbb{C}\langle\mathbf{x}\rangle = \mathbb{C}\langle x_1, \dots, x_d \rangle$ for the free algebra in d variables. For either $f \in \mathbb{C}\langle\mathbf{x}\rangle$ or $\mathbb{C}\langle\mathbf{x}\rangle$ write f in terms of its homogeneous pieces as

$$f = f_0 + f_1 + f_2 + f_3 + f_4 + \dots,$$

and define the *order* of f to be $\text{ord}(f) = \min\{i \mid f_i \neq 0\}$, where by convention $\text{ord}(0) = \infty$. For any $t \geq 0$ set $\mathbb{C}\langle\mathbf{x}\rangle_{\geq t} = \{f \in \mathbb{C}\langle\mathbf{x}\rangle \mid f_i = 0 \text{ if } i < t\} = \{f \in \mathbb{C}\langle\mathbf{x}\rangle \mid \text{ord}(f) \geq t\}$, and note that this contains the zero element.

2.2. Complete Completions. To fix notation, let $\mathfrak{m} = (x_1, \dots, x_d)$ denote the two-sided maximal ideal of the free algebra $\mathbb{C}\langle\mathbf{x}\rangle$. The \mathfrak{m} -adic completion of $\mathbb{C}\langle\mathbf{x}\rangle$ is

$$\varprojlim \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^n$$

which is the set of sequences $(a_n)_{n \geq 1}$ of $a_n \in \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^n$ that satisfy $a_{n+1} + \mathfrak{m}^n = a_n + \mathfrak{m}^n$ for all n , sometimes called *coherent sequences*.

On the other hand, consider the formal power series ring $\mathbb{C}\langle\mathbf{x}\rangle$ in noncommutative variables x_1, \dots, x_d , with two-sided maximal ideal \mathfrak{n} containing those power series with zero constant term. There is an isomorphism

$$\mathbb{C}\langle\mathbf{x}\rangle \cong \varprojlim \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^n$$

which sends a formal power series f to the coherent sequence $(f_{<n} + \mathfrak{m}^n)_{n \geq 1}$. Below we will freely make this identification, and further that the following diagrams for all $i \geq j$ form an inverse limit system

$$\begin{array}{ccc} & \mathbb{C}\langle\mathbf{x}\rangle & \\ \pi_i \swarrow & & \searrow \pi_j \\ \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^j \end{array}$$

where the map π_i sends $f \mapsto f_{<i} + \mathfrak{m}^i$, and the horizontal map is the natural one.

Given a sequence $(f_i)_{i \geq 1}$ of elements of $\mathbb{C}\langle\mathbf{x}\rangle$, and $f \in \mathbb{C}\langle\mathbf{x}\rangle$, recall the following:

- (f_i) *converges to* f if $\forall n \geq 1, \exists N$ such that $f_i - f \in \mathfrak{n}^n$ for all $i \geq N$.
- (f_i) is *Cauchy* if $\forall n \geq 1, \exists N$ such that $f_i - f_j \in \mathfrak{n}^n$ for all $i, j \geq N$.

Taking completions of non-noetherian rings in general can be subtle. However, in the situation here, since for all i ,

$$\mathfrak{n}^i = \text{Ker}(\pi_i) = \{f \in \mathbb{C}\langle\mathbf{x}\rangle \mid f_0 = \dots = f_{i-1} = 0\},$$

it is clear that $\mathbb{C}\langle\mathbf{x}\rangle$ is complete with respect to its \mathfrak{n} -adic topology. That is, every Cauchy sequence in $\mathbb{C}\langle\mathbf{x}\rangle$ converges.

The algebra $\mathbb{C}\langle\mathbf{x}\rangle$ is a *topological algebra* with basis of the topology given by the ideals $\{\mathfrak{n}^i\}$, where \mathfrak{n}^i is both open and closed. The free algebra $\mathbb{C}\langle\mathbf{x}\rangle$ embeds as a dense subalgebra of $\mathbb{C}\langle\mathbf{x}\rangle$, and the ideal \mathfrak{n}^n is the closure of \mathfrak{m}^n , or equivalently, \mathfrak{n}^n is the smallest closed ideal that contains all monomials $x_1^{a_1} \dots x_d^{a_d}$ of degree $\sum_{k=1}^d a_k = n$.

2.3. Formal Automorphisms. As input, consider a sequence of algebra isomorphisms $(\phi_i: \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i \rightarrow \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i)_{i \geq 1}$ for which

$$\begin{array}{ccc} \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^j \\ \phi_i \downarrow & & \downarrow \phi_j \\ \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^j \end{array} \quad (2.A)$$

commutes for all $i \geq j$. Then the universal property for the \mathfrak{m} -adic completion lifts these to an algebra automorphism $\phi: \mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}\langle\mathbf{x}\rangle$ such that the following diagram commutes:

$$\begin{array}{ccc}
& \mathbb{C}\langle\mathbf{x}\rangle & \\
\pi_i \swarrow & \downarrow & \searrow \pi_j \\
\mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i & \xrightarrow{\phi} & \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^j \\
\downarrow \phi_i & & \downarrow \phi_j \\
& \mathbb{C}\langle\mathbf{x}\rangle & \\
\pi_i \swarrow & \downarrow & \searrow \pi_j \\
\mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^j
\end{array} \tag{2.B}$$

The following special case will be important later. For any fixed $f_1, \dots, f_d \in \mathfrak{n}^2 \subset \mathbb{C}\langle\mathbf{x}\rangle$, consider the algebra homomorphisms

$$\varphi_i: \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i \rightarrow \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i$$

defined by sending $x_k + \mathfrak{m}^i \mapsto x_k + (f_k)_{<i} + \mathfrak{m}^i$ for each $1 \leq k \leq d$. On the truncated finite dimensional algebras $\mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i$, clearly each φ_i is an algebra isomorphism, and further since the truncation of a truncation is itself a truncation, (2.A) applied to the φ_i commutes. As a consequence, (2.B) induces an automorphism $\phi: \mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}\langle\mathbf{x}\rangle$.

Definition 2.1. Given $f_1, \dots, f_d \in \mathfrak{n}^2$, the above $\phi: \mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}\langle\mathbf{x}\rangle$ is called a *unitriangular automorphism*. We will abuse notation slightly and write

$$\begin{array}{l}
\mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}\langle\mathbf{x}\rangle \\
x_k \mapsto x_k + f_k
\end{array}$$

for ϕ , since indeed ϕ is induced by such morphisms on the truncations $\mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i$. For $e \geq 1$ we say that ϕ has *depth* e provided that $f_1, \dots, f_d \in \mathfrak{n}^{e+1}$.

Lemma 2.2. *With notation as above, the following statements hold.*

- (1) A \mathbb{C} -algebra homomorphism $\phi: \mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}\langle\mathbf{x}\rangle$ is a unitriangular automorphism of depth $e \geq 1$ if and only if $\phi(f)_{\leq e} = f_{\leq e}$ for every $f \in \mathbb{C}\langle\mathbf{x}\rangle$.
- (2) If ϕ and ψ are unitriangular automorphisms of $\mathbb{C}\langle\mathbf{x}\rangle$ of depth $e_1 \geq 1$ and $e_2 \geq 1$ respectively, then their composition $\psi \circ \phi$ is a unitriangular automorphism, of depth $\min\{e_1, e_2\}$.

Remark 2.3. Any homomorphism $\phi: \mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}\langle\mathbf{x}\rangle$ is continuous. Indeed, $\phi^{-1}(\mathfrak{n})$ is the kernel of the surjective composition

$$\mathbb{C}\langle\mathbf{x}\rangle \xrightarrow{\phi} \mathbb{C}\langle\mathbf{x}\rangle \longrightarrow \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{n}$$

hence $\mathbb{C}\langle\mathbf{x}\rangle/\phi^{-1}(\mathfrak{n}) \cong \mathbb{C}$ and so $\phi^{-1}(\mathfrak{n}) = \mathfrak{n}$ since \mathfrak{n} is the unique maximal ideal. In particular, in the language of [Wa, 5.10], any algebra automorphism of $\mathbb{C}\langle\mathbf{x}\rangle$ is automatically a topological isomorphism, since its inverse is automatically continuous.

2.4. Limits of unitriangular automorphisms. Under specific situations, it is possible to build a sequence of automorphisms $\varphi^1, \varphi^2, \dots$ of $\mathbb{C}\langle\mathbf{x}\rangle$, and take their limit.

For this, consider any d sequences $(\mathbf{g}_i^1)_{i \geq 1}, \dots, (\mathbf{g}_i^d)_{i \geq 1}$, where each $\mathbf{g}_i^k \in \mathfrak{n}^{i+1}$. By 2.1 these give rise to a sequence of unitriangular automorphisms $\varphi^1, \varphi^2, \dots$ where

$$\begin{array}{l}
\varphi^i: \mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}\langle\mathbf{x}\rangle \\
x_k \mapsto x_k + \mathbf{g}_i^k.
\end{array}$$

Again, the above are induced from the corresponding maps $x_k + \mathfrak{m}^j \mapsto x_k + (\mathbf{g}_i^k)_{<j} + \mathfrak{m}^j$ on the truncations $\mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^j$, and where each φ^i has depth i . To ease the subscripts in the notation below, we will also write φ^i for these morphisms viewed on the truncations.

Given this abuse of notation, for all $i \geq j \geq 1$ we claim that the following diagram commutes, where if $i = 1$ or $j = 1$ then the corresponding vertical map is the identity.

$$\begin{array}{ccc}
\mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^j \\
\downarrow \varphi^{i-1} \circ \dots \circ \varphi^1 & & \downarrow \varphi^{j-1} \circ \dots \circ \varphi^1 \\
\mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{m}^j
\end{array} \tag{2.C}$$

To see this, note that since each $\mathbf{g}_i^k \in \mathfrak{n}^{i+1}$, it follows (in the case $i > j$) that the bottom square in the following diagram commutes:

$$\begin{array}{ccc}
 \mathbb{C}\langle x \rangle / \mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle x \rangle / \mathfrak{m}^j \\
 \varphi^1 \downarrow & & \downarrow \varphi^1 \\
 \mathbb{C}\langle x \rangle / \mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle x \rangle / \mathfrak{m}^j \\
 \varphi^2 \downarrow & & \downarrow \varphi^2 \\
 \vdots & & \vdots \\
 \varphi^{j-1} \downarrow & & \downarrow \varphi^{j-1} \\
 \mathbb{C}\langle x \rangle / \mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle x \rangle / \mathfrak{m}^j \\
 \varphi^{i-1} \circ \dots \circ \varphi^j \downarrow & & \downarrow \text{Id} \\
 \mathbb{C}\langle x \rangle / \mathfrak{m}^i & \longrightarrow & \mathbb{C}\langle x \rangle / \mathfrak{m}^j
 \end{array}$$

Since we are abusing notation, the higher squares commute simply since the truncation of a truncation is itself a truncation. Thus all squares commute, establishing (2.C).

Setting $\vartheta_i := \varphi^{i-1} \circ \dots \circ \varphi^1: \mathbb{C}\langle x \rangle / \mathfrak{m}^i \rightarrow \mathbb{C}\langle x \rangle / \mathfrak{m}^i$, again with the convention that $\vartheta_1 = \text{Id}$, then each ϑ_i is an automorphism since each φ^t is. Thus (2.A) induces, through (2.B), an automorphism of $\mathbb{C}\langle\langle x \rangle\rangle$ such that for all $i \geq j$ the following diagram commutes.

$$\begin{array}{ccc}
 \mathbb{C}\langle\langle x \rangle\rangle & \xrightarrow{\exists} & \mathbb{C}\langle\langle x \rangle\rangle \\
 \pi_i \downarrow & & \downarrow \pi_i \\
 \mathbb{C}\langle x \rangle / \mathfrak{m}^i & \xrightarrow{\vartheta_i} & \mathbb{C}\langle x \rangle / \mathfrak{m}^i
 \end{array} \tag{2.D}$$

Write $\varprojlim \varphi^n \dots \varphi^1$ for this induced automorphism.

Lemma 2.4. *With notation and assumptions as directly above, for any $f \in \mathbb{C}\langle\langle x \rangle\rangle$ the sequence $(\varphi^n \dots \varphi^1(f))_{n \geq 1}$ has limit $\varprojlim \varphi^n \dots \varphi^1(f)$.*

Proof. Set $F = \varprojlim \varphi^n \dots \varphi^1$, then it suffices to prove that for all $n \geq 1$, there exists N such that $\varphi^t \dots \varphi^1(f) - F(f) \in \mathfrak{n}^n$ for all $t \geq N$. This follows since for all $i > n$

$$F(f) + \mathfrak{n}^n \stackrel{(2.D)}{=} \varphi^{n-1} \dots \varphi^1(f) + \mathfrak{n}^n \stackrel{(2.C)}{=} \varphi^{i-1} \dots \varphi^1(f) + \mathfrak{n}^n. \quad \square$$

2.5. Closure and Cyclic Permutation.

Definition 2.5. For any subset $\mathcal{S} \subset \mathbb{C}\langle\langle x \rangle\rangle$, its *closure* is defined to be

$$\bar{\mathcal{S}} = \bigcap_{i=0}^{\infty} (\mathcal{S} + \mathfrak{n}^i).$$

That is, $b \in \bar{\mathcal{S}}$ if and only if for all $i \geq 0$, there exists $s_i \in \mathcal{S}$ such that $b - s_i \in \mathfrak{n}^i$.

Notation 2.6. For $\mathcal{A} := \mathbb{C}\langle\langle x \rangle\rangle$, consider $\{\mathcal{A}, \mathcal{A}\}$, the commutator vector space of $\mathbb{C}\langle\langle x \rangle\rangle$. That is, elements of $\{\mathcal{A}, \mathcal{A}\}$ are finite sums

$$\sum_{i=1}^n \lambda_i (a_i b_i - b_i a_i)$$

for elements $a_i, b_i \in \mathbb{C}\langle\langle x \rangle\rangle$ and $\lambda_i \in \mathbb{C}$. Write $\{\{\mathcal{A}, \mathcal{A}\}\}$ for the closure of the commutator vector space $\{\mathcal{A}, \mathcal{A}\}$. Note that $\{\{\mathcal{A}, \mathcal{A}\}\}$ is only a vector space, not an ideal.

Definition 2.7. Two elements $f, g \in \mathbb{C}\langle\langle x \rangle\rangle$ are called *cyclically equivalent*, or f is said to *cyclically permute* to g , if $f - g \in \{\{\mathcal{A}, \mathcal{A}\}\}$. We write $f \sim g$ in this case.

Remark 2.8. This notion of cyclic equivalence applied a pair of polynomials is finite and elementary: it is generated over \mathbb{C} by commutators $[m_1, m_2]$ of monomials $m_i \in \mathbb{C}\langle x \rangle$. With that in mind, 2.7 is then the natural notion for formal power series, as $f \sim g$ means precisely that $f_d \sim g_d$ in every degree d , and no more: the closure merely handles the possibility that f and g may differ by infinitely many such operations.

2.6. Chasing into Higher Degrees. The following will be one of our main techniques for producing normal forms of potentials in $\mathbb{C}\langle\langle x \rangle\rangle$. The basic idea is to start with a given f , then produce an infinite sequence of automorphisms which chase terms into higher and higher degrees. Taking limits then gives a single automorphism which takes f to the desired normal form. The subtle point is that at each stage the automorphisms in (2) below only give the desired elements up to cyclic permutation. As such, the content in the following is that, with care, limits interact well with cyclic permutation.

Theorem 2.9. *Let $f \in \mathbb{C}\langle\langle x \rangle\rangle$, and set $f_1 = f$. Suppose that there exist elements f_2, f_3, \dots and automorphisms $\varphi^1, \varphi^2, \dots$ such that*

- (1) *Every φ^i is unitriangular, of depth of $\geq i$, and*
- (2) *$\varphi^i(f_i) - f_{i+1} \in \{\{\mathcal{A}, \mathcal{A}\} \cap \mathfrak{n}^{i+1}\}$, for all $i \geq 1$.*

Then $\lim f_i$ exists, and there exists an automorphism F such that $F(f) \sim \lim f_i$.

Proof. The proof follows the strategy used in [DWZ, 4.7], but as the axiomatics are different here, we give the full proof. By §2.4 there is an automorphism $F := \varprojlim \varphi^n \cdots \varphi^1$.

Since the depth of φ^i is $\geq i$, by 2.2(1) $\varphi^i(f_i)$ differs from f_i only in degrees $> i$. By (2), $\varphi^i(f_i)$ differs from f_{i+1} only in degrees $> i$. Hence f_{i+1} differs from f_i only in degrees $> i$, from which it easily follows that (f_n) is a Cauchy sequence. Since Cauchy sequences converge in $\mathbb{C}\langle\langle x \rangle\rangle$, the limit $\lim f_i$ exists.

Set $c_i = \varphi^i(f_i) - f_{i+1} \in \{\{\mathcal{A}, \mathcal{A}\} \cap \mathfrak{n}^{i+1}\}$. Since $f = f_1$, it is easy to see that

$$\begin{aligned} \varphi^n \cdots \varphi^1(f) &= f_{n+1} + \sum_{t=1}^n \varphi^n \cdots \varphi^{t+1}(c_t) \\ &= f_{n+1} + \varphi^n \cdots \varphi^1 \left(\sum_{t=1}^n (\varphi^t \cdots \varphi^1)^{-1}(c_t) \right) \end{aligned} \quad (2.E)$$

where $\varphi^n \cdots \varphi^{t+1}$ is the identity when $t = n$. By 2.4 the left hand side has limit $F(f)$. The first part of the right hand side has limit $\lim f_i$, which exists by above. We next claim that the rightmost term has limit $F(g)$, where g is the limit of the sequence $(\sum_{t=1}^n (\varphi^t \cdots \varphi^1)^{-1}(c_t))_{n \geq 1}$.

First, g exists, since by (2) $c_i \in \mathfrak{n}^{i+1}$, and so since automorphisms preserve the maximal ideal, $(\varphi^t \cdots \varphi^1)^{-1}(c_t) \in \mathfrak{n}^{t+1}$ for all t . It follows easily that the sequence $(\sum_{t=1}^n (\varphi^t \cdots \varphi^1)^{-1}(c_t))_{n \geq 1}$ is Cauchy, and so its limit g exists in $\mathbb{C}\langle\langle x \rangle\rangle$. Given this, the fact that the sequence $(\varphi^n \cdots \varphi^1(\sum_{t=1}^n (\varphi^t \cdots \varphi^1)^{-1}(c_t)))_{n \geq 1}$ has limit $F(g)$ follows, since for all $i > n$

$$\begin{aligned} F(g) + \mathfrak{n}^{n+1} &= \varphi^n \cdots \varphi^1(\pi_{n+1}(g)) + \mathfrak{n}^{n+1} && \text{(by (2.D))} \\ &= \varphi^n \cdots \varphi^1 \left(\sum_{t=1}^n (\varphi^t \cdots \varphi^1)^{-1}(c_t) + \mathfrak{n}^{n+1} \right) + \mathfrak{n}^{n+1} \\ &&& \text{(since } (\varphi^t \cdots \varphi^1)^{-1}(c_t) \in \mathfrak{n}^{t+1}\text{)} \\ &= \varphi^n \cdots \varphi^1 \left(\sum_{t=1}^i (\varphi^t \cdots \varphi^1)^{-1}(c_t) + \mathfrak{n}^{n+1} \right) + \mathfrak{n}^{n+1} && \text{(add zero)} \\ &= \varphi^i \cdots \varphi^1 \left(\sum_{t=1}^i (\varphi^t \cdots \varphi^1)^{-1}(c_t) \right) + \mathfrak{n}^{n+1}. && \text{(by (2.C))} \end{aligned}$$

Combining with (2.E) and taking limits it follows that

$$F(f) = \lim f_i + F(g). \quad (2.F)$$

Now, it is easy to check that automorphisms preserve $\{\{\mathcal{A}, \mathcal{A}\}\}$, so each term in the sequence $(\varphi^n \cdots \varphi^1(\sum_{t=1}^n (\varphi^t \cdots \varphi^1)^{-1}(c_t)))_{n \geq 1}$ belongs to $\{\{\mathcal{A}, \mathcal{A}\}\}$. But since $\mathbb{C}\langle\langle x \rangle\rangle$ is complete, every Cauchy sequence within a closed set has limit in that closed set. It follows that the limit $g \in \{\{\mathcal{A}, \mathcal{A}\}\}$. One final application of the fact that automorphisms preserve $\{\{\mathcal{A}, \mathcal{A}\}\}$ shows that $F(g) \in \{\{\mathcal{A}, \mathcal{A}\}\}$, and so $F(f) \sim \lim f_i$. \square

2.7. Elementary Properties of Closed Ideals. We finish this section with some technical results on closed ideals that are used throughout §6–§8.

Notation 2.10. When I is an ideal, write $\langle\langle I \rangle\rangle$ for its closure (in the sense of 2.5), which is again an ideal since the ring operations are continuous. Note that $\langle\langle I \rangle\rangle$ need not be finitely generated, even if I is.

For a finite set of elements \mathcal{S} in $\mathbb{C}\langle\langle x \rangle\rangle$, consider the closed ideal $\langle\langle \mathcal{S} \rangle\rangle = \langle\langle s \mid s \in \mathcal{S} \rangle\rangle$.

Lemma 2.11. *Let \mathcal{S} be a finite subset of elements in $\mathbb{C}\langle\langle x \rangle\rangle$, and $f_1, \dots, f_s \in \mathbb{C}\langle\langle x \rangle\rangle$. Then the following statements hold.*

- (1) $\langle\langle f_1, \dots, f_s \rangle\rangle = \langle\langle f_1 u_1, \dots, f_s u_s \rangle\rangle$ for any units $u_1, \dots, u_s \in \mathbb{C}\langle\langle x \rangle\rangle$.
- (2) $\langle\langle f + \langle\langle \mathcal{S} \rangle\rangle \rangle\rangle = \langle\langle f, s \mid s \in \mathcal{S} \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle$ in $\mathbb{C}\langle\langle x \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle$.
- (3) If $\psi: \mathbb{C}\langle\langle x \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle \rightarrow \mathbb{C}\langle\langle x \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle$ is a topological isomorphism which sends $f + \langle\langle \mathcal{S} \rangle\rangle \mapsto g + \langle\langle \mathcal{S} \rangle\rangle$, for two elements $f, g \in \mathbb{C}\langle\langle x \rangle\rangle$, then there is an induced topological isomorphism

$$\frac{\mathbb{C}\langle\langle x \rangle\rangle}{\langle\langle f, s \mid s \in \mathcal{S} \rangle\rangle} \xrightarrow{\cong} \frac{\mathbb{C}\langle\langle x \rangle\rangle}{\langle\langle g, s \mid s \in \mathcal{S} \rangle\rangle}.$$

Proof. (1) $\langle\langle f_1, \dots, f_s \rangle\rangle$ is the smallest closed ideal containing all f_i . Since $f_i = (f_i u_i) u_i^{-1} \in \langle\langle f_1 u_1, \dots, f_s u_s \rangle\rangle$ for each i , by minimality $\langle\langle f_1, \dots, f_s \rangle\rangle \subseteq \langle\langle f_1 u_1, \dots, f_s u_s \rangle\rangle$. Repeating the same argument to $f_i u_i \in \langle\langle f_1, \dots, f_s \rangle\rangle$, the converse inclusion also holds.

(2) Certainly $\langle\langle f + \langle\langle \mathcal{S} \rangle\rangle \rangle\rangle = I / \langle\langle \mathcal{S} \rangle\rangle$ for some ideal I , given it is an ideal of the quotient. This ideal I is closed by [Wa, 5.2], since the map to the quotient is continuous, and hence the inverse image of a closed set is closed. This closed ideal I contains both f and \mathcal{S} , and so $\langle\langle f, s \mid s \in \mathcal{S} \rangle\rangle \subseteq I$.

On the other hand $\langle\langle f + \langle\langle \mathcal{S} \rangle\rangle \rangle\rangle$ is the smallest closed ideal containing $f + \langle\langle \mathcal{S} \rangle\rangle$. Setting $A = \mathbb{C}\langle\langle x \rangle\rangle$, $J = \langle\langle \mathcal{S} \rangle\rangle$, and $H = \langle\langle f, s \mid s \in \mathcal{S} \rangle\rangle$, the third isomorphism theorem for topological rings [Wa, 5.13] asserts that there is a topological isomorphism

$$(A/J)/(H/J) \cong A/H.$$

In particular, A/H is Hausdorff, since H is closed in A , by [Wa, 5.7(1)] applied to A . This being the case, H/J is closed in A/J , by [Wa, 5.7(1)] applied to A/J . Hence $\langle\langle f, s \mid s \in \mathcal{S} \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle$ is a closed ideal, which clearly contains $f + \langle\langle \mathcal{S} \rangle\rangle$. By minimality $\langle\langle f + \langle\langle \mathcal{S} \rangle\rangle \rangle\rangle = I / \langle\langle \mathcal{S} \rangle\rangle \subseteq \langle\langle f, s \mid s \in \mathcal{S} \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle$ and thus $I \subseteq \langle\langle f, s \mid s \in \mathcal{S} \rangle\rangle$. Combining inclusions, the required equality holds.

(3) Since ψ is a continuous isomorphism, the closed ideal generated by $f + \langle\langle \mathcal{S} \rangle\rangle$ corresponds to the closed ideal generated by $g + \langle\langle \mathcal{S} \rangle\rangle$. Thus there is a topological isomorphism

$$\frac{\mathbb{C}\langle\langle x \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle}{\langle\langle f + \langle\langle \mathcal{S} \rangle\rangle \rangle\rangle} \xrightarrow{\cong} \frac{\mathbb{C}\langle\langle x \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle}{\langle\langle g + \langle\langle \mathcal{S} \rangle\rangle \rangle\rangle}.$$

Now by (2) we have $\langle\langle f + \langle\langle \mathcal{S} \rangle\rangle \rangle\rangle = \langle\langle f, s \mid s \in \mathcal{S} \rangle\rangle / \langle\langle \mathcal{S} \rangle\rangle$, and likewise for g . The statement follows by the third isomorphism theorem for topological rings [Wa, 5.13]. \square

3. JACOBI ALGEBRAS

3.1. Differentiation. Consider the \mathbb{C} -linear maps $\partial_i: \mathbb{C}\langle\langle x \rangle\rangle \rightarrow \mathbb{C}\langle\langle x \rangle\rangle$ which simply ‘strike off’ the leftmost x_i of each monomial, in other words act on monomials via the rule

$$\partial_i(m) = \begin{cases} n & \text{if } m = x_i n \\ 0 & \text{otherwise.} \end{cases} \quad (3.A)$$

The \mathbb{C} -linear cyclic symmetrisation map $\text{cyc}: \mathbb{C}\langle\langle x \rangle\rangle \rightarrow \mathbb{C}\langle\langle x \rangle\rangle$ on monomials sends

$$x_{i_1} \dots x_{i_t} \mapsto \sum_{j=1}^t x_{i_j} x_{i_{j+1}} \dots x_{i_t} \cdot x_{i_1} \dots x_{i_{j-1}}.$$

Combining these two gives the cyclic derivatives. These are the \mathbb{C} -linear maps $\delta_i: \mathbb{C}\langle\langle x \rangle\rangle \rightarrow \mathbb{C}\langle\langle x \rangle\rangle$ which on monomials send

$$x_{i_1} \dots x_{i_t} \mapsto \partial_i \text{cyc}(x_{i_1} \dots x_{i_t}) = \sum_{j=1}^t \partial_i(x_{i_j} x_{i_{j+1}} \dots x_{i_t} \cdot x_{i_1} \dots x_{i_{j-1}}). \quad (3.B)$$

Definition 3.1. For $f \in \mathbb{C}\langle\mathbf{x}\rangle$, the Jacobi algebra is defined to be

$$\mathcal{J}ac(f) = \frac{\mathbb{C}\langle\mathbf{x}\rangle}{\langle\langle\delta_1 f, \dots, \delta_d f\rangle\rangle}$$

where $\langle\langle\delta_1 f, \dots, \delta_d f\rangle\rangle = \langle\langle\delta f\rangle\rangle$ is the closure of the two-sided ideal $(\delta_1 f, \dots, \delta_d f)$.

In general, the quotient of a complete topological ring by a closed ideal is always separated, but it need not be complete.

Notation 3.2. For any ring R , write $\mathfrak{J}(R)$ for its Jacobson radical. If I is any ideal of R contained in $\mathfrak{J}(R)$, then $\mathfrak{J}(R/I) = \mathfrak{J}(R)/I$ (see e.g. [L1, 4.6]).

- (1) It is clear that $\mathfrak{J}(\mathbb{C}\langle\mathbf{x}\rangle) = \mathfrak{n}$. If $f \in \mathbb{C}\langle\mathbf{x}\rangle_{\geq 2}$, then $\langle\langle\delta f\rangle\rangle$ is contained in \mathfrak{n} , and so $\mathfrak{J}(\mathcal{J}ac(f)) = \mathfrak{n}/\langle\langle\delta f\rangle\rangle$, and furthermore $\mathfrak{J}(\mathcal{J}ac(f))^n = (\mathfrak{n}^n + \langle\langle\delta f\rangle\rangle)/\langle\langle\delta f\rangle\rangle$ for $n \geq 2$.
- (2) The topology on $\mathbb{C}\langle\mathbf{x}\rangle$ is an ideal topology generated by powers of \mathfrak{n} , so the natural quotient topology on the quotient $\mathcal{J}ac(f)$ is induced by powers of the image of \mathfrak{n} in the quotient [Wa, 5.5]. Thus, by (1), provided $f \in \mathbb{C}\langle\mathbf{x}\rangle_{\geq 2}$ then the topology on both $\mathbb{C}\langle\mathbf{x}\rangle$ and $\mathcal{J}ac(f)$ is the radical-adic topology. Since $\langle\langle\delta f\rangle\rangle$ is closed, $\mathcal{J}ac(f)$ is Hausdorff [Wa, 5.7(1)]. Under extra assumptions it is also complete; see 8.4(3).

Remark 3.3. A (polynomial or) power series $f = \sum f_i \in \mathbb{C}\langle\mathbf{x}\rangle$ is called *cyclically symmetric* if $\text{cyc}(f_i) = if_i$ for each graded piece $f_i \in \mathbb{C}\langle\mathbf{x}\rangle$. It is possible to phrase the whole paper using only cyclically symmetric potentials, however this becomes notationally unmanageable in §4–§6, since the property of being cyclically symmetric is not preserved under change variables. Thus from the viewpoint of noncommutative singularity theory, it is much more natural to work with plain old elements of $\mathbb{C}\langle\mathbf{x}\rangle$. There are times when passing to cyclically symmetric potentials is convenient, but this is confined entirely to §A.2.

3.2. Dimension. Being a quotient of formal noncommutative power series, determining which dimension to use for $\mathcal{J}ac(f)$ is a subtle point.

Definition 3.4. For $f \in \mathbb{C}\langle\mathbf{x}\rangle_{\geq 2}$, we say that $\mathcal{J}ac(f)$ has *polynomial growth (with respect to \mathfrak{J})* if there exist $c, r \in \mathbb{R}$ such that $\dim \mathcal{J}ac(f)/\mathfrak{J}^n \leq cn^r$ for all $n \in \mathbb{N}$. In the case that $\mathcal{J}ac(f)$ has polynomial growth, then the *\mathfrak{J} -dimension of $\mathcal{J}ac(f)$* is the degree of that growth, precisely

$$\text{Jdim } \mathcal{J}ac(f) := \inf \{r \in \mathbb{R} \mid \text{for some } c \in \mathbb{R}, \dim \mathcal{J}ac(f)/\mathfrak{J}^n \leq cn^r \text{ for every } n \in \mathbb{N}\},$$

and $\text{Jdim } \mathcal{J}ac(f) = \infty$ otherwise.

The \mathfrak{J} -dimension is analogous to the usual dimension of a commutative noetherian local ring (A, \mathfrak{m}) , defined as the degree of the characteristic polynomial $\chi_{\mathfrak{m}}(n) = \ell(A/\mathfrak{m}^n)$, where, in that context, the dimension is necessarily an integer [AM, 11.4, 11.14].

Lemma 3.5. *If $\text{Jdim } \mathcal{J}ac(f) \leq 1$, then either $\text{Jdim } \mathcal{J}ac(f) = 0$ or $\text{Jdim } \mathcal{J}ac(f) = 1$.*

Proof. Certainly $\mathcal{J}ac(f)/\mathfrak{J}^{n+1} \twoheadrightarrow \mathcal{J}ac(f)/\mathfrak{J}^n$ for all $n \geq 1$, with equality if and only if $\mathfrak{J}^{n+1} = \mathfrak{J}^n$. If each such map has nontrivial kernel, then $\dim \mathcal{J}ac(f)/\mathfrak{J}^n \geq n$ and so $\text{Jdim } \mathcal{J}ac(f) \geq 1$. Otherwise, by Nakayama's Lemma, $\mathfrak{J}^n = 0$ for some n , hence $\mathfrak{n}^n \subset \langle\langle\delta f\rangle\rangle$ and so $\dim_{\mathbb{C}} \mathcal{J}ac(f) \leq \dim_{\mathbb{C}} \mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{n}^n = 2^n - 1$ and $\text{Jdim } \mathcal{J}ac(f) = 0$. \square

Remark 3.6. The \mathfrak{J} -dimension is used throughout, since it is better suited to the complete local situation than the GK dimension [KL]. Indeed, it is well-known that the GK dimension does not behave well with respect to completions. For example, $\text{GKdim } \mathbb{C}[[x]] = \infty$ whereas $\text{Jdim } \mathbb{C}[[x]] = 1$. Compare [AB, §3.4], and in particular [AB, §5.6]. Furthermore, $\text{Jdim } \mathcal{J}ac(f) = 0$ if and only if $\dim_{\mathbb{C}} \mathcal{J}ac(f) < \infty$, a property which does not hold for GK dimension since $\mathcal{J}ac(f)$ is not finitely generated.

3.3. Equivalences and Isomorphisms. In what follows, recognising and producing isomorphisms of Jacobi algebras will be key. The following techniques will be used extensively. The first is trivial, but worth recording since it gives great flexibility in proofs; the second two are more substantial with Part (2) being [DWZ, 3.7], and Part (3) following from (2), together with 2.9. Recall the notation $f \sim g$ from 2.7.

Summary 3.7. Suppose that $f \in \mathbb{C}\langle\mathbf{x}\rangle$.

- (1) If f cyclically permutes to g , so $f \sim g$, then $\mathcal{J}ac(f) \cong \mathcal{J}ac(g)$.
- (2) If $\varphi \in \text{Aut } \mathbb{C}\langle\mathbf{x}\rangle$ then $\mathcal{J}ac(f) \cong \mathcal{J}ac(g)$, where $g = \varphi(f)$.
- (3) Set $f_1 = f$. If there exist f_2, f_3, \dots and automorphisms $\varphi^1, \varphi^2, \dots$ such that
 - (a) every φ^i is unitriangular of depth of $\geq i$, and
 - (b) $\varphi^i(f_i) - f_{i+1} \in \{\{\mathcal{A}, \mathcal{A}\}\} \cap \mathfrak{n}^{i+1}$, for all $i \geq 1$,
 then the sequence $(f_i)_{i \geq 1}$ converges and $\mathcal{J}ac(f) \cong \mathcal{J}ac(g)$ where $g = \lim f_i$.

Lemma 3.8. *Let $f \in \mathbb{C}\langle\mathbf{x}\rangle$, and $m \in \mathbb{C}\langle\mathbf{x}\rangle$ be a monomial. Then the following hold.*

- (1) $\text{cyc}(m) \sim \text{deg}(m)m$.
- (2) If f contains λm , then $f \sim f + \lambda \left(\frac{1}{\text{deg } m} \text{cyc}(m) - m \right)$.
- (3) Let h be the sum of terms of f whose monomials appear in $\text{cyc}(m)$. Then

$$f \sim f - h + \alpha \text{cyc}(m) \sim f - h + \alpha \text{deg}(m) m,$$

for some $\alpha \in \mathbb{C}$.

Proof. Writing $m = m_1 m_2 \dots m_r$, where each m_i is a variable $x_{j(i)}$, we have

$$\begin{aligned} rm - \text{cyc}(m) &= (r-1)m_1 \dots m_r - m_2 \dots m_r m_1 \\ &\quad - m_3 \dots m_r m_1 m_2 - \dots - m_r m_1 \dots m_{r-1} \\ &= [m_1, m_2 \dots m_r] + [m_1 m_2, m_3 \dots m_r] + \dots + [m_1 \dots m_{r-1}, m_r] \end{aligned}$$

and (1) follows. (2) follows at once from (1). The final claim (3) follows by applying (2) to each monomial of h in turn. \square

Below it will be convenient to work with the following three equivalence relations.

Definition 3.9. For elements $f, g \in \mathbb{C}\langle\mathbf{x}\rangle$, (recall and) define

- (1) $f \sim g$ if $f - g \in \{\{\mathbb{C}\langle\mathbf{x}\rangle, \mathbb{C}\langle\mathbf{x}\rangle\}\}$ (see 2.7).
- (2) $f \simeq g$ if there is an equality of ideals $(\delta_1 f, \dots, \delta_d f) = (\delta_1 g, \dots, \delta_d g)$.
- (3) $f \cong g$ if there is an isomorphism of algebras $\mathcal{J}ac(f) \cong \mathcal{J}ac(g)$.

Clearly $f \sim g$ implies $f \simeq g$ implies $f \cong g$, but the converse implications do not hold. The relation \sim is additive by definition, but \simeq is not: $x^2 + y^3 \simeq x^2 + 2y^3$ but $x^2 \not\simeq x^2 + y^3$.

The Jacobi isomorphism relation \cong is the equivalence relation that we will classify up to, but the others help understand the structure of the various arguments. For example, by 2.9, the symmetrisation relation \sim behaves well in limits. It appears to permit creation from the void, in the sense that $0 \sim xy - yx$, but of course this form has all derivatives zero, so does not contribute to Jacobi ideals. The relation \simeq is useful for cancelling high order terms in potentials (see e.g. the proof of 6.5), whereas \cong is most suited to, and is often a by-product of, analytic changes in coordinates.

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4.1. Corank and the Splitting Lemma. The closed vector subspace of commutators $\{\{\mathbb{C}\langle\mathbf{x}\rangle, \mathbb{C}\langle\mathbf{x}\rangle\}\}$ generates the much larger closed ideal of commutators, and the quotient of $\mathbb{C}\langle\mathbf{x}\rangle$ by this ideal is the ring of commutative power series $\mathbb{C}[[x_1, \dots, x_d]]$. The quotient, or ‘abelianisation’, map $\mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}[[x_1, \dots, x_d]]$ written $g \mapsto g^{\text{ab}}$ simply takes the expression for g to the same expression in the commutative ring.

Lemma 4.1. *With notation as above, the following hold:*

- (1) The abelianisation map $\mathbb{C}\langle\mathbf{x}\rangle \rightarrow \mathbb{C}[[x_1, \dots, x_d]]$ is continuous and surjective.
- (2) For any $f \in \mathbb{C}\langle\mathbf{x}\rangle$, the map $f \mapsto f^{\text{ab}}$ descends to a surjection

$$\mathcal{J}ac(f) \twoheadrightarrow \frac{\mathbb{C}[[x_1, \dots, x_d]]}{\langle\langle \partial f^{\text{ab}} / \partial x_i \mid i = 1, \dots, n \rangle\rangle}. \quad (4.A)$$

Proof. (1) At the level of ideals, $\mathfrak{n}^k \twoheadrightarrow \mathfrak{n}_{\text{ab}}^k$ for every $k \geq 0$, since abelianisation is a ring homomorphism mapping each x_i to x_i .

(2) Since $(\delta_i f)^{\text{ab}} = \partial f^{\text{ab}} / \partial x_i$, where $\partial / \partial x_i$ is the usual differentiation of commutative functions, surjectivity at the level of (unclosed) Jacobian ideals follows. Since the abelianisation map is continuous and surjective by (1), this passes to their closures, as claimed. \square

Below we will consider

$$\mathcal{J}ac(f)^{\text{ab}} := \frac{\mathbb{C}\langle x_1, \dots, x_d \rangle}{\langle \partial f^{\text{ab}} / \partial x_i \mid i = 1, \dots, n \rangle} = \frac{\mathbb{C}\langle x_1, \dots, x_d \rangle}{\langle \partial f^{\text{ab}} / \partial x_i \mid i = 1, \dots, n \rangle}$$

where, since $\mathbb{C}\langle x_1, \dots, x_d \rangle$ is commutative noetherian, all ideals are closed [M1, 8.1(1)].

Remark 4.2. In classical singularity theory, for $g \in \mathbb{C}\langle x_1, \dots, x_d \rangle$ both the Milnor algebra $\mathbb{C}\langle x_1, \dots, x_d \rangle / \langle \delta_1 g, \dots, \delta_d g \rangle$ and the Tjurina algebra $\mathbb{C}\langle x_1, \dots, x_d \rangle / \langle g, \delta_1 g, \dots, \delta_d g \rangle$ are defined, and play a major role. In the noncommutative setting, the analogous Tjurina algebra is not well defined on \sim classes. For example, the potentials $0 \sim xy - yx$ determine the same Jacobi algebra, but their naively-defined Tjurina algebras are $\mathbb{C}\langle x, y \rangle$ and $\mathbb{C}\langle x, y \rangle$ respectively. To have any hope of classifying elements in the completed free algebra, some identification is required, and for us identifying \sim classes is essential for applications. Compare [HZ], where the lack of a noncommutative Tjurina algebra motivates the use of Hochschild classes to generalise Saito's theorem on homogeneous potentials.

Definition 4.3. For $f \in \mathbb{C}\langle x \rangle_{\geq 2}$, the *corank* of f is defined to be

$$\text{Crk}(f) = \dim_{\mathbb{C}} \left(\frac{\mathfrak{J}}{\mathfrak{J}^2} \right)$$

where \mathfrak{J} is the Jacobson radical of $\mathcal{J}ac(f)$.

Remark 4.4. Clearly $0 \leq \text{Crk}(f) \leq d$. Since $\mathfrak{J}/\mathfrak{J}^2 \cong (\mathfrak{n} + I)/(\mathfrak{n}^2 + I)$, where $I = \langle \delta f \rangle$, the exactness of the sequence of $\mathbb{C}\langle x \rangle / \mathfrak{n} = \mathbb{C}$ -vector spaces

$$0 \longrightarrow \frac{\mathfrak{n}^2 + I}{\mathfrak{n}^2} \longrightarrow \frac{\mathfrak{n}}{\mathfrak{n}^2} \longrightarrow \frac{\mathfrak{n} + I}{\mathfrak{n}^2 + I} \longrightarrow 0$$

shows that $\text{Crk}(f) = d - \dim_{\mathbb{C}} \left(\frac{\mathfrak{n}^2 + I}{\mathfrak{n}^2} \right)$, so that the corank is determined by the linear conditions imposed by derivatives, and is therefore uniquely determined by f_2 .

Theorem 4.5 (Splitting Lemma). *Let $f \in \mathbb{C}\langle x \rangle$. Then $f \cong x_1^2 + \dots + x_r^2 + g$ for some $g \in \mathbb{C}\langle x_{r+1}, \dots, x_d \rangle_{\geq 3}$, where $d - r = \text{Crk}(f)$. In particular,*

$$\mathcal{J}ac(f) \cong \frac{\mathbb{C}\langle x_{r+1}, \dots, x_d \rangle}{\langle \delta_{x_{r+1}} g, \dots, \delta_{x_d} g \rangle}.$$

Proof. This is [DWZ, 4.5] for the d -loop quiver. Since $\text{ord } g \geq 3$, the derivatives of g impose no linear conditions, so necessarily $d - r = \text{Crk}(f)$. \square

4.2. Golod–Shafarevich–Vinberg. The classical approach to growth of algebras comes from the Golod–Shafarevich theorem [GS], adapted by Vinberg [V3] to power series; see also [E2]. This result constrains f to achieve $\text{Jdim } \mathcal{J}ac(f) < \infty$, and we develop a stronger version in 4.7 adapted to Jacobi algebras.

Theorem 4.6 (Golod–Shafarevich, Vinberg). *Let $I = (g_1, \dots, g_s) \subset \mathbb{C}\langle x \rangle$ be an ideal, set $r_i = \text{ord } g_i$ for each $i = 1, \dots, s$, and write $h = 1 - dt + t^{r_1} + \dots + t^{r_s} \in \mathbb{R}\langle t \rangle$. If the coefficients of $(1 - t)/h$ are non negative, then $\dim_{\mathbb{C}} \mathbb{C}\langle x \rangle / (I) = \infty$.*

In most cases where the result applies, one can in fact show exponential growth. The Golod–Shafarevich–Vinberg estimates readily show that $\text{Jdim } \mathbb{C}\langle x \rangle / \langle g_1, \dots, g_d \rangle = \infty$ in the following cases:

- (1) $d = 2$ with either $r_1 \geq 3, r_2 \geq 8$, or $r_1 \geq 4, r_2 \geq 5$.
- (2) $d = 3$ with either $r_1 \geq 2, r_2, r_3 \geq 3$, or $r_1 = r_2 = 2, r_3 \geq 5$.
- (3) $d \geq 4$ with $r_i \geq 2$ for every i .

For example, in the case $d = 4$, it is sufficient to observe the exponential growth of

$$(1 - 4t + 4t^2)^{-1} = 1 + 3t + 8t^2 + 20t^3 + 48t^4 + \dots + (1 + k)2^k t^k + \dots$$

as this bounds the growth of the algebra from below in the case of an order 3 potential with four order 2 derivatives.

Setting aside quadratic terms by the Splitting Lemma, this then puts constraints on the motivating problem 1.1. Indeed, if $f \in \mathbb{C}\langle x \rangle_{\geq 3}$ and $\text{Jdim } \mathcal{J}ac(f) < \infty$, then either

- (1) $d = 2, \text{ord } f \leq 5$ and $f_{\leq 5} \not\sim \ell^5$ for a linear form $\ell = \ell(x_1, x_2)$, or

(2) $d = 3$, $\text{ord } f = 3$, and $f_3 \not\sim \ell^3$ for a linear form $\ell = \ell(x_1, x_2, x_3)$.

It turns out that these estimates can be substantially improved, but this requires much more work. Iyudu and collaborators [ISm, IS2] introduce several new ideas that exploit the Jacobi structure; in Appendix A we extend their techniques into the power series context, and establish the following. Recall that $\mathbf{x} = x_1, \dots, x_d$.

Theorem 4.7 (A.13). *Suppose that $d = 2$ and $k \geq 4$, or $d \geq 3$ and $k \geq 3$. If $f \in \mathbb{C}\langle\mathbf{x}\rangle$ has order k , then $\text{Jdim } \mathcal{J}\text{ac}(f) \geq 3$.*

Remark 4.8. Together with the Splitting Lemma, the above 4.7 reduces the classification of those f satisfying $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$ to the case of two variables ($d = 2$).

4.3. Higher Coranks. Higher-degree versions of the corank exist, and contain more detailed information about Jacobi algebras.

Definition 4.9. Let $f \in \mathbb{C}\langle\mathbf{x}\rangle_{\geq 2}$. For $m \geq 1$, the m th corank of f is defined to be

$$\text{Crk}_m(f) = \dim_{\mathbb{C}} \left(\frac{\mathfrak{J}^m}{\mathfrak{J}^{m+1}} \right),$$

where \mathfrak{J} is the Jacobson radical of $\mathcal{J}\text{ac}(f)$. We also define $\text{Crk}_0(f) = \dim_{\mathbb{C}} \mathcal{J}\text{ac}(f)/\mathfrak{J} = 1$ and note that $\text{Crk}_1(f) = \text{Crk}(f)$.

Remark 4.10. Since $\mathfrak{J}^m/\mathfrak{J}^{m+1} \cong (\mathfrak{n}^m + I)/(\mathfrak{n}^{m+1} + I)$, the exactness of the sequence of $\mathbb{C}\langle\mathbf{x}\rangle/\mathfrak{n} = \mathbb{C}$ -vector spaces

$$0 \rightarrow \frac{\mathfrak{n}^m \cap I}{\mathfrak{n}^{m+1} \cap I} \cong \frac{(\mathfrak{n}^m \cap I) + \mathfrak{n}^{m+1}}{\mathfrak{n}^{m+1}} \rightarrow \frac{\mathfrak{n}^m}{\mathfrak{n}^{m+1}} \rightarrow \frac{\mathfrak{n}^m + I}{\mathfrak{n}^{m+1} + I} \rightarrow 0$$

shows that $\text{Crk}_m(f) = d^m - \dim_{\mathbb{C}} \left(\frac{\mathfrak{n}^m \cap I}{\mathfrak{n}^{m+1} \cap I} \right)$; compare [V3, (4)]. Thus the m th corank is determined by the conditions imposed on the leading terms of elements of the Jacobian ideal of order exactly m . In particular, $0 \leq \text{Crk}_m(f) \leq d^m$. If $\text{ord}(f) \geq m + 1$, then

$$\frac{(\mathfrak{n}^m \cap I) + \mathfrak{n}^{m+1}}{\mathfrak{n}^{m+1}} \cong \frac{\mathfrak{n}^{m+1} + I}{\mathfrak{n}^{m+1}}$$

matching (1.A), (1.B), and 4.4.

By definition, the \mathfrak{J} -dimension is the growth of the sum of coranks. Calculating the m th corank is not necessarily straightforward: essentially it amounts to calculating a Gröbner basis of the Jacobian ideal with a local monomial order to at least order m .

To study Jacobi algebras $\mathcal{J}\text{ac}(f)$ of \mathfrak{J} -dimension ≤ 1 , 4.7 constrains the number of variables to $d \leq 2$ and $k = \text{ord}(f) \leq 3$. The corank controls the rank of f_2 . The main case is when $f_2 = 0$, when it is clear that $2 \leq \text{Crk}_2(f) \leq 4$. The two derivatives $\delta_x f_3$ and $\delta_y f_3$ are linearly independent when $\text{Crk}_2(f) = 2$ and they are dependent when $\text{Crk}_2(f) = 3$. The case $\text{Crk}_2(f) = 4$ holds only when $f_3 = 0$, which is ruled out by 4.7.

This provides a numerical characterisation of the ADE types. The first Type A case is when $\text{Crk}(f) = 0$, in which case $\mathcal{J}\text{ac}(f) \cong \mathbb{C}$. In addition to this, if $f \in \mathbb{C}\langle\mathbf{x}\rangle$ has $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$, we say f has Type A, D or E according to the following table.

Type	$\text{Crk}(f)$	$\text{Crk}_2(f)$
A	1	1
D	2	2
E	2	3

The higher coranks provide much more detail. In Type A they provide enough information to classify up to isomorphism, however in Type D this is not true.

Example 4.11. Consider the families $D_{n,\infty}$ and $D_{n,m}$ from the introduction. The higher coranks are given by the following table.

Type	f	Conditions	$\text{Crk}_i(f), i = 0, 1, \dots$
$D_{n,\infty}$	$xy^2 + x^{2n}$	$n \geq 2$	$\overbrace{1, 2, \dots, 2}^{2n-1}, \overbrace{1, \dots, 1}^{2n-2}$
$D_{n,m}$	$xy^2 + x^{2n} + x^{2m+1}$	$2n - 2 \geq m \geq n \geq 2$	$\overbrace{1, 2, \dots, 2}^{2n-1}, \overbrace{1, \dots, 1}^{2m-1}$
$D_{n,m}$	$xy^2 + x^{2m+1} + x^{2n}$	$n > m \geq 1$	$\overbrace{1, 2, \dots, 2}^{2n-1}, \overbrace{1, \dots, 1}^{2m-1}$

In particular $\dim_{\mathbb{C}} \mathcal{J}ac(f) = 6n - 3$ in the first families, which is independent of m , whilst $\dim_{\mathbb{C}} \mathcal{J}ac(f) = 4n + 2m - 2$ in the final family.

4.4. Linear Changes in Coordinates and Discriminants. In light of 4.8, from §6 onwards we work in two non-commuting variables x and y .

The following is an immediate consequence of the Splitting Lemma and abelianisation.

Lemma 4.12. *Let $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$ with $f_2 = ax^2 + b_1xy + b_2yx + cy^2 \not\sim 0$. Set $b = b_1 + b_2$ and consider the discriminant $\Delta = b^2 - 4ac$. Then $f \cong g$, for some $g \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$ with*

$$g_2 = \begin{cases} x^2 + y^2 & \text{if } \Delta \neq 0 \\ x^2 & \text{if } \Delta = 0. \end{cases}$$

As for the quadratic forms above, up to \sim we may commute variables appearing in cubic forms in $\mathbb{C}\langle\langle x, y \rangle\rangle$, and we use this to simplify the statement of the following lemma, writing bx^2y rather than $b_1x^2y + b_2xyx + \dots$, and so on. Note that, in general, cyclic equivalence no longer simulates commutativity in higher degree, as $xyxy \sim x^2y^2$.

Lemma 4.13. [I] *Let $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ with $f_3 \sim ax^3 + bx^2y + cxy^2 + dy^3$ for some $a, b, c, d \in \mathbb{C}$, not all zero. Let $\Delta = -27a^2d^2 + 18abcd - 4ac^3 - 4b^3d + b^2c^2 \in \mathbb{C}$ be the cubic discriminant. Then $f \cong g$, for some $g \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ with*

$$g_3 = \begin{cases} x^3 + y^3 & \text{if } \Delta \neq 0 \\ x^2y & \text{if } \Delta = 0 \text{ and } \begin{cases} a(b^2 - 3ac) \neq 0 \text{ or} \\ (c^2 - 3bd)d \neq 0 \text{ or} \\ a = d = 0 \end{cases} \\ x^3 & \text{otherwise.} \end{cases}$$

Thus these three leading cubic normal forms are characterised by whether f_3^{ab} has three, two or one distinct factors respectively.

Proof. Consider the linear automorphism of $\mathbb{C}[x, y]$

$$x \mapsto \alpha x + \beta y, \quad y \mapsto \gamma x + \delta y \quad \text{for } \alpha, \beta, \gamma, \delta \in \mathbb{C} \quad (4.B)$$

that maps $(f_3)^{\text{ab}} \in \mathbb{C}[x, y]$ to one of the normal forms $x^3 + y^3$, xy^2 or x^3 . The choice of normal form is determined by the cubic determinant. The additional conditions on the coefficients in the statement are simply that $p \neq 0$ in the depressed form after completing the cube $x^3 + px^2 + qy^3$, in which $\Delta = -4p^3 - 27q^2$, and accounting for the fact that $a = 0$ or $d = 0$ or both are possible.

Let φ be the linear automorphism of $\mathbb{C}\langle\langle x, y \rangle\rangle$ defined by the same formula (4.B) and $g_3 \in \mathbb{C}\langle\langle x, y \rangle\rangle$ be the corresponding cubic normal form. Then $\varphi(f_3) \sim g_3$, although they are not equal, so $\varphi(f) \sim g_3 + \varphi(f_{\geq 4})$. Thus $f \cong \varphi(f) \cong g_3 + \varphi(f_{\geq 4})$, as claimed. \square

5. TYPE A AND COMMUTATIVITY

This section considers the most elementary situation, namely $f \in \mathbb{C}\langle\langle x \rangle\rangle_{\geq 2}$ with large quadratic part. Normal forms are established in §5.1. Together with the linear coordinate changes from §4.4, this proves in §5.2 that for any $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$, the algebra $\mathcal{J}ac(f)$ is commutative if and only if f has corank at most 1. This fact is used in later sections.

5.1. Normal Forms of Type A. It is notationally convenient to identify $y = x_d$ and work in the ring $\mathbb{C}\langle x \rangle = \mathbb{C}\langle x_1, \dots, x_{d-1}, y \rangle$.

Theorem 5.1. *If $f \in \mathbb{C}\langle x \rangle_{\geq 2}$ with $\text{Crk}(f) \leq 1$, then there is a unique polynomial g of the form $g = x_1^2 + \dots + x_{d-1}^2 + \varepsilon y^n$ for some $n \geq 2$ and $\varepsilon \in \{0, 1\}$ such that $f \cong g$.*

- (1) *If $\varepsilon = 1$, then $\mathcal{J}\text{ac}(f)$ is commutative with $\dim_{\mathbb{C}} \mathcal{J}\text{ac}(f) = n - 1$.*
- (2) *If $\varepsilon = 0$, then $\mathcal{J}\text{ac}(f)$ is commutative with $\text{Jdim} \mathcal{J}\text{ac}(f) = 1$.*

Proof. By the Splitting Lemma 4.5, there is $f \in \mathbb{C}\langle x_1, \dots, x_{d-1}, y \rangle$ with $f \cong f$ and either

$$f = \begin{cases} x_1^2 + \dots + x_{d-1}^2 + y^2 & \text{if } \text{Crk}(f) = 0 \\ x_1^2 + \dots + x_{d-1}^2 + q(y) & \text{if } \text{Crk}(f) = 1 \end{cases}$$

for some $q \in \mathbb{C}[y]$ with $\text{ord}(q) \geq 3$.

If q is zero we are done, else after pulling out the lowest term, we can write $q = y^n u$ for some $u = c_n + c_{n+1}y + \dots \in \mathbb{C}[y]$ with $c_n \neq 0$. The homomorphism $\mathbb{C}\langle x_1, \dots, x_{d-1}, y \rangle \rightarrow \mathbb{C}\langle x_1, \dots, x_{d-1}, y \rangle$ which sends $x_k \mapsto x_k$ and $y \mapsto y \sqrt[n]{u}$ is an automorphism. Since $\sqrt[n]{u}$ is a power series only in y , it commutes with y , and so this automorphism sends $\sum x_i^2 + y^n$ to $\sum x_i^2 + y^n u = f$. Hence $\mathcal{J}\text{ac}(\sum x_i^2 + y^n) \cong \mathcal{J}\text{ac}(f) \cong \mathcal{J}\text{ac}(f)$, as required.

Parts (1)–(2) are obvious, since $\mathcal{J}\text{ac}(\sum x_i^2 + y^n) \cong \mathbb{C}[y]/(y^{n-1})$ and $\mathcal{J}\text{ac}(\sum x_i^2) \cong \mathbb{C}[y]$, and uniqueness then follows since $\sum x_i^2 + y^{n_1} \cong \sum x_i^2 + y^{n_2}$ if and only if $n_1 = n_2$. \square

Recall from the introduction our geometric applications in the setting of cDV singularities. These correspond to Jacobi algebras in two non-commuting variables, so we set $d = 2$ and write the variables as x, y .

Corollary 5.2. *Every $\mathcal{J}\text{ac}(f)$, where $f \in \mathbb{C}\langle x, y \rangle_{\geq 2}$ with $f_2 \neq 0$, is geometric.*

Proof. Consider $\mathcal{R} = \mathbb{C}\langle x, y, z \rangle^{\frac{1}{2}(1,1,0)}$, and its unique crepant resolution $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$. This has contraction algebra $\mathbb{C}[y] \cong \mathcal{J}\text{ac}(x^2)$, realising the second case in 5.1. On the other hand, by [DW1, 3.10] the Type A m -Pagoda flop (with $m \geq 1$) has contraction algebra $\mathbb{C}[y]/y^m \cong \mathcal{J}\text{ac}(x^2 + y^{m+1})$, which realises the infinite family in 5.1. \square

Example 5.3. Consider $f = x^2 + \frac{2}{3}(xy^2 + yxy + y^2x) \in \mathbb{C}\langle x, y \rangle$. This has 3-dimensional Jacobi algebra

$$\mathcal{J}\text{ac}(f) \cong \mathbb{C}\langle x, y \rangle / \langle (x + y^2, xy + yx) \rangle \cong \mathbb{C}[y]/y^3$$

so, by 5.1 or Reid's Pagoda [R2], f gives the same Jacobi algebra as $g = x^2 + y^4$. Commutatively, one would see this by completing the square, but that automorphism does not work directly in the noncommutative context: $x \mapsto x - \frac{2}{3}y^2$ gives $f \mapsto x^2 + \frac{2}{3}yxy - \frac{8}{9}y^4$, and we cannot attack the yxy term by coordinate changes that preserve $f_2 = x^2$. But $f \sim x^2 + xy^2 + y^2x$, which then allows us to complete the square (and a scalar on y) to conclude. This exemplifies the way \sim helps to navigate the Jacobi isomorphism classes.

5.2. Commutativity. The following characterisation of commutative Jacobi algebras in $d = 2$ variables will be used later.

Proposition 5.4. *$f \in \mathbb{C}\langle x, y \rangle_{\geq 2}$, then $\mathcal{J}\text{ac}(f)$ is commutative if and only if $\text{Crk}(f) \leq 1$.*

Proof. (\Leftarrow) is clear from 5.1. For (\Rightarrow), we prove the contrapositive. If $\text{Crk}(f) \not\leq 1$, then $f_2 = 0$, and we need to prove that $\mathcal{J}\text{ac}(f)$ is not commutative. For this, it suffices to exhibit a factor that is not commutative. By 4.13 without loss of generality we can assume that f equals

$$x^3 + y^3 + \mathcal{O}_4, \quad xy^2 + \mathcal{O}_4, \quad x^3 + \mathcal{O}_4 \quad \text{or} \quad \mathcal{O}_4.$$

Write \mathcal{M}_3 for the set of all noncommutative monomials of degree 3, then factor by the ideal $(\delta_x f, \delta_y f, \mathcal{M}_3) / (\delta_x f, \delta_y f)$ in $\mathcal{J}\text{ac}(f)$. But in the four cases above, by differentiating then using the third isomorphism theorem it follows that $\mathcal{J}\text{ac}(f)$ is one of

$$\frac{\mathbb{C}\langle x, y \rangle}{(x^2, y^2, \mathcal{M}_3)}, \quad \frac{\mathbb{C}\langle x, y \rangle}{(y^2, xy + yx, \mathcal{M}_3)}, \quad \frac{\mathbb{C}\langle x, y \rangle}{(x^2, \mathcal{M}_3)}, \quad \text{or} \quad \frac{\mathbb{C}\langle x, y \rangle}{(\mathcal{M}_3)}.$$

None of these factors is commutative, and so $\mathcal{J}\text{ac}(f)$ is not commutative. \square

6. TYPE D NORMAL FORMS

This section considers the next case, namely those $f \in \mathbb{C}\langle\langle x \rangle\rangle_{\geq 2}$ with $\text{Crk}(f) = 2$ and $\text{Crk}_2(f) = 2$. Reducing to two variables by the Splitting Lemma 4.5, the assumption $\text{Crk}_2(f) = 2$ is then equivalent to the first two cases in 4.13, namely those $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ with $f_3 \neq 0$ for which f_3^{ab} has either two or three distinct linear factors. Full normal forms are obtained in both situations, and are then merged into a unified form in §6.4. These are the Type D normal forms in the tables in §1.2.

Throughout this section, it will be convenient to adopt the following language.

Definition 6.1. We say that a monomial $m \in \mathbb{C}\langle\langle x, y \rangle\rangle$ contains x^2 if $m \sim nx^2$ for some monomial n , else m does not contain x^2 . Similarly, an element $f \in \mathbb{C}\langle\langle x, y \rangle\rangle$ contains x^2 if for some (nonzero) term λm of f , the monomial m contains x^2 , else f does not contain x^2 . We also use the analogous expressions for y^2 .

6.1. Abelianized Cubic with Three Factors. This subsection considers the case of 4.13 where f_3^{ab} has three distinct factors, that is $f \cong x^3 + y^3 + \mathcal{O}_4$, and in 6.5 and 6.6 provides two different, but equivalent, normal forms.

Recall from 2.2(1) that a unitriangular automorphism φ of $\mathbb{C}\langle\langle x \rangle\rangle$ has depth $e \geq 1$ if and only if $\varphi(f)_{\leq e} = f_{\leq e}$ for all $f \in \mathbb{C}\langle\langle x \rangle\rangle$.

Lemma 6.2. Fix $t \geq 4$, and let $f = x^3 + y^3 + f_4 + \cdots + f_t + \mathcal{O}_{t+1}$. For any $h_1, h_2 \in \mathbb{C}\langle\langle x, y \rangle\rangle$ with $\text{ord}(h_i) \geq t - 2$, there is a unitriangular automorphism ψ of depth $\geq t - 3$ such that

$$\psi(f) \sim x^3 + y^3 + f_4 + \cdots + f_{t-1} + (f_t - h_1x^2 - h_2y^2) + \mathcal{O}_{t+1}$$

and $\psi(f) - (x^3 + y^3 + f_4 + \cdots + f_{t-1}) \in \mathfrak{n}^t$.

Proof. Consider the unitriangular automorphism ψ which sends $x \mapsto x - \frac{1}{3}h_1$, $y \mapsto y - \frac{1}{3}h_2$. The result follows since

$$\psi(x^3 + y^3) \sim x^3 - h_1x^2 + \frac{1}{3}h_1^2x - \frac{1}{27}h_1^3 + y^3 - h_2y^2 + \frac{1}{3}h_2^2y - \frac{1}{27}h_2^3,$$

and $\psi(m) \equiv m \pmod{\mathfrak{n}^{t+1}}$ whenever $\deg(m) \geq 4$. \square

With the preparatory lemma in place, the strategy is to first find a standard power series form of each potential, and then distill that down to a polynomial normal form.

Proposition 6.3. Suppose that $f = f_3 + \mathcal{O}_4$ where f_3^{ab} has three distinct factors. Then $f \cong x^3 + y^3 + p(xy)$ for some power series $p(z) \in \mathbb{C}[[z]]$ with $\text{ord}(p) \geq 2$.

Recall the Conventions 1.8 on denoting graded pieces of sequence elements: we denote sequence elements $\mathfrak{f}_n \in \mathbb{C}\langle\langle x, y \rangle\rangle$ in Greek font, and we write $(\mathfrak{f}_n)_t$ for its degree t piece, and $(\mathfrak{f}_n)_{<t}$ and $(\mathfrak{f}_n)_{>t}$ for sub- and super-degree t portions respectively.

Proof. We construct a sequence of power series $\mathfrak{f}_1, \mathfrak{f}_2, \dots$ and unitriangular automorphisms $\varphi^1, \varphi^2, \dots$ inductively, with each \mathfrak{f}_t having the form of the target power series $x^3 + y^3 + p(xy)$ in small degree. Summary 3.7(3) then constructs $\mathfrak{f} = \lim \mathfrak{f}_i$ of the required form with $\mathcal{J}\text{ac}(f) \cong \mathcal{J}\text{ac}(\mathfrak{f})$.

By 4.13, $f \cong g$ where $g_3 = x^3 + y^3$. After grouping together terms containing x^2 or y^2 and cyclically permuting, we may write

$$g \sim x^3 + y^3 + \mathfrak{h}_2 \cdot x^2 + \mathfrak{h}'_2 \cdot y^2 + \mu_4(xy)^2 + \mathcal{O}_5$$

for $\mathfrak{h}_2, \mathfrak{h}'_2 \in \mathbb{C}\langle\langle x, y \rangle\rangle_2$ and $\mu_4 \in \mathbb{C}$.

Hence we begin the induction by setting

$$\mathfrak{f}_1 = x^3 + y^3 + \mathfrak{h}_2 \cdot x^2 + \mathfrak{h}'_2 \cdot y^2 + \mu_4(xy)^2 + g_{\geq 5}$$

and note that $\mathfrak{f}_1 \sim g \cong f$. Thus \mathfrak{f}_1 is in the desired form in degrees ≤ 3 and has its degree 4 piece prepared in standard form for further analysis.

For the inductive step more generally, we may suppose that $\mathfrak{f}_t \in \mathbb{C}\langle\langle x, y \rangle\rangle$ has been constructed of the form

$$\mathfrak{f}_t = (x^3 + y^3 + \mathfrak{p}_{t+2}(xy)) + (\mathfrak{h}_{t+1} \cdot x^2 + \mathfrak{h}'_{t+1} \cdot y^2 + \mu_{t+3}(xy)^{\lfloor (t+3)/2 \rfloor}) + \mathcal{O}_{t+4}$$

with $\mathfrak{p}_3 = 0$ and by convention $\mu_{t+3} = 0$ for even t , where

- (1) $(f_t)_{\leq t+2} = x^3 + y^3 + \mathbf{p}_{t+2}(xy)$, for some polynomial $\mathbf{p}_{t+2} \in \mathbb{C}[z]_{\geq 2}$ of degree $\leq (t+2)/2$, where the polynomials $\mathbf{p}_3, \dots, \mathbf{p}_{t+2}$ satisfy $\mathbf{p}_{i+1} = \mathbf{p}_i$ for even i and $\mathbf{p}_{i+1} - \mathbf{p}_i = \mu_{i+1}z^{(i+1)/2}$ for odd i , and
- (2) $(f_t)_{t+3} = \mathbf{h}_{t+1} \cdot x^2 + \mathbf{h}'_{t+1} \cdot y^2 + \mu_{t+3}(xy)^{\lfloor (t+3)/2 \rfloor}$ for some homogeneous forms $\mathbf{h}_{t+1}, \mathbf{h}'_{t+1}$ of degree $t+1$.

Applying 6.2 with $h_1 = \mathbf{h}_{t+1}$ and $h_2 = \mathbf{h}'_{t+1}$, there exists a unitriangular φ^t of depth $\geq t$ such that

$$\varphi^t(f_t) \sim (x^3 + y^3 + \mathbf{p}_{t+2}(xy)) + \mu_{t+3}(xy)^{\lfloor (t+3)/2 \rfloor} + \mathcal{O}_{t+4}.$$

In degree $t+4$, again grouping together the terms containing x^2 or y^2 and cyclically permuting, we may write

$$\varphi^t(f_t)_{t+4} \sim \mathbf{h}_{t+2} \cdot x^2 + \mathbf{h}'_{t+2} \cdot y^2 + \mu_{t+4}(xy)^{\lfloor (t+4)/2 \rfloor}$$

for homogeneous forms $\mathbf{h}_{t+2}, \mathbf{h}'_{t+2}$ of degree $t+2$ and some $\mu_{t+4} \in \mathbb{C}$, where again $\mu_{t+4} = 0$ for odd t . Thus, after setting $\mathbf{p}_{t+3}(xy) = \mathbf{p}_{t+2}(xy) + \mu_{t+3}(xy)^{\lfloor (t+3)/2 \rfloor}$, define

$$\mathbf{f}_{t+1} = x^3 + y^3 + \mathbf{p}_{t+3}(xy) + (\mathbf{h}_{t+2} \cdot x^2 + \mathbf{h}'_{t+2} \cdot y^2 + \mu_{t+4}(xy)^{\lfloor (t+4)/2 \rfloor}) + \varphi^t(f_t)_{\geq t+5}.$$

Note that $\varphi^t(f_t) \sim \mathbf{f}_{t+1}$, and $\varphi^t(f_t) - \mathbf{f}_{t+1} \in \mathfrak{n}^{t+3} \subset \mathfrak{n}^{t+1}$ using the last statement of 6.2.

Thus we have constructed a sequence of power series $\mathbf{f}_1, \mathbf{f}_2, \dots$ and unitriangular automorphisms $\varphi^1, \varphi^2, \dots$ to which 3.7(3) applies. For $s \geq 3$ either s is even, in which case $\mathbf{p}_{s+1} = \mathbf{p}_s$, or s is odd, in which case $\mathbf{p}_{s+1} = \mathbf{p}_s + \mu_{s+1}z^{(s+1)/2}$, thus it is clear that $p := \lim \mathbf{p}_s = \sum_{s=2}^{\infty} \mu_{2s}z^s$. Further, $\mathbf{f} = \lim \mathbf{f}_i = x^3 + y^3 + p(xy)$, since the difference $(\mathbf{f}_i - (x^3 + y^3 + p(xy)))_{i \geq 1}$ converges to zero.

It follows from 3.7(3) that $\mathcal{J}ac(f) \cong \mathcal{J}ac(x^3 + y^3 + p(xy))$, as required. \square

The next step is to replace the power series $p(xy)$ by its leading term, without changing the Jacobi algebra.

Lemma 6.4. *If $f = x^3 + y^3 + p(xy) \in \mathbb{C}\langle x, y \rangle$ for some $0 \neq p(z) \in \mathbb{C}[z]$ for which $s = \text{ord}(p) \geq 2$, then the following statements hold.*

- (1) $yx^2, xy^2 \in ((\delta_x f, \delta_y f))$.
- (2) $x^2y, y^2x \in ((\delta_x f, \delta_y f))$.
- (3) $(xy)^s x, (yx)^s y \in ((\delta_x f, \delta_y f))$.

Proof. (1) Write $J_f = (\delta_x f, \delta_y f)$, so that $((\delta_x f, \delta_y f))$ is the closure of J_f . Differentiating and pulling out the lowest terms, write

$$\delta_x f = 3x^2 + y(xy)^{s-1}q(xy) \quad \text{and} \quad \delta_y f = 3y^2 + q(xy)(xy)^{s-1}x \quad (6.A)$$

for some $q = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \dots \in \mathbb{C}[z]$ with $\lambda_0 \neq 0$. Writing $A \equiv B$ for $A - B \in J_f$, then in particular

$$x^2 \equiv -\frac{1}{3}y(xy)^{s-1}q(xy) \quad \text{and} \quad y^2 \equiv -\frac{1}{3}q(xy)(xy)^{s-1}x. \quad (6.B)$$

Substituting for x^2 or y^2 at each step, we see that

$$\begin{aligned} y \cdot x^2 &\equiv \frac{-1}{3}y^2 \cdot (xy)^{s-1}q(xy) && \in \mathfrak{n}^{2s} \\ &\equiv \frac{\pm 1}{3^2}q(xy)(xy)^{s-1}x \cdot (xy)^{s-1}q(xy) \\ &= \frac{\pm 1}{3^2}q(xy)(xy)^{s-1} \cdot x^2 \cdot y(xy)^{s-2}q(xy) && \in \mathfrak{n}^{4s-3} \\ &\equiv \frac{-1}{3^3}q(xy)(xy)^{s-1} \cdot y(xy)^{s-1}q(xy) \cdot y(xy)^{s-2}q(xy) \\ &= \frac{-1}{3^3}q(xy)(xy)^{s-2}x \cdot y^2 \cdot (xy)^{s-1}q(xy)y(xy)^{s-2}q(xy) && \in \mathfrak{n}^{6s-6} \\ &= \dots \end{aligned}$$

At each substitution, the resulting power series has order $2s - 3 \geq 1$ higher than the previous one. It follows from the above that for all $t \geq 2s$ there exists $n_t \in \mathfrak{n}^t$ such that $yx^2 - n_t \in J_f$. Hence $yx^2 \in \bigcap_{t \geq 2s} (J_f + \mathfrak{n}^t)$, which is precisely the closure $((\delta_x f, \delta_y f))$. By symmetry in x and y , the analogous statement $xy^2 \in ((\delta_x f, \delta_y f))$ also follows.

(2) This follows in an analogous way: start by writing

$$\delta_x f = 3x^2 + r(yx)(yx)^{s-1}y \quad \text{and} \quad \delta_y f = 3y^2 + x(yx)^{s-1}r(yx)$$

then consider $x^2 \cdot y \equiv \frac{-1}{3}r(yx)(yx)^{s-1}y^2$, etc.

(3) Now write $A \equiv B$ for $A - B \in \langle \delta_x f, \delta_y f \rangle$. Separating off the lowest term of $q(xy)$ in (6.A), we may write

$$\delta_y f = 3y^2 + \lambda_0(xy)^{s-1}x + (q(xy) - \lambda_0)(xy)^{s-1}x$$

and so $\lambda_0(xy)^{s-1}x \equiv -3y^2 - (q(xy) - \lambda_0)(xy)^{s-1}x$ where $q(xy) - \lambda_0 \in \mathfrak{n}^2$. Then

$$\begin{aligned} \lambda_0(xy)^s x &= (\lambda_0(xy)^{s-1}x) yx \\ &\equiv (-3y^2 - (q(xy) - \lambda_0)(xy)^{s-1}x) (yx) \\ &= -3y \cdot y^2 \cdot x - (q(xy) - \lambda_0)(xy)^s x \\ &\equiv y \cdot q(xy)(xy)^{s-1}x \cdot x - (q(xy) - \lambda_0)(xy)^s x && \text{(by (6.B))} \\ &\equiv -(q(xy) - \lambda_0)(xy)^s x && (yx^2 \equiv 0 \text{ by (1)}) \end{aligned}$$

The $\lambda_0(xy)^s x$ on each side cancel, showing that $q(xy)(xy)^s x \in \langle \delta_x f, \delta_y f \rangle$. Since $q(xy)$ is a unit, it follows that $(xy)^s x \in \langle \delta_x f, \delta_y f \rangle$. Again, appealing to symmetry in x and y proves the final statement. \square

Proposition 6.5. *Suppose that $f = x^3 + y^3 + p(xy)$ where $p(z) \in \mathbb{C}[[z]]$ with $s = \text{ord}(p) \geq 2$. Then*

$$f \cong \begin{cases} x^3 + y^3 & \text{when } p = 0 \\ x^3 + y^3 + (xy)^s & \text{when } p \neq 0. \end{cases}$$

Furthermore

- (1) $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$, with equality if and only if $p = 0$.
- (2) If $p \neq 0$, then $\dim_{\mathbb{C}} \mathcal{J}\text{ac}(f) = 4s$, and $\dim_{\mathbb{C}} \mathcal{J}\text{ac}(f)^{\text{ab}} = 4$.

Therefore the expressions $x^3 + y^3 + (xy)^s$ with $s \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ form a set of normal forms.

Proof. For the first statement, if $p = 0$ we are done, so suppose $p \neq 0$. Continuing the notation in the proof of 6.4 above, after differentiating and pulling out the lowest terms, we may write

$$\delta_x f = 3x^2 + y(xy)^{s-1}q(xy) \quad \text{and} \quad \delta_y f = 3y^2 + q(xy)(xy)^{s-1}x \quad (6.C)$$

for some $q = \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \dots \in \mathbb{C}[[z]]$ with $\lambda_0 \neq 0$. Set $g = x^3 + y^3 + (\lambda_0/s)(xy)^s$. Now 6.4 applies equally well to both f and g , hence both $(xy)^s x$ and $(yx)^s y$ belong to both the Jacobi ideals associated to f and g . Consequently

$$\begin{aligned} \langle \delta_x f, \delta_y f \rangle &= \langle \delta_x f, \delta_y f, (xy)^s x, (yx)^s y \rangle \\ &= \langle \delta_x g, \delta_y g, (xy)^s x, (yx)^s y \rangle && \text{(cancel higher terms from } \delta_x f \text{ and } \delta_y f) \\ &= \langle \delta_x g, \delta_y g \rangle. \end{aligned}$$

It follows that $f \simeq g$. The coordinate change $x \mapsto ax$, $y \mapsto ay$ for $a = \sqrt[2s-3]{s/\lambda_0}$ then normalises the constant factor $\lambda_0/s \neq 0$, as required.

(1) Consider the case $p = 0$. As in 3.2, if \mathfrak{J} is the Jacobson radical of $\mathcal{J}\text{ac}(f)$, then

$$\frac{\mathfrak{J}^d}{\mathfrak{J}^{d+1}} = \frac{\mathfrak{n}^d + \langle (x^2, y^2) \rangle}{\mathfrak{n}^{d+1} + \langle (x^2, y^2) \rangle}.$$

This is always a two-dimensional vector space, since if d is even it has basis $(xy)^{d/2}$ and $(yx)^{d/2}$, whilst if d is odd it has basis $x(yx)^{(d-1)/2}$ and $y(xy)^{(d-1)/2}$. It follows that $\mathcal{J}\text{ac}(f)$ is an infinite-dimensional \mathbb{C} -algebra, with $\text{Jdim } \mathcal{J}\text{ac}(f) = 1$.

When $p \neq 0$, 6.4 shows at once that $\mathcal{J}\text{ac}(f)$ is finite dimensional: indeed any monomial of degree $t \geq 2s + 1$ either contains one of the monomials listed in 6.4(1–3) or is x^t or y^t . But by (6.C) $x^t = x^{t-4} \cdot x^2 \cdot x^2$ and $y^t = y^2 \cdot y^2 \cdot y^{t-4}$ is equivalent, modulo $\langle \delta_x f, \delta_y f \rangle$, to a monomial that contains one of those listed, and thus is equivalent to zero. Consequently, the entire graded piece of degree t is zero, and so $\mathcal{J}\text{ac}(f)$ is finite dimensional.

(2) We compute a suitable Gröbner basis with a local graded monomial order, that is, the leading terms are those of lowest order (there is no need here for lexicographical considerations to break ties). The proof of part (1) above introduces a key simplifying

factor: since all monomials of degree $t = 2s + 1$ lie in the closed Jacobian ideal, it is sufficient to work in the quotient

$$\mathcal{J}ac(f) \cong \frac{\mathbb{C}\langle x, y \rangle}{(\delta_x f, \delta_y f, \mathcal{M}_t)}$$

where \mathcal{M}_t for the set of all noncommutative monomials of degree t , where now closure is no longer an issue.

We compile a Gröbner basis $\{g_1, g_2, \dots\}$ of $(\delta_x f, \delta_y f, \mathcal{M}^t)$ starting with

$$g_1 = 3x^2 + sy(xy)^{s-1} \quad \text{and} \quad g_2 = 3y^2 + sx(yx)^{s-1}$$

which have leading terms $3x^2$ and $3y^2$ respectively, as $s \geq 2$. The computation proceeds by resolving non-trivial overlaps among leading terms. The leading term of g_1 overlaps non-trivially with itself to produce

$$g_3 = xg_1 - g_1x = s(xy)^s - s(yx)^s$$

which does not reduce further modulo existing leading terms. The analogous overlap $yg_2 - g_2y$ gives the same g_3 , and all other overlaps have order $\geq t$, so are zero in the ideal. Hence the Gröbner basis is $\{g_1, g_2, g_3\}$.

This Gröbner basis provides a monomial basis for the quotient as follows. Among all monomials of degree $< t$, any of the form $m_1x^2m_2$, for monomials m_1, m_2 , reduce modulo g_1 to zero at once, and similarly with y^2 . Thus a basis consists of all other monomials, with the single relation g_3 . That is, in each pair of degrees $2e, 2e + 1$ for $1 \leq e \leq s - 1$ there are 4 monomials

$$(xy)^e, (yx)^e, (xy)^e x, (yx)^e y,$$

and then $1, x, y$ in degrees ≤ 1 and finally $(xy)^s \equiv (yx)^s$ in degree $2s$. Summing up, this basis has size $4s$, as claimed.

In the abelianisation, if $s > 2$ we may rewrite the derivatives as $x^2(\text{unit})$ and $y^2(\text{unit})$. Hence $\mathcal{J}ac(f)^{\text{ab}} \cong \mathbb{C}\llbracket x, y \rrbracket / (x^2, y^2)$, which is four dimensional. When $s = 2$, it is also easy to verify that $\mathcal{J}ac(f)^{\text{ab}} \cong \mathbb{C}\llbracket x, y \rrbracket / (x^2, y^2)$, so in all cases the dimension is four. \square

In order to state a unified theorem with the xy^2 case in §6.4 below, it is convenient to mildly change basis. This is rather cheap, largely because there are no moduli.

Corollary 6.6. *Suppose that $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ where f_3^{ab} has three roots. Then either*

$$f \cong \begin{cases} xy^2 + x^3 \\ xy^2 + x^3 + x^{2n} \end{cases} \quad \text{for some } n \geq 2.$$

Furthermore, the above are normal forms.

Proof. Each of the forms f listed satisfies the condition that f_3^{ab} has three roots. Thus there exists some g from the list in 6.5 with $g \cong h$. Furthermore, $\dim_{\mathbb{C}} \mathcal{J}ac(xy^2 + x^3) = \infty$, whereas $\dim_{\mathbb{C}} \mathcal{J}ac(xy^2 + x^3 + x^{2n}) = 4n$ (see e.g. [vG, §5] and [Ka, §5], or 4.11), and so all options are uniquely covered. Since the g listed in 6.5 are normal forms, it follows that the f listed here are normal forms. \square

6.2. Isomorphisms on the Quantum Plane. The following, which may be of independent interest, is one of the key reduction steps that will be used in §6.3.

Lemma 6.7. *For any units $v, w \in \mathbb{C}\llbracket x^2 \rrbracket$, the unitriangular automorphism φ of $\mathbb{C}\langle\langle x, y \rangle\rangle$ sending $x \mapsto xv$, $y \mapsto yw$ descends to a topological isomorphism*

$$\frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{\langle\langle xy + yx \rangle\rangle} \xrightarrow{\cong} \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{\langle\langle xy + yx \rangle\rangle}.$$

Proof. The inverse of φ , as an automorphism of $\mathbb{C}\langle\langle x, y \rangle\rangle$, is clearly given by the unitriangular automorphism $\psi: x \mapsto xv^{-1}$, $y \mapsto yw^{-1}$. Set $I = \langle\langle xy + yx \rangle\rangle$, then since φ is a topological isomorphism by 2.3, we just need to prove that $\varphi(I) = I$.

Now x commutes with v and w , being power series in x^2 , and also $vw = wv$. But, modulo $I = \langle\langle xy + yx \rangle\rangle$, y commutes with x^2 , thus since the ideal is closed y commutes with both v and w . It follows that

$$\varphi(xy + yx) = xvyw + yw xv \equiv (xy + yx)vw \equiv 0 \pmod{I}. \quad (6.D)$$

Since φ is a continuous isomorphism, and I is the smallest closed ideal containing $xy + yx$, $\varphi(I)$ is the smallest closed ideal containing $\varphi(xy + yx)$. But by (6.D) $\varphi(xy + yx)$ also belongs to the closed ideal I , so by minimality $\varphi(I) \subseteq I$.

Since v^{-1} and w^{-1} are also units in $\mathbb{C}[[x^2]]$, exactly the same logic applied to ψ shows that $\psi(I) \subseteq I$. Applying φ to this inclusion, we see that $I = \varphi\psi(I) \subseteq \varphi(I)$. Combining inclusions gives $\varphi(I) = I$. \square

6.3. Abelianized Cubic with Two Factors. This subsection considers the case of 4.13 where f_3^{ab} has two distinct factors, that is $f \cong x^2y + \mathcal{O}_4$, and in 6.17 provides normal forms. This is substantially harder than in §6.1.

Lemma 6.8. *Fix $t \geq 4$, and let $f = xy^2 + f_4 + \cdots + f_t + \mathcal{O}_{t+1}$. For any $h \in \mathbb{C}\langle x, y \rangle$ with $\text{ord}(h) = t - 2$, the unitriangular automorphism $x \mapsto x - h, y \mapsto y$ sends*

$$f \mapsto xy^2 + f_4 + \cdots + f_{t-1} + (f_t - hy^2) + \mathcal{O}_{t+1}.$$

Proof. Write ψ for the stated automorphism. The result follows since $\psi(xy^2) = xy^2 - hy^2$ and $\psi(m) \equiv m \pmod{\mathfrak{n}^{t+1}}$ whenever $\deg(m) \geq 4$. \square

The next lemma is much less elementary.

Lemma 6.9. *Fix $t \geq 4$, and let $f = xy^2 + f_4 + \cdots + f_t + \mathcal{O}_{t+1}$, where furthermore*

$$f_t = \mathfrak{h}_{t-2} \cdot y^2 + \sum \lambda_a x^{a_1} y \cdots x^{a_r} y + \alpha x^t \quad (6.E)$$

for some homogeneous form \mathfrak{h}_{t-2} of degree $t - 2$, each $a_i \geq 1$, $r \geq 1$ and $r + \sum a_i = t$, $\alpha \in \mathbb{C}$ and each $\lambda_a = \lambda_{a_1 \cdots a_r} \in \mathbb{C}$. Then there exists a unitriangular automorphism φ of depth $\geq t - 3$ such that

$$\varphi(f) = xy^2 + f_4 + \cdots + f_{t-1} + (g_t + \alpha x^t) + \mathcal{O}_{t+1}.$$

where $g_t \in \mathbb{C}\langle x, y \rangle_t$ satisfies $g_t \sim 0$.

Example 6.10. It is worth considering an example to make the notation of both the statement and proof more transparent. Consider

$$f = xy^2 + \left(\lambda_{51} x^5 y x y + \lambda_{42} x^4 y x^2 y + \lambda_{33} x^3 y x^3 y \right) + f_{\geq 9}$$

which has f_8 of the form (6.E). Applying $\varphi_1: x \mapsto x$ and $y \mapsto y - \lambda_{33} x^3 y x^2$, where we cancelled xy from the right of the target λ_{33} term to obtain the subtracted term, gives

$$\varphi_1(f) = xy^2 + \lambda_{51} x^5 y x y + (\lambda_{42} - \lambda_{33}) x^4 y x^2 y + g_1 + \mathcal{O}_9$$

where $g_1 = \lambda_{33}(x^3 y x^3 y - x y x^3 y x^2) \sim 0$. Ignoring g_1 , the summation in degree 8 symbolically now has only two terms, which is progress.

An analogous automorphism φ_2 sending $x \mapsto x$ and $y \mapsto -(\lambda_{42} - \lambda_{33}) x^4 y x$, where we cancelled xy from the right of the next target term, gives

$$\varphi_2 \varphi_1(f) = xy^2 + (\lambda_{51} - \lambda_{42} + \lambda_{33}) x^5 y x y + g_2 + \mathcal{O}_9$$

for some $g_2 \sim 0$, and again the number of terms in degree 8 (outside g_2) has not increased. Repeating again with an analogous automorphism φ_3 gives

$$\varphi_3 \varphi_2 \varphi_1(f) = xy^2 - (\lambda_{51} - \lambda_{42} + \lambda_{33}) x^6 y^2 + g_3 + \mathcal{O}_9$$

with $g_3 \sim 0$. We are now in a position to apply 6.8 to leave only g_3 in degree 8.

The proof below confirms that this inductive idea works more generally.

Proof. If the middle sum in the expression for f_t is zero, we are done by 6.8 (with $g_t = 0$), so we may assume that the sum is nonzero.

Suppose that the middle sum contains a term $\mathfrak{t}_1 = \lambda_b x^{b_1} y \cdots x^{b_r} y$ with $r > 1$. In this case, consider the unitriangular automorphism φ defined by $x \mapsto x, y \mapsto y - \lambda_b x^{b_1} y \cdots y x^{b_r - 1}$, where we have simply cancelled xy from the right-hand side of the target term \mathfrak{t}_1 . As in 6.8, $\varphi(m) \equiv m \pmod{\mathfrak{n}^{t+1}}$ whenever $\deg(m) \geq 4$, so any change in degree $\leq t$ comes from $\varphi(xy^2)$, and thus

$$\begin{aligned} \varphi(f) &= xy^2 + f_4 + \cdots + f_{t-1} \\ &\quad + (f_t - \lambda_b x y x^{b_1} y \cdots y x^{b_r - 1} - \lambda_b x^{b_1 + 1} y \cdots y x^{b_r - 1} y) + \mathcal{O}_{t+1} \end{aligned} \quad (6.F)$$

Writing $\mathbf{g}_1 = \mathbf{t}_1 - \lambda_b x y x^{b_1} y \cdots y x^{b_r-1} \sim 0$, then the degree t term of (6.F) equals

$$\mathbf{g}_1 + h \cdot y^2 + \sum \lambda_a x^{a_1} y \cdots x^{a_r} y + \alpha x^t$$

where, under the summand, the target term \mathbf{t}_1 has been replaced by a term of the form $\mathbf{t}_2 = -\lambda_b x^{b_1+1} y \cdots y x^{b_r-1} y$, so the sum has the same number of terms or fewer (depending on whether \mathbf{t}_2 cancels with existing terms or not).

If $b_r = 1$, then the new term \mathbf{t}_2 equals $h y^2$ for $h = -\lambda_b x^{b_1+1} y \cdots y x^{b_r-1}$, and we so may apply 6.8 to (6.F) to find ψ such that

$$\psi \varphi(f) = x y^2 + f_4 + \cdots + f_{t-1} + (f_t - \lambda_b x y x^{b_1} y \cdots y x^{b_r-1}) + \mathcal{O}_{t+1}$$

where the degree t term is equal to

$$\begin{aligned} f_t - \lambda_b x y x^{b_1} y \cdots y x^{b_r-1} &= \mathbf{g}_1 + f_t - \lambda_b x^{b_1} y \cdots x^{b_r} y \\ &= \mathbf{g}_1 + h \cdot y^2 + \sum_{a \neq b} \lambda_a x^{a_1} y \cdots x^{a_r} y + \alpha x^t \end{aligned}$$

and the number of terms under the summand is now strictly reduced.

Otherwise, $b_r > 1$. Set $\varphi_1 = \varphi$, and repeating the original construction of a unitriangular automorphism by cancelling xy from the right, we can construct φ_2 such that

$$\begin{aligned} \varphi_2 \varphi_1(f) &= x y^2 + f_4 + \cdots + f_{t-1} \\ &\quad + (\mathbf{g}_2 + h \cdot y^2 + \sum \lambda_a x^{a_1} y \cdots x^{a_r} y + \alpha x^t) + \mathcal{O}_{t+1} \end{aligned} \quad (6.G)$$

where $\mathbf{g}_2 \sim 0$ is the sum of \mathbf{g}_1 and another binomial ~ 0 , and in the sum we have replaced the term $-\lambda_b x^{b_1+1} y \cdots y x^{b_r-1} y$ by $\lambda_b x^{b_1+2} y \cdots y x^{b_r-2} y$. Repeating this, we find unitriangular automorphisms $\varphi_1, \dots, \varphi_{b_r-1}$ so that $\varphi_{b_r-1} \cdots \varphi_2 \varphi_1(f)$ has the form of (6.G), and the sum has the same number of terms or fewer, but in which the target monomial we are focussing on has become $x^{b_1+b_r-1} y x^{b_2} \cdots y x y$. A further repetition with a unitriangular automorphism φ_{b_r} replaces that term by one that contains y^2 , and once again we may apply 6.8 to find a unitriangular automorphism ψ that moves this term into higher degree. Thus after applying the single unitriangular automorphism $\psi \varphi_{b_r} \cdots \varphi_1$ to f , the number of terms in the summation when parsed in the form (6.E) has strictly reduced.

We repeat this process inductively, and it will terminate when there are no terms under the summation sign of the form $\lambda_a x^{a_1} y \cdots x^{a_r} y$ with $r > 1$. Each step was achieved by a single unitriangular automorphism (itself built as a composition of unitriangular automorphisms), and composing each of these gives a single unitriangular automorphism ϑ such that

$$\vartheta(f) = x y^2 + f_4 + \cdots + f_{t-1} + (\mathbf{g} + h \cdot y^2 + \lambda x^{t-1} y + \alpha x^t) + \mathcal{O}_{t+1}$$

for some \mathbf{g} with $\mathbf{g} \sim 0$.

To conclude, the unitriangular automorphism φ defined by $x \mapsto x-h, y \mapsto y - \frac{\lambda}{2} x^{a_1-1} = y - \frac{\lambda}{2} x^{t-2}$ has depth $t-3$, so again $\varphi(m) \equiv m \pmod{\mathfrak{n}^{t+1}}$ whenever $\deg(m) \geq 4$ and thus

$$\varphi \vartheta(f) = x y^2 + f_4 + \cdots + f_{t-1} + (\mathbf{g} + \underbrace{\lambda x^{t-1} y - \frac{\lambda}{2} x y x^{t-2} - \frac{\lambda}{2} x^{t-2} y}_{\mathbf{h}} + \alpha x^t) + \mathcal{O}_{t+1}.$$

Set $g_t = \mathbf{g} + \mathbf{h}$, then since both $\mathbf{g} \sim 0$ and $\mathbf{h} \sim 0$, we are done. \square

From here, the strategy of §6.1 remains: first find a standard power series form of each potential, then simplify into polynomial normal form.

Proposition 6.11. *Suppose that $f = f_3 + \mathcal{O}_4$ where f_3^{ab} has two distinct linear factors. Then $f \cong xy^2 + q(x)$ for some power series $q(x) \in \mathbb{C}[[x]]$ with $\text{ord}(q) \geq 4$.*

Recall the Conventions 1.8, used in 6.3, on graded pieces of sequence elements: namely sequence elements $f_n \in \mathbb{C}\langle\langle x, y \rangle\rangle$ are in Greek font, whilst $(f_n)_t$ is the degree t piece of f_n .

Proof. We construct a sequence of power series f_1, f_2, \dots and unitriangular automorphisms $\varphi^1, \varphi^2, \dots$ inductively, with each f_t having the form of the target power series $xy^2 + q(x)$ in low degree. Summary 3.7(3) will then construct $f = \lim f_i$ of the required form with $\mathcal{J}\text{ac}(f) \cong \mathcal{J}\text{ac}(f)$.

By 4.13, $f \cong g$ where $g_3 = xy^2$. After grouping together the terms containing y^2 , then the terms that contain y but not y^2 , and cyclically permuting, we may write

$$g_4 \sim h_2 \cdot y^2 + \sum \lambda_a x^{a_1} y \dots x^{a_r} y + \mu_4 x^4$$

for $h_2 \in \mathbb{C}\langle x, y \rangle_2$, $r \geq 1$ and each $a_i \geq 1$ and $\mu_4 \in \mathbb{C}$ and where we use the abbreviated notation $\lambda_a := \lambda_{a_1 \dots a_r} \in \mathbb{C}$. It is convenient to write the sum as $\sum \lambda_a x^{a_1} y \dots x^{a_r} y$ by analogy with the general case, noting that here it is nothing more than $\lambda_{11} xyxy + \lambda_3 x^3 y$.

Hence we begin the induction by setting

$$f_1 = xy^2 + (h_2 \cdot y^2 + \sum \lambda_a x^{a_1} y \dots x^{a_r} y + \mu_4 x^4) + g_{\geq 5}$$

and note that $f_1 \sim g \cong f$. Thus f_1 is in the desired form in degrees ≤ 3 and has its degree 4 piece prepared in standard form for further analysis.

For the inductive step more generally, we may suppose that $f_t \in \mathbb{C}\langle x, y \rangle$ has been constructed of the form

$$f_t = (xy^2 + \mathbf{q}_{t+2}(x)) + (h_{t+1} \cdot y^2 + \sum \lambda_a x^{a_1} y \dots x^{a_r} y + \mu_{t+3} x^{t+3}) + \mathcal{O}_{t+4}$$

with $\mathbf{q}_3 = 0$ and

- (1) $(f_t)_{\leq t+2} = xy^2 + \mathbf{q}_{t+2}(x)$, for some polynomial $\mathbf{q}_{t+2} \in \mathbb{C}[x]_{\geq 4}$ of degree $\leq t+2$, where the polynomials $\mathbf{q}_3, \dots, \mathbf{q}_{t+2}$ satisfy $\mathbf{q}_{i+1} - \mathbf{q}_i = \mu_{i+1} x^{i+1}$ for $\mu_{i+1} \in \mathbb{C}$, and
- (2) $(f_t)_{t+3} = h_{t+1} \cdot y^2 + \sum \lambda_a x^{a_1} y \dots x^{a_r} y + \mu_{t+3} x^{t+3}$ for some homogeneous form h_{t+1} of degree $t+1$, each $a_i \geq 1$, $r \geq 1$ and $r + \sum a_i = t+3$, and $\mu_{t+3} \in \mathbb{C}$.

By 6.9 there exists a unitriangular φ^t of depth t such that

$$\varphi^t(f_t) = xy^2 + q_{t+2}(x) + (k_{t+3} + \mu_{t+3} x^{t+3}) + \mathcal{O}_{t+4}.$$

where $k_{t+3} \sim 0$. In degree $t+4$, again grouping together the terms containing y^2 , then the terms that contain y but not y^2 , and cyclically permuting, we may write

$$\varphi^t(f_t)_{t+4} \sim h_{t+2} \cdot y^2 + \sum \lambda_a x^{a_1} y \dots x^{a_r} y + \mu_{t+4} x^{t+4}.$$

Thus after setting $\mathbf{q}_{t+3}(x) = \mathbf{q}_{t+2}(x) + \mu_{t+3} x^{t+3}$, define

$$f_{t+1} = xy^2 + \mathbf{q}_{t+3}(x) + (h_{t+2} \cdot y^2 + \sum \lambda_a x^{a_1} y \dots x^{a_r} y + \mu_{t+4} x^{t+4}) + \varphi^t(f_t)_{\geq t+5}.$$

Note that $\varphi^t(f_t) \sim f_{t+1}$, and that $\varphi^t(f_t) - f_{t+1} \in \mathfrak{n}^{t+3} \subset \mathfrak{n}^{t+1}$.

Thus we have constructed a sequence of power series f_1, f_2, \dots and unitriangular automorphisms $\varphi^1, \varphi^2, \dots$ to which 3.7(3) applies. Since at each stage $\mathbf{q}_s = \mathbf{q}_{s-1} + \mu_s x^s$, it is clear that $q := \lim \mathbf{q}_s = \sum_{s=4}^{\infty} \mu_s x^s$, and that $f = \lim f_i = xy^2 + q$, since the difference $(f_i - (xy^2 + q))_{i \geq 1}$ converges to zero. Hence $\text{Jac}(f) \cong \text{Jac}(xy^2 + q)$, as required. \square

The next step is to reduce the options for $q(x)$, using the following preliminary lemma.

Lemma 6.12. *Let $u \in \mathbb{C}\llbracket x \rrbracket$ be an even power series: that is, u is a power series in x^2 .*

- (1) *If u is a unit, then u^{-1} and $\sqrt[n]{u}$ are also even power series for any $n \geq 2$.*
- (2) *Let $U \in \mathbb{C}\llbracket x \rrbracket$ be a unit and $n \in \mathbb{Z}$ a nonzero integer. Then there is a unit $t \in \mathbb{C}\llbracket x \rrbracket$ with $t^n = U(xt)$. Furthermore, if U is even then t is even.*

Proof. (1) Consider $v \in \mathbb{C}\llbracket z \rrbracket$ with $u(x) = v(x^2)$. If u is a unit, then v is a unit and v^{-1} and $\sqrt[n]{v} \in \mathbb{C}\llbracket z \rrbracket$ for all $n \geq 2$. Then $u^{-1}(x) = v^{-1}(x^2)$ and $\sqrt[n]{u}(x) = \sqrt[n]{v}(x^2)$.

(2) Write $U = a_0 + a_1 x + a_2 x^2 + \dots$ with $a_0 \neq 0$. Consider the case $n > 0$. We show that we may solve inductively for the coefficients b_d of the expansion $t = b_0 + b_1 x + b_2 x^2 + \dots$ in the equation $t^n = U(xt)$.

It is clear that the coefficient of x^d in t^n is a sum of $nb_0^{n-1} b_d$ with terms involving only coefficients b_i with $i < d$. On the other side of the equation, the coefficient of x^d in $U(xt)$ is a sum of terms involving a_i and b_j with $i \leq d$ and $j < d$. Putting these together, b_d does not appear in the coefficient of x^i for any $i < d$, and it appears linearly with nonzero coefficient for the first time in the coefficient of x^d , and so we may solve for it. Working inductively in increasing $d \geq 0$, and taking the limit, determines t as claimed. For $n < 0$, the same argument proves the existence of the unit t^{-1} , which is equivalent.

Suppose that U is even, and let b_{2n+1} be the smallest nonzero odd-degree coefficient of t . Then the odd-degree term with smallest degree in t^n is $nb_0^{n-1} b_{2n+1} x^{2n+1}$, while in

$U(xt)$ it is $2a_2b_0b_{2n+1}x^{2n+3}$ which appears in the summand $a_2(xt)^2$, a contradiction. So t must be even. \square

The point is now simple: y is under control, and so there is a relation $xy + yx$ in the Jacobi algebra. Then 6.7 yields the following key preparation result.

Proposition 6.13. *If $f = xy^2 + p(x)$ where $p(x) \in \mathbb{C}\llbracket x \rrbracket$ with $\text{ord}(p) \geq 4$, then*

$$f \cong xy^2 + \alpha x^{2n} + \beta x^{2m+1}$$

for some $n, m \geq 2$, and some $\alpha, \beta \in \{0, 1\}$. Furthermore, $\alpha = 1$ if and only if p has a nonzero even-degree term, in which case $2n$ is the least even degree appearing in p , and similarly the analogous criterion for $\beta = 1$ and least odd degree term in p .

Proof. First note that

$$\mathcal{J}\text{ac}(f) = \mathbb{C}\langle\langle x, y \rangle\rangle / ((xy + yx, y^2 + \delta_x p)).$$

We exhibit an automorphism of $\mathbb{C}\langle\langle x, y \rangle\rangle$ that takes the two generators of the Jacobian ideal to $(xy + yx)(\text{unit})$ and $(y^2 + \beta x^{2m} + \alpha x^{2n-1})(\text{unit})$, respectively, where either $\alpha = 0$ or $2n \geq 4$ is the least even degree appearing in p , and either $\beta = 0$ or $2m + 1 \geq 5$ is the least odd degree appearing in p . This proves all the claims.

Parsing $\delta_x p$ into even and odd terms, write the Jacobi algebra relations as

$$xy + yx \quad \text{and} \quad y^2 + ax^{2N}u + bx^{2M-1}v$$

where $u, v \in \mathbb{C}\llbracket x^2 \rrbracket$ are each either a unit or zero, $N, M \geq 2$, and $a, b \in \mathbb{C}$ are any nonzero numbers that carry through the calculation undisturbed; we choose $a = 2N + 1$ and $b = 2M$ at the end.

Suppose in the first place that $u \neq 0$, then fix a square root $s = \sqrt{u} \in \mathbb{C}\llbracket x^2 \rrbracket$ and consider the unitriangular automorphism φ sending $x \mapsto x$, $y \mapsto ys$. By 6.7 this induces a topological isomorphism

$$\bar{\varphi}: \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((xy + yx))} \xrightarrow{\sim} \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((xy + ys))}.$$

In the codomain of this map, y commutes with x^2 and thus commutes with $s \in \mathbb{C}\llbracket x^2 \rrbracket$. It follows that $\bar{\varphi}(y^2) = ysys = y^2s^2 = yu$ and thus

$$\bar{\varphi}(y^2 + \delta_x p) = (y^2 + ax^{2N})u + bx^{2M-1}v.$$

By 2.11(1)(3) after right multiplying by the unit u^{-1} , we obtain an isomorphism

$$\frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((xy + yx, y^2 + \delta_x p))} \xrightarrow{\sim} \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((xy + yx, y^2 + ax^{2N} + bx^{2M-1}\frac{v}{u}))}. \quad (6.H)$$

If $v = 0$, then (6.H) asserts that $\mathcal{J}\text{ac}(f) \cong \mathcal{J}\text{ac}(xy^2 + x^{2N+1})$, and so we are done. Hence we may assume that $v \neq 0$.

As u and v are both unit power series in x^2 , so is $\frac{v}{u}$. By 6.12(2), since $2N - 2M + 1$ is nonzero, we may choose a unit $t \in \mathbb{C}\llbracket x^2 \rrbracket$ such that $t^{2N} = t^{2M-1}v(xt)/u(xt)$. Consider the unitriangular automorphism ψ sending $x \mapsto xt$, $y \mapsto yt^N$. Again by 6.7 there is an induced topological isomorphism

$$\bar{\psi}: \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((xy + yx))} \xrightarrow{\sim} \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((xy + yx))}.$$

Clearly x commutes with $t, t^N \in \mathbb{C}\llbracket x^2 \rrbracket$, and further in the codomain of $\bar{\psi}$ the element y commutes with x^2 and thus commutes with $t, t^N \in \mathbb{C}\llbracket x^2 \rrbracket$. Thus

$$\begin{aligned} \bar{\psi}(y^2 + ax^{2N} + bx^{2M-1}\frac{v}{u}) &= y^2t^{2N} + ax^{2N}t^{2N} + bx^{2M-1}t^{2M-1}u(xt)/v(xt) \\ &= (y^2 + ax^{2N} + bx^{2M-1})t^{2N}. \end{aligned}$$

Again 2.11(1)(3) then induces an isomorphism

$$\frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((xy + yx, y^2 + ax^{2N} + bx^{2M-1}\frac{v}{u}))} \xrightarrow{\sim} \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((xy + yx, y^2 + ax^{2N} + bx^{2M-1}))}. \quad (6.I)$$

Setting $a = 2N + 1$ and $b = 2M$, the right hand side is $\mathcal{J}\text{ac}(xy^2 + x^{2N+1} + x^{2M})$. Hence composing (6.H) with (6.I) gives an isomorphism $\mathcal{J}\text{ac}(f) \cong \mathcal{J}\text{ac}(xy^2 + x^{2N+1} + x^{2M})$.

The case $u = 0$ and $v \neq 0$ works in exactly the same way as the case $u \neq 0$ and $v = 0$ above, applying an automorphism with $s = \sqrt{v}$, while the case $u = v = 0$ is trivial. \square

The above is not quite yet in normal form, since some of the polynomial potentials in 6.13 have isomorphic Jacobi algebras. The next step is to discard cases where the odd term in $p(x)$ has significantly greater degree than the even term.

Lemma 6.14. *If $f = xy^2 + x^{2n} + \varepsilon x^{2m+1}$ where $\varepsilon \in \{0, 1\}$ and $m \geq n$, then $y^3 \in (\delta_x f, \delta_y f)$. In particular $y^3 \in ((\delta_x f, \delta_y f))$.*

Proof. Set $I = (\delta_x f, \delta_y f) = (y^2 + 2nx^{2n-1} + \varepsilon(2m+1)x^{2m}, xy + yx)$, and below write \equiv for an equality mod I . Since $y^2 \equiv -2nx^{2n-1} - \varepsilon(2m+1)x^{2m}$, multiplying on the left by y and on the right by y gives

$$\begin{aligned} -2nx^{2n-1}y - \varepsilon(2m+1)x^{2m}y &\equiv y^3 \equiv -2nyx^{2n-1} - \varepsilon(2m+1)yx^{2m} \\ &\equiv 2nx^{2n-1}y - \varepsilon(2m+1)x^{2m}y \end{aligned}$$

where the last line holds since $xy \equiv -yx$ using the second generator of I . Inspecting the right and lefthand sides, the $x^{2m}y$ terms cancel, and so $4nx^{2n-1}y \equiv 0$, thus $x^{2n-1}y \equiv 0$. Finally, since $m \geq n$, taking out the common factor we see that

$$y^3 \equiv (2n - \varepsilon(2m+1))x^{2m-2n-1}x^{2n-1}y \equiv 0.$$

Thus $y^3 \in I$. The final statement follows immediately. \square

Corollary 6.15. *If $f = xy^2 + x^{2n} + \varepsilon x^{2m+1}$ where $m \geq 2n-1$, then $x^{4n-2} \in ((\delta_x f, \delta_y f))$.*

Proof. Continue to write \equiv for an equality mod $(\delta_x f, \delta_y f)$. Then

$$\begin{aligned} x^{4n-2} &= (-x^{2n-1})^2 \equiv \frac{1}{(2n)^2}(y^2 + \varepsilon(2m+1)x^{2m})^2 && \text{(since } \delta_x f \equiv 0) \\ &\equiv \frac{\varepsilon(2m+1)}{(2n)^2}(y^2x^{2m} + x^{2m}y^2 + \varepsilon(2m+1)x^{4m}) && (y^3 \equiv 0 \text{ by 6.14}) \\ &\equiv \frac{\varepsilon(2m+1)}{(2n)^2}(2x^{2m}y^2 + \varepsilon(2m+1)x^{4m}). && (xy \equiv -yx) \end{aligned}$$

Taking out the x^{2m} common factor from the front, we may write $x^{4n-2} \equiv x^{2m}g$ for some g with no constant term. Then, since $2m \geq 4n-2$ by assumption, we see that $x^{4n-2} \equiv x^{4n-2}(x^{2m-(4n-2)}g)$, and so $x^{4n-2}(1 - x^{2m-(4n-2)}g) \equiv 0$.

Given this statement holds mod $(\delta_x f, \delta_y f)$, it also holds mod $((\delta_x f, \delta_y f))$, and hence $x^{4n-2}(1 - x^{2m-(4n-2)}g) = 0$ in $\mathcal{J}ac(f)$. But there, $1 - x^{2m-(4n-2)}g$ is a unit, and so it follows that $x^{4n-2} = 0$ in $\mathcal{J}ac(f)$, as required. \square

The above two results combine to remove the case when the odd-degree x term is sufficiently larger than the even-degree x term, as follows.

Corollary 6.16. *If $f = xy^2 + x^{2n} + x^{2m+1}$ where $m \geq 2n-1$, then $f \cong xy^2 + x^{2n}$.*

Proof. By 6.15 we have $x^{4n-2} \in ((\delta_x f, \delta_y f))$ and $x^{4n-2} \in ((y^2 + 2nx^{2n-1}, xy + yx))$. Since $2m \geq 4n-2$, it follows that x^{2m} belongs to *both* of the ideals above, and thus

$$\begin{aligned} ((\delta_x f, \delta_y f)) &:= ((y^2 + 2nx^{2n-1} + (2m+1)x^{2m}, xy + yx, x^{2m})) \\ &= ((y^2 + 2nx^{2n-1}, xy + yx, x^{2m})) \\ &= ((y^2 + 2nx^{2n-1}, xy + yx)). \end{aligned}$$

As this final ideal is obtained from $xy^2 + x^{2n}$ by differentiation, the result follows. \square

Summarising the above gives the following, which is the main result of this subsection.

Corollary 6.17. *Suppose that $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ where $(f_3)^{\text{ab}}$ has two roots. Then either*

$$f \cong \begin{cases} xy^2 & \\ xy^2 + x^{2m+1} & m \geq 2 \\ xy^2 + x^{2n} & n \geq 2 \\ xy^2 + x^{2m+1} + x^{2n} & 2 \leq m \leq n-1 \\ xy^2 + x^{2n} + x^{2m+1} & 2 \leq n \leq m \leq 2(n-1) \end{cases}$$

All of the above are mutually non-isomorphic.

Proof. The fact that the stated list covers all cases follows from 6.13, using 6.16 to discount the case when the odd-degree x term is sufficiently larger than the even-degree x term. We now claim that the potentials listed give pairwise non-isomorphic Jacobi algebras.

The first two families both have infinite dimensional Jacobi algebras, whereas the bottom three are all finite dimensional. As such, the only possibilities for isomorphisms are between members in families one and two, or between members in families three, four and five. But $\dim_{\mathbb{C}} \mathcal{J}ac(xy^2)^{ab} = \infty$, whereas $\dim_{\mathbb{C}} \mathcal{J}ac(xy^2 + x^{2m+1})^{ab} = 2m + 2$, and so all members of families one and two are mutually non-isomorphic.

For the final three families, all members of families three and four are mutually non-isomorphic, as can be seen by extending the method of [BW1, 4.7], or by using [Ka, 5.10] directly. Further, all members of family five are also mutually non-isomorphic for dimension reasons, since for f in family five $\dim_{\mathbb{C}} \mathcal{J}ac(f)^{ab} = 2m + 2$ and $\dim_{\mathbb{C}} \mathcal{J}ac(f) = (2m + 2) + 4(n - 1)$ by either [vG, §5] or §4.3, and thus we can distinguish between all different m and n . The only remaining possibility is an isomorphism between a member of family five, and a member of family three or four. But by above the dimension of $\mathcal{J}ac(f)$ for f in family five is even, and the dimension of $\mathcal{J}ac(g)$ for g in families three and four is odd [Ka, 5.10], so there can be no such isomorphisms. \square

6.4. Overview of Type D normal forms. The previous subsections combine to give the following, which is the main result of this section.

Theorem 6.18. *Let $f \in \mathbb{C}\langle\langle x \rangle\rangle_{\geq 2}$ with $\text{Crk}(f) = 2$ and $\text{Crk}_2(f) = 2$. Then either*

$$f \cong \begin{cases} z_1^2 + \cdots + z_{d-2}^2 + xy^2 & \text{with } m \geq 1 \\ z_1^2 + \cdots + z_{d-2}^2 + xy^2 + x^{2m+1} & \text{with } n \geq 2 \\ z_1^2 + \cdots + z_{d-2}^2 + xy^2 + x^{2n} & \text{with } n \geq 2, n \leq m \leq 2n - 2 \\ z_1^2 + \cdots + z_{d-2}^2 + xy^2 + x^{2n} + x^{2m+1} & \text{with } m \geq 1, n \geq m + 1. \end{cases}$$

The Jacobi algebras of these potentials are all mutually non-isomorphic, and furthermore the following statements hold.

- (1) *Every f in the first two families satisfies $\text{Jdim } \mathcal{J}ac(f) = 1$.*
- (2) *Every f in the last three families satisfies $\text{Jdim } \mathcal{J}ac(f) = 0$.*
 - (a) *For any fixed $n \geq 2$, the algebras in families three and four combine to give $n - 1$ non-isomorphic Jacobi algebras, all of which satisfy $\dim_{\mathbb{C}} \mathcal{J}ac(f)^{ab} = 2n + 1$ and $\dim_{\mathbb{C}} \mathcal{J}ac(f) = (2n + 1) + 4(n - 1) = 6n - 3$.*
 - (b) *In the fifth family, $\dim_{\mathbb{C}} \mathcal{J}ac(f)^{ab} = 2m + 2$ and $\dim_{\mathbb{C}} \mathcal{J}ac(f) = (2m + 2) + 4(n - 1)$.*

Proof. By the Splitting Lemma 4.5 the condition $\text{Crk}(f) = 2$ implies that $f \cong z_1^2 + \cdots + z_{d-2}^2 + g$ for some $g \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$. The condition $\text{Crk}_2(f) = 2$ is then equivalent to the first two cases in 4.13, namely those $g \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ with $g_3 \neq 0$ for which g_3^{ab} has either two or three distinct linear factors. The options for all such g thus follow from combining 6.6 and 6.17

The fact that $\text{Jdim } \mathcal{J}ac(xy^2 + x^3) = 1$ follows since $\mathcal{J}ac(xy^2 + x^3) \cong \mathcal{J}ac(x^3 + y^3)$ by linear change in coordinates, and $\text{Jdim } \mathcal{J}ac(x^3 + y^3) = 1$ by 6.5(1). The statements that $\text{Jdim } \mathcal{J}ac(xy^2 + x^{2m-1}) = 1$ for all $m \geq 2$ and $\text{Jdim } \mathcal{J}ac(xy^2) = 1$ can be shown by a very similar explicit method as in the proof of 6.5(1), or alternatively by using 8.5 below, once we know (in 8.16) that all such Jacobi algebras are contraction algebras. The stated vector space dimensions of the Jacobi algebras in all remaining cases have already been justified in the proofs of 6.6 and 6.17 respectively.

The fact that the above are all mutually non-isomorphic, and thus a list of normal forms, then follows. Indeed, by inspecting \mathfrak{J} -dimension, the only possible isomorphisms are between members of families one and two, or between members of families three, four and five. Given we have just added the normal forms of 6.6 to the normal forms of 6.17, the only remaining possible isomorphisms are between these two cases. But again, either the dimension of the abelianisation, or the dimension of the contraction algebra itself, distinguishes in all cases. \square

Definition 7.4. Given $f \in \mathbb{C}\langle x \rangle_{\geq 2}$, we say that $\mathcal{J}ac(f)$ has Type X if for all finite dimensional vector spaces $V \subset \mathfrak{m}_{\mathcal{Z}}$ such that $V \rightarrow \mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$, there exists a Zariski open subset U of V such that $\mathcal{J}ac(f)/(u) \cong e\Pi e$ for all $u \in U$, where Π is the preprojective algebra of Type X , and e is an idempotent marked \circ in (7.A).

Equivalently, in the language of [R1, 2.5], $\mathcal{J}ac(f)$ has Type X provided that a general hyperplane section u of \mathcal{Z} satisfies $\mathcal{J}ac(f)/(u) \cong e\Pi e$ where Π is the preprojective algebra of Type X , and e is an idempotent marked \circ in (7.A). We also remark that there are two different Type E_8 's in 7.4, corresponding to the two different choices of \circ in E_8 in (7.A). This feature matches the two different E_8 cases in the length classification of flops [KM].

Much like the definition of cDV singularities, 7.4 is only designed to be useful in specific situations. Indeed, for general $f \in \mathbb{C}\langle x \rangle_{\geq 2}$, it is not clear whether the centre \mathcal{Z} of $\mathcal{J}ac(f)$ is noetherian, nor whether its maximal ideal $\mathfrak{m}_{\mathcal{Z}}$ is finitely generated as a \mathcal{Z} -module. Consequently, work is required to establish that $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$ is finite dimensional, which is needed for there to exist a finite dimensional vector space V surjecting onto it.

When $\mathcal{J}ac(f)$ is finite dimensional, these difficulties disappear, since $\mathcal{J}ac(f)$ and thus \mathcal{Z} , $\mathfrak{m}_{\mathcal{Z}}$ and $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$ are all finite dimensional vector spaces. Other cases are more tricky, but for our purposes the following suffices.

Lemma 7.5. *If f is a normal form from 6.18, then the following statements hold.*

- (1) *If $u \in \mathfrak{m}_{\mathcal{Z}}$, then $u \equiv \lambda x^2 + h$ in $\mathcal{J}ac(f)$ for some $h \in \mathbb{C}\langle x, y \rangle_{\geq 3}$.*
- (2) *If further f has Type $D_{\infty, \infty}$ or $D_{\infty, m}$, then $\mathcal{Z} \cong \mathbb{C}[x^2]$.*

In particular, in all cases $\dim_{\mathbb{C}}(\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2) < \infty$.

Proof. (1) In all cases $\delta_y f = xy + yx$, and so certainly x^2 commutes with y in $\mathcal{J}ac(f)$. Obviously x^2 commutes with x , thus since we are considering closed ideals, it follows that x^2 is central in $\mathcal{J}ac(f)$. Similarly y^2 is central.

We next claim that there are no elements in $\mathfrak{m}_{\mathcal{Z}}$ that contain linear terms. Write $u \in \mathfrak{m}_{\mathcal{Z}}$ as $u = k + g$ for some $k = \lambda_1 x + \lambda_2 y$ and $g \in \mathbb{C}\langle x, y \rangle_{\geq 2}$. Now u remains central after factoring by \mathfrak{J}^3 , so set $I = (\delta_x f, \delta_y f, \mathfrak{n}^3)$, and observe that

$$0 + I = [x + I, u + I] = 2\lambda_2 xy + I$$

For any (finite-dimensional) $\mathbb{C}\langle x, y \rangle / (\delta_x f, \delta_y f, \mathfrak{n}^r)$, $r \geq 3$, we may choose a basis of the form $1, x, \dots, x^i, xy, x^2 y, \dots, x^j y$ for some $i, j \geq 2$. Since xy forms part of this basis it follows that $\lambda_2 = 0$. Repeating using the commutator $[y + I, u + I]$ shows that $\lambda_1 = 0$.

Thus $u = g$ for some $g \in \mathbb{C}\langle x, y \rangle_{\geq 2}$. Using the relation $xy + yx$ to move x 's to the left, and the other relation to move y^2 either to zero, to x^2 , or into higher degree, write

$$u \equiv \lambda_1 x^2 + \lambda_2 xy + h$$

in $\mathcal{J}ac(f)$, for some $h \in \mathbb{C}\langle x, y \rangle_{\geq 3}$. We next claim that $\lambda_2 = 0$. Since u is central, and x^2 is central, it follows that $v := \lambda_2 xy + h$ is also central in $\mathcal{J}ac(f)$. In particular it is still central after factoring by \mathfrak{J}^4 . Set $I = (\delta_x f, \delta_y f, \mathfrak{n}^4)$, so that

$$0 + I = [x + I, v + I] = 2\lambda_2 x^2 y + I$$

and thus $2\lambda_2 x^2 y \in I$. But $x^2 y$ forms part of a basis of $\mathbb{C}\langle x, y \rangle / I$, so $\lambda_2 = 0$.

(2) Either set $I = (xy + yx, y^2)$, or $(xy + yx, y^2 + x^{2m})$, so by assumption $\mathcal{J}ac(f) \cong \mathbb{C}\langle x, y \rangle / (I)$. Consider an arbitrary element $u \in \mathcal{Z}$. By using the first relation to move all the x 's to the left, and the second relation to either move y^2 to zero or to higher powers of x , we may write $u \equiv p + qy$ in $\mathcal{J}ac(f)$, where by (1) $p \in \mathbb{C}[x]_{\geq 2}$ and $q \in \mathbb{C}[x]_{\geq 2}$. Observe that in $\mathcal{J}ac(f)$

$$0 \equiv [x, u] \equiv [x, p + qy] = 2xqy$$

and so $xqy \in (I)$. Thus $xq_{\leq t-3}y \in I + \mathfrak{n}^t$ for all $t \geq 3$. Now $\mathbb{C}\langle x, y \rangle / (I + \mathfrak{n}^t)$ has basis $1, x, \dots, x^{t-1}, y, xy, \dots, x^{t-2}y$. Write $q_{\leq t-3} = \sum_{i=2}^{t-3} \lambda_i x^i$, then the second part of this basis on the equation $xq_{\leq t-3}y \in I + \mathfrak{n}^t$ shows that $\lambda_2 = \dots = \lambda_{t-3} = 0$. This holds for all t , and so $q = 0$.

Thus the central element $u \equiv p$. Splitting into even and odd terms, write $u \equiv P(x^2) + xQ(x^2)$ in $\mathcal{J}ac(f)$ for some $P, Q \in \mathbb{C}[x]_{\geq 1}$. Then, in $\mathcal{J}ac(f)$, since x^2 is central

$$0 \equiv [y, u] \equiv [y, P + xQ] \equiv -2(xQ)y.$$

Using the same argument as above, $Q = 0$, and so $u \equiv P(x^2)$, as claimed. This shows that $\mathcal{Z} \subseteq \mathbb{C}[[x^2]]$, but since x^2 is central by (1), equality holds, proving (2).

For the very last statement, all the finite-dimensional $\mathcal{J}\text{ac}(f)$ satisfy $\dim_{\mathbb{C}}(\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2) < \infty$. Since the only other potentials in 6.18 are those in (2), where visibly $\dim_{\mathbb{C}}(\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2) = 1$, the final statement follows. \square

It follows from 7.5 that $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$ is finite dimensional for any $f \in \mathbb{C}\langle\langle x \rangle\rangle_{\geq 2}$ with $\text{Crk}(f) = 2$ and $\text{Crk}_2(f) = 2$. This is a rather remarkable use of normal forms: we have no method to prove such a result without using 6.18.

7.3. ADE preliminaries. The next problem is to exhibit a single element of the centre \mathcal{Z} that gives an ADE quotient. For Type *A* and *D* this turns out to be easy, but Type *E* requires the following preparation. Consider the elements

$$g_{6,n} := \begin{cases} x^2 + xyx + yx^2 & \text{if } n = 3t + 1 \text{ with } t \geq 1 \\ x^2 + xyx + yx^2 + (-1)^t 3^t (3t + 2)x^{2t+1} & \text{if } n = 3t + 2 \text{ with } t \geq 1 \\ x^2 + xyx + yx^2 + (-1)^{t+1} 3^t t (xyx^{2t-2} + yx^{2t-1}) & \text{if } n = 3t \text{ with } t \geq 2. \end{cases}$$

The following establishes, in the cases $E_{6,n}$, that the centre of $\mathcal{J}\text{ac}(f)$ is non-trivial, and that $\mathfrak{m}_{\mathcal{Z}}$ is at least two dimensional as a vector space.

Lemma 7.6. *If f has Type $E_{6,n}$, then x^2 is central in $\mathcal{J}\text{ac}(f)$, as is $g_{6,n}$.*

Proof. The first statement follows from the relation $3x^2 + y^3 \equiv 0$, which implies that $yx^2 \equiv -\frac{1}{3}y^3 \equiv x^2y$ and thus x^2 commutes with y . Since x^2 clearly commutes with x and we are considering closed ideals, it follows that x^2 is central in $\mathcal{J}\text{ac}(f)$.

For the second statement, we establish the first case, with the proofs of all other cases being similar. For this, it suffices to show that $xyx + yx^2$ is also central in $\mathcal{J}\text{ac}(f)$ when $n = 3t + 1$ with $t \geq 1$. We first claim that $xyxy - yxyx \in ((\delta_x f, \delta_y f))$. This follows since $n - 1 = 3t$, then $y^{n-1} \equiv (-3x^2)^t$ is central, and thus the commutator

$$((\delta_x f, \delta_y f)) \ni [x, \delta_y f] \equiv (xyxy - yxyx) + n(xy^{n-1} - y^{n-1}x) \equiv xyxy - yxyx.$$

Using this, again with the fact that x^2 is central, it follows that

$$\begin{aligned} [x, xyx + yx^2] &\equiv (yx^3 + x^3y) - (x^3y + yx^3) = 0 \\ [y, xyx + yx^2] &\equiv (yxyx + y^2x^2) - (xyxy + yx^2y) \equiv 0. \end{aligned}$$

Thus $xyx + yx^2$ commutes with both x and y , and so is central in $\mathcal{J}\text{ac}(f)$. \square

7.4. Extracting ADE. We are now in a position to extract ADE using general hyperplane sections of the centre. In what follows, in the case that $\mathcal{J}\text{ac}(f)$ is finite dimensional, all ideals are automatically closed. In the cases when $\text{Jdim } \mathcal{J}\text{ac}(f) = 1$ this fact is also true for Type *A* by inspection, and for Type *D* by e.g. 8.4 and 8.17 below. As such, in the following technically we should temporarily write $\mathcal{J}\text{ac}(f)/((g))$ when considering Type $D_{\infty,m}$ and $D_{\infty,\infty}$ until we have established 8.4 and 8.17. However, since 8.4 and 8.17 are logically independent of what follows, we refrain from doing so, and drop the double bracket to ease notation.

Theorem 7.7. *Consider the normal forms $A_n, D_{n,m}, D_{n,\infty}, E_{6,n}, A_{\infty}, D_{\infty,m}, D_{\infty,\infty}$ and $E_{6,\infty}$ from §1.2. In each case, define an element s as follows*

Type	Normal form	Conditions	s
A	$z_1^2 + \dots + z_{d-2}^2 + x^2 + \varepsilon_1 y^n$	$n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$	y
D	$z_1^2 + \dots + z_{d-2}^2 + xy^2 + \varepsilon_2 x^{2n} + \varepsilon_3 x^{2m-1}$	$m, n \in \mathbb{N}_{\geq 2} \cup \{\infty\}$	x^2
E	$z_1^2 + \dots + z_{d-2}^2 + x^3 + xy^3 + \varepsilon_4 y^n$	$n \in \mathbb{N}_{\geq 4}$	$g_{6,n}$

where $g_{6,n}$ is defined in §7.3 above. Then the following statements hold.

- (1) The element s is central in $\mathcal{J}\text{ac}(f)$, and $\mathcal{J}\text{ac}(f)/(s) \cong e\Pi e$, where Π is the preprojective algebra of Type A_1, D_4 , or E_6 , and e is the idempotent marked \circ .
- (2) Normal forms of Type *A* and *D* give rise to Jacobi algebras which have Type *A* and *D* respectively, in the sense of 7.4.

Proof. For (1), Type A is clear, since $\mathcal{J}ac(f) \cong \mathbb{C}\llbracket y \rrbracket / y^{n-1}$ or $\mathbb{C}\llbracket y \rrbracket$ depending on whether ε_1 is 1 or 0. In both cases y is central, and the quotient is $\mathcal{J}ac(f)/y \cong \mathbb{C} \cong e\Pi e$ where Π is the preprojective algebra of Type A_1 .

Type D is similar. The fact x^2 is central was justified in 7.5(1). But then since (x^2) is a closed ideal of $\mathcal{J}ac(f)$, setting $\lambda_2 = (2n)\varepsilon_2$ and $\lambda_3 = (2m-1)\varepsilon_3$ it follows that

$$\begin{aligned} \mathcal{J}ac(f)/(x^2) &\cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{\langle\langle x^2, xy + yx, y^2 + \lambda_2 x^{2n-1} + \lambda_3 x^{2m-2} \rangle\rangle} && \text{(by 2.11)} \\ &\cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{\langle\langle x^2, xy + yx, y^2 \rangle\rangle} && \text{(since } 2n-1 \geq 2 \text{ and } 2m-2 \geq 2\text{)} \\ &\cong e\Pi e && \text{(by 7.2)} \end{aligned}$$

where Π is the preprojective algebra of Type D_4 , and e is the central vertex.

Type E is more involved. All proofs turn out to be similar, so here we illustrate the technique by considering the case f of Type $E_{6,n}$ with $n = 3t + 1$ and $t \geq 2$. Certainly $g_{6,n}$ is central by 7.6. After rescaling the x and y appropriately,

$$\mathcal{J}ac(f) \cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{\langle\langle -x^2 + y^3, xy^2 + yxy + y^2x + y^{3t} \rangle\rangle} =: \Gamma$$

and we work on the right-hand side. Under this identification, the element $g_{6,n}$ becomes $\lambda x^2 + \mu(xy x + yx^2)$ for some non-zero scalars λ and μ . We now claim that for any non-zero scalars λ and μ the factor

$$A := \frac{\Gamma}{(\lambda x^2 + \mu(xy x + yx^2))} = \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{\langle\langle -x^2 + y^3, xy^2 + yxy + y^2x + y^{3t}, \lambda x^2 + \mu(xy x + yx^2) \rangle\rangle}$$

is isomorphic to the model algebra $B := \mathbb{C}\langle\langle x, y \rangle\rangle / \langle\langle x^2, y^3, (x+y)^3 \rangle\rangle \cong e\Pi e$ in 7.2. The result will then follow, since for particular λ, μ , there is an isomorphism $A \cong \mathcal{J}ac(f)/(g_{6,n})$.

To establish the claim, note first that $x^3 \equiv 0$ in A for any $t \geq 2$, as follows. The additional relation gives $-\lambda x^3 \equiv \mu(x^2 y x + x y x^2)$, which equals $\mu(y x^3 + x^3 y)$ since x^2 is central. Repeating, we may push x^3 into higher and higher degrees, and so x^3 belongs to the closed ideal defining A , as claimed. Given $x^3 \equiv 0$ in A , it follows from the first relation that $y^6 \equiv x^4 \equiv 0$ in A , and since $t \geq 2$ also that $y^{3t} \equiv 0$ in A . Consequently

$$A \cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{\langle\langle -x^2 + y^3, xy^2 + yxy + y^2x, \lambda x^2 + \mu(xy x + yx^2), y^{3t} \rangle\rangle} \quad (7.C)$$

$$= \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{\langle\langle -x^2 + y^3, xy^2 + yxy + y^2x, \lambda x^2 + \mu(xy x + yx^2) \rangle\rangle}. \quad (7.D)$$

where the last equality holds since y^6 belongs to the closed ideal in (7.C), and $t \geq 2$. This latter presentation has no dependence on t .

Now, composing the automorphism $\phi: \mathbb{C}\langle\langle x, y \rangle\rangle \rightarrow \mathbb{C}\langle\langle x, y \rangle\rangle$ defined by

$$x \mapsto x - (xy + yx) + yxy, \quad y \mapsto (x + 2y) - y^2 - yx$$

with $\mathbb{C}\langle\langle x \rangle\rangle \rightarrow B$ gives a surjective homomorphism $\mathbb{C}\langle\langle x \rangle\rangle \rightarrow B$. It is elementary to check that the three relations of A in (7.D) map to zero, and hence since ϕ is continuous, it extends to the closure of ideals and thus induces a surjection $\phi: A \rightarrow B$.

Using the same method, it is also elementary to check that the automorphism $\mathbb{C}\langle\langle x, y \rangle\rangle \rightarrow \mathbb{C}\langle\langle x, y \rangle\rangle$ given by

$$x \mapsto -\frac{2}{3}x(3-2x) + (y-2)xy, \quad y \mapsto x(1 + \frac{25}{48}x) + y(1 + \frac{1}{4}y + x)$$

descends to a surjective map $B \rightarrow A$. Thus $\dim A \leq \dim B = 12$, and in particular $\dim A$ is also finite. The previous surjection $A \rightarrow B$ then implies that $\dim A = \dim B$, so that $\phi: A \rightarrow B$, being a surjective map between algebras of the same dimension, is necessarily an isomorphism.

(2) For potentials of Type A , clearly $\mathcal{J}ac(f)$ is either $\mathbb{C}\llbracket y \rrbracket / (y^{n-1})$ or $\mathbb{C}\llbracket y \rrbracket$, both of which are commutative, so $\mathcal{Z} = \mathcal{J}ac(f)$ and further $\mathfrak{m}_{\mathcal{Z}} / \mathfrak{m}_{\mathcal{Z}}^2$ is spanned by the image of y . Given any finite dimensional vector space $V \subset \mathfrak{m}_{\mathcal{Z}}$ such that $\pi: V \rightarrow \mathfrak{m}_{\mathcal{Z}} / \mathfrak{m}_{\mathcal{Z}}^2$, set $\mathcal{U}_1 = \{\lambda \in \mathbb{A}^1 \mid \lambda \neq 0\}$, $\mathcal{U} = \pi^{-1}(\mathcal{U}_1)$, and let $u \in \mathcal{U}$. Then for all $u \in \mathcal{U}$, write $u = \lambda y + p$ in $\mathcal{J}ac(f)$ for some $\lambda \neq 0$ and some $p \in \mathbb{C}\llbracket y \rrbracket_{\geq 2}$. In particular u equals y multiplied by a unit, and so $\mathcal{J}ac(f)/(u) \cong \mathcal{J}ac(f)/(y) \cong \mathbb{C} \cong e\Pi e$, where Π is the preprojective algebra of Type A .

Lastly, consider Type D . By 7.5 all potentials f in 6.18 satisfy $\dim_{\mathbb{C}}(\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2) < \infty$. Hence we can again consider a finite dimensional vector space $V \subset \mathfrak{m}_{\mathcal{Z}}$, such that $\pi: V \rightarrow \mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$. Since $x^2 \in \mathfrak{m}_{\mathcal{Z}}$ by (1), and $\mathfrak{m}_{\mathcal{Z}}$ contains no linear terms (as justified in 7.5(1)), x^2 is non-zero in $\mathfrak{m}_{\mathcal{Z}}/\mathfrak{m}_{\mathcal{Z}}^2$. Thus set $b_1 = x^2 + \mathfrak{m}_{\mathcal{Z}}^2$, and extend to a basis b_1, \dots, b_t . Set $\mathcal{U}_1 = \{\sum \lambda_i b_i \mid \lambda_1 \neq 0\}$, and $\mathcal{U} = \pi^{-1}(\mathcal{U}_1)$.

Let $u \in \mathcal{U}$, then by 7.5(1) $u \equiv \lambda x^2 + h$ in $\mathcal{J}\text{ac}(f)$, for some $h \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$. The assumption $u \in \mathcal{U}$ ensures that $\lambda \neq 0$. By the relation $xy \equiv -yx$, we may pull all the x 's in h to the left, and since h has order at least three, afterwards each term either starts with x^2 , or ends with y^2 . Thus in $\mathcal{J}\text{ac}(f)$

$$h \equiv x^2 r + xy^2 p + y^2 q$$

for some $r \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 1}$, $q \in \mathbb{C}\langle\langle y \rangle\rangle_{\geq 1}$ and $p \in \mathbb{C}\langle\langle y \rangle\rangle$. Consequently $u \equiv x^2(\lambda + r) + xy^2 p + y^2 q$, and further using the relation $\delta_x f$, it follows that

$$u \equiv x^2(\lambda + r - (\lambda_2 x^{2n-2} + \lambda_3 x^{2m-3})p - (\lambda_2 x^{2n-3} + \lambda_3 x^{2m-4})q)$$

where again $\lambda_2 = (2n)\varepsilon_2$ and $\lambda_3 = (2m-1)\varepsilon_3$. The term in brackets is a unit: since $n, m \geq 2$ its only degree zero term is λ , which by assumption is non-zero. It follows that $\mathcal{J}\text{ac}(f)/(u) \cong \mathcal{J}\text{ac}(f)/(x^2)$, and so the result follows by (1). \square

Remark 7.8. The need for taking a generic, or at least well chosen, central element in 7.7 is essential. Indeed, by 7.6, for f of Type $E_{6,n}$ the element x^2 is central in $\mathcal{J}\text{ac}(f)$. However

$$\mathcal{J}\text{ac}(f)/(x^2) \cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((x^2, y^3, (x+y)^3 - xyx))} \not\cong \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((x^2, y^3, (x+y)^3))} = e\Pi e \quad (7.E)$$

even although both sides have dimension twelve. Write \mathcal{U} for the open set given by the non-vanishing of the co-efficient of x^2 , then (7.E) together with 7.7(1) assert that the isomorphism class of $\mathcal{J}\text{ac}(f)/(u)$ is not constant along $u \in \mathcal{U}$. Consequently, for Type E a smaller generic open set is required.

Remark 7.9. In the proof of 7.7(1) above, the inverse of $\phi: A \rightarrow B$ is not the constructed map $B \rightarrow A$, rather ϕ^{-1} is induced by the much more non-obvious automorphism

$$\begin{aligned} x &\mapsto x + \frac{1}{2}(xy + yx) - \frac{1}{8}yxy \\ y &\mapsto \frac{1}{2}(-x + y) + \frac{1}{8}(x^2 - 3xy - yx + y^2) + \\ &\quad \frac{1}{64}(5yx^2 + 12yxy + 16y^2x) + \frac{3}{64}y^2xy + \frac{7}{128}y^2x^2. \end{aligned}$$

8. GEOMETRIC COROLLARIES

The previous results have geometric consequences. Section 8.1 classifies contraction algebras, up to isomorphism, from all Type A and D flopping contractions. This immediate gives, in §8.2, a classification of Type A and D flops, and it also has consequences to GV invariants. Then §8.3 constructs the first, and conjecturally only, infinite family of Type D divisor-to-curve contractions. Using this, and known results from flops, we then prove that the Realisation Conjecture (1.11) is true, except possibly for some exceptional cases, establishing 1.12 in the introduction. The last subsection classifies contraction algebras that can arise from Type A and Type D_4 divisor-to-curve contractions.

8.1. Classification of contraction algebras for A and D flops. In this section we classify contraction algebras that can arise from Type A and D flops, in both cases without referring to any classification of such flops (noting that a Type D classification arises as a consequence, in §8.2). We will show that the only possible options are those finite dimensional Jacobi algebras in 5.1 and 6.18 respectively. We use the notation of §1.5 freely; in particular, \mathcal{R} is commutative noetherian and in applications $\text{Spec } \mathcal{R}$ is the base of a simple 3-fold flop.

Remark 8.1. In both 8.2 and 8.9 below we classify the contraction algebras within a given type, but in fact more is true. By [HT, 4.6] if B_{con} is the contraction algebra of a Type X flopping contraction which is isomorphic to a contraction algebra A_{con} of a Type Y flopping contraction, then $X = Y$. Hence, the algebras in 8.2 below are the contraction

algebras of all possible Type A flops, and only of Type A flops, and the algebras in 8.9 are the contraction algebras of all possible Type D flops, and only of Type D flops.

The classification in Type A is elementary.

Proposition 8.2. *If A_{con} is a contraction algebra of a Type A flopping contraction, then $A_{\text{con}} \cong \mathcal{J}\text{ac}(x^2 + y^n)$ for some $n \geq 2$. Furthermore, any other contraction algebra of any other type cannot be isomorphic to such a Jacobi algebra.*

Proof. Consider A_{con} from an arbitrary Type A flop $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$. By a now standard argument of Van den Bergh [V2, A.1], any indecomposable CM \mathcal{R} -module necessarily has rank one. Further, since \mathcal{R} is normal the endomorphism ring of any rank one CM module is isomorphic to \mathcal{R} . Hence A_{con} , being defined to be a factor of $\text{End}_{\mathcal{R}}(N)$ for some indecomposable CM \mathcal{R} -module N , is thus a factor of \mathcal{R} , and hence is commutative.

Since $A_{\text{con}} \cong \mathcal{J}\text{ac}(f)$ for some f , combining 5.1 and 5.4 we see that $A_{\text{con}} \cong \mathcal{J}\text{ac}(x^2 + y^n)$ or $A_{\text{con}} \cong \mathcal{J}\text{ac}(x^2)$. The last case is impossible, since $\dim_{\mathbb{C}} A_{\text{con}} < \infty$ given that $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ is a flop [DW3]. \square

Remark 8.3. A more geometric proof of 8.2 uses Reid's Pagoda classification of Type A flops [R2] and then applies [DW1, 3.10] to conclude that $A_{\text{con}} \cong \mathcal{J}\text{ac}(x^2 + y^n)$ for some n .

Type D is much more involved, and requires multiple preliminary results. In the following, by an \mathcal{R} -algebra Γ we simply mean that there exists a homomorphism $\mathcal{R} \rightarrow Z(\Gamma)$, where $Z(\Gamma)$ is the centre. From 3.2, let $\mathfrak{J}(\Gamma)$ denote the Jacobson radical of Γ .

Lemma 8.4. *Let Γ be an \mathcal{R} -algebra, where $(\mathcal{R}, \mathfrak{m})$ is commutative local noetherian, and suppose that Γ is finitely generated as an \mathcal{R} -module.*

- (1) *The $\mathfrak{J}(\Gamma)$ -adic topology coincides with the \mathfrak{m} -adic topology on Γ .*
- (2) *Every ideal of Γ is closed with respect to the $\mathfrak{J}(\Gamma)$ -adic and the \mathfrak{m} -adic topologies.*
- (3) *If $(\mathcal{R}, \mathfrak{m})$ is complete local, then Γ is complete with respect to both the $\mathfrak{J}(\Gamma)$ -adic and the \mathfrak{m} -adic topologies.*

Proof. (1) Since $(\mathcal{R}, \mathfrak{m})$ is local, and Γ is finitely generated as an \mathcal{R} -module, it follows immediately from e.g. [L1, 20.6] that there exists $n \geq 1$ such that

$$\mathfrak{J}(\Gamma)^n \subseteq \mathfrak{m}\Gamma \subseteq \mathfrak{J}(\Gamma). \quad (8.A)$$

From this, it is clear that the $\mathfrak{J}(\Gamma)$ -adic and the \mathfrak{m} -adic topologies coincide.

(2) We show that for any finitely generated \mathcal{R} -module M , with submodule N , then N is closed in M . Given this, applying (1) to $M = \Gamma$ and $N = I$ proves the result. But since M is finitely generated, and \mathcal{R} is noetherian, Krull's intersection theorem [M1, 8.1(1)] immediately shows that M/N is separated, and hence N is closed in M .

(3) Again, this is well known. Since Γ is a finitely generated \mathcal{R} -module, and $(\mathcal{R}, \mathfrak{m})$ is complete local and noetherian, it follows from e.g. [L1, 21.34(1)] that Γ is \mathfrak{m} -adically complete, and hence the result follows by (1). \square

Corollary 8.5. *If A_{con} is a contraction algebra associated to a crepant $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ as above, then $\text{Jdim } A_{\text{con}} = 0, 1$. Furthermore, the following statements hold.*

- (1) *$\text{Jdim } A_{\text{con}} = 0$ if and only if $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ is a flop.*
- (2) *$\text{Jdim } A_{\text{con}} = 1$ if and only if $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$ is a divisorial contraction to a curve.*

Proof. A_{con} is module finite over \mathcal{R} , being a factor of an NCCR [DW1].

Now if M is any finitely generated \mathcal{R} -module (e.g. $M = A_{\text{con}}$), since $(\mathcal{R}, \mathfrak{m})$ is local $\dim_{\mathcal{R}}(M)$ can be defined using the \mathfrak{m} -adic topology, as the growth rate of the function $\text{length}(M/\mathfrak{m}^i M)$, see e.g. [E1, §12.1]. Taking suitable powers of the inclusions in (8.A), it is elementary to see that the two sets

$$\mathcal{S}_1 = \{r \in \mathbb{R} \mid \text{for some } c \in \mathbb{R}, \dim_{\mathbb{C}}(A_{\text{con}}/\mathfrak{J}^n) \leq cn^r \text{ for every } n \in \mathbb{N}\}$$

$$\mathcal{S}_2 = \{r \in \mathbb{R} \mid \text{for some } c \in \mathbb{R}, \dim_{\mathbb{C}}(A_{\text{con}}/\mathfrak{m}^n A_{\text{con}}) \leq cn^r \text{ for every } n \in \mathbb{N}\}$$

are equal, so $\text{Jdim } A_{\text{con}} = \inf \mathcal{S}_1 = \inf \mathcal{S}_2 = \dim_{\mathcal{R}}(A_{\text{con}})$.

The main result of [DW2] shows that $\text{Supp}_{\mathcal{R}}(A_{\text{con}})$ equals the contracted locus in $\text{Spec } \mathcal{R}$. Hence $\dim_{\mathcal{R}}(A_{\text{con}})$ is either 0 for flops, or 1 for divisor-to-curves respectively. It follows that $\text{Jdim}(A_{\text{con}})$ is either 0 or 1, respectively. \square

Theorem 8.6. *If $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ and $\mathcal{J}\text{ac}(f)$ is complete in its \mathfrak{J} -adic topology, then every central element $g + (\delta_x f, \delta_y f) \in \mathcal{J}\text{ac}(f)$ which is not a unit satisfies $g \in \mathfrak{n}^2$.*

Proof. Set $R = \mathcal{J}\text{ac}(f)$, and $\mathfrak{J} = \mathfrak{J}(R)$. Note that $I = (\delta f) \subseteq \mathfrak{n}^2$, hence by 3.2(1)

$$\mathfrak{J}^2 = (\mathfrak{n}^2 + I)/I = \mathfrak{n}^2/I. \quad (8.B)$$

Consider the central element $g' = g + I$ in R . Since g' is not a unit in R , certainly $g' \in \mathfrak{J}$. Further g cannot be a unit in $\mathbb{C}\langle\langle x, y \rangle\rangle$, or it would descend to a unit, so $g \in \mathfrak{n}$.

We now suppose that $g \notin \mathfrak{n}^2$, and aim for a contradiction. Since $I \subseteq \mathfrak{n}^2$ we see that $g' = g + I \notin \mathfrak{J}^2$, and hence $0 \neq g' + \mathfrak{J}^2 \in \mathfrak{J}/\mathfrak{J}^2$. But by (8.B) we have

$$\frac{\mathfrak{J}}{\mathfrak{J}^2} = \frac{\mathfrak{n}/I}{\mathfrak{n}^2/I} \cong \frac{\mathfrak{n}}{\mathfrak{n}^2}$$

and so $\dim_{\mathbb{C}} \mathfrak{J}/\mathfrak{J}^2 = 2$. Pick $0 \neq h + \mathfrak{J}^2$ to complete $g' + \mathfrak{J}^2$ to a basis of $\mathfrak{J}/\mathfrak{J}^2$.

But now since by assumption R is complete, and local, we may use [BIRS, 3.1] to present R . Consider the two-loop quiver Q , and map the trivial path to the identity of R , one of the loops ℓ_1 to h and the other loop ℓ_2 to g' . Then by [BIRS, 3.1] the completeness of R extends this to a surjective homomorphism

$$\varphi: \mathbb{C}\langle\langle \ell_1, \ell_2 \rangle\rangle \rightarrow R$$

and the kernel is a closed ideal. Since the kernel contains the relation $\ell_1\ell_2 - \ell_2\ell_1$, given g' is central and so commutes with h , it follows that $(\ell_1\ell_2 - \ell_2\ell_1) \subseteq \text{Ker } \varphi$. In particular φ induces a surjection $\mathbb{C}[[\ell_1, \ell_2]] \rightarrow R$, and so R is commutative. Given this would contradict 5.4, we conclude that $g \in \mathfrak{n}^2$. \square

We will also require the following fact.

Proposition 8.7. *Suppose that A_{con} is the contraction algebra associated to a D_4 contraction, then there is a central element $g \in A_{\text{con}}$ such that*

$$A_{\text{con}}/(g) \cong \frac{\mathbb{C}\langle x, y \rangle}{(x^2, xy + yx, y^2)}.$$

Proof. Consider the 3-fold contraction $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$, and for generic $g \in \mathcal{R}$ consider the pullback diagram

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \text{Spec } \mathcal{R}/g & \longrightarrow & \text{Spec } \mathcal{R} \end{array}$$

By assumption, \mathcal{R}/g is a D_4 Kleinian singularity. Now let $A = \text{End}_{\mathcal{R}}(N)$ be the NCCR associated to $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$, and view $g \in \mathcal{R} = Z(A) \subset A$. Since g is generic, we can find such a g which is not contained in any associated prime ideal of $\text{Ext}_{\mathcal{R}}^1(N, N)$, from which [IW2, 5.19] (the assumptions there are Type A, but the method is general) shows that there is an isomorphism

$$A/g \cong \text{End}_{\mathcal{R}/g}(N/gN).$$

From this isomorphism [DW1, (3.C)] establishes that A_{con}/g is isomorphic to the contraction algebra associated to the surfaces contraction $\mathcal{Y} \rightarrow \text{Spec } \mathcal{R}/g$. The fact that the surfaces contraction algebra for this particular D_4 contraction is $\mathbb{C}\langle x, y \rangle/(x^2, xy + yx, y^2)$ can be deduced from [DW1, 8.7]; see also [M2]. \square

We obtain the following remarkable consequence.

Corollary 8.8. *No $\mathcal{J}\text{ac}(f)$ with $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 3}$ such that either $f_3 = 0$ or $f_3 \cong x^3$, can arise as a contraction algebra of a D_4 flop, or a D_4 divisor-to-curve contraction.*

Proof. Given such an f , suppose that $\mathcal{J}\text{ac}(f) \cong A_{\text{con}}$ for a contraction algebra of a D_4 flop, or a D_4 divisor-to-curve contraction $\mathcal{X} \rightarrow \text{Spec } \mathcal{R}$. Since A_{con} is module finite over \mathcal{R} , being a factor of a NCCR [DW1, 8.4] shows that A_{con} and hence $\mathcal{J}\text{ac}(f)$ is complete with respect to its radical-adic topology, and further every ideal is closed.

Further, by 8.7 we can find a central g such that $\dim_{\mathbb{C}} \mathcal{J}ac(f)/(g) = 4$, and since $\mathcal{J}ac(f) \cong A_{\text{con}}$ is complete, we can use 8.6 to write $g = g' + ((\delta_x f, \delta_y f))$ where $g' \in \mathfrak{n}^2$. But since all ideals in $\mathcal{J}ac(f)$ are closed, it follows that

$$\frac{\mathcal{J}ac(f)}{(g)} = \frac{\mathcal{J}ac(f)}{(g)} \stackrel{2.11}{\cong} \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((\delta_x f, \delta_y f, g'))}$$

has dimension four. We claim that this is impossible, by exhibiting a factor with higher dimension. Reusing the notation in 5.4, write \mathcal{M}_3 for the set of all noncommutative monomials of degree 3, and then we will factor by $((\mathcal{M}_3))$. In the two cases $f_3 = x^3$ and $f_3 = 0$, the factors are, respectively

$$\frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((x^2, g', \mathcal{M}_3))} \quad \text{and} \quad \frac{\mathbb{C}\langle\langle x, y \rangle\rangle}{((g', \mathcal{M}_3))}.$$

The right hand algebra surjects onto the left hand algebra, so it suffices to prove that $\dim_{\mathbb{C}} \mathbb{C}\langle\langle x, y \rangle\rangle / ((x^2, g', \mathcal{M}_3)) > 4$. But since $g' \in \mathfrak{n}^2$ by 8.6, inside the ideal we can replace g' by $\lambda_1 xy + \lambda_2 yx + \lambda_3 y^2$, which gives at most one linear relation between xy , yx and y^2 . From this, the statement is clear. \square

The above gives rise to the following, which is the main result in this subsection.

Corollary 8.9. *If A_{con} is a contraction algebra of a Type D flopping contraction, then $A_{\text{con}} \cong \mathcal{J}ac(x^2 y + x^{2n})$ for some $n \geq 2$, or $A_{\text{con}} \cong \mathcal{J}ac(xy^2 + x^{2n} + x^{2m-1})$ for some $m, n \geq 2$ with $m \leq 2n - 1$.*

Proof. Consider A_{con} from an arbitrary Type D flop. By [KM] necessarily the elephant is D_4 , so A_{con} is not commutative by 8.7, since A_{con} has a factor which is not commutative. As $A_{\text{con}} \cong \mathcal{J}ac(f)$ for some $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$, appealing to 5.4 then gives $f_2 = 0$.

From this, 8.8 asserts that $f_3 \neq 0$, and $f_3 \not\cong \ell^3$. Hence by 6.18 $A_{\text{con}} \cong \mathcal{J}ac(f)$ for some f in the list stated there. Only the bottom three families are possible, since $\dim_{\mathbb{C}} A_{\text{con}} < \infty$ given the contraction is a flop [DW3]. \square

8.2. Classification of Flops. The above results for contraction algebras classify flops, given that the Donovan–Wemyss conjecture is true [DW1, A3, JKM].

Theorem 8.10. *With notation as above, the following statements hold.*

- (1) *Type A flops are classified by Type A normal forms in Table 1.*
- (2) *Type D flops are classified by Type D normal forms in Table 1.*

Furthermore, Type E flops are classified by Type E normal forms.

Proof. The fact that the isomorphism class of the contraction algebra classifies flops is established in [JKM]. Thus (1) follows from the fact in 8.2 that the contraction algebras of Type A flops are precisely those Jacobi algebras coming from the Type A normal forms. Similarly, (2) follows from 8.9 and the fact that each normal form is realisable from geometry [vG, Ka] (see also 8.11 below). Since the remaining Type E normal forms cannot correspond to either Type A or Type D geometry (by either 8.2 or 8.8), it follows that the contraction algebras of the remaining Type E flops must be isomorphic to Jacobi algebras of Type E normal forms. Note that the Type E normal forms stated in Table 1 are genuine examples; full details of others will appear elsewhere [BW2]. \square

The classification of contraction algebras in 8.9 then has the following consequence.

Theorem 8.11. *There is a one-to-one correspondence between lattice points in the diagram in 1.6 and the base singularities $0 \in \text{Spec } \mathcal{R}$ of Type D flops, given by*

$$(n, m) \mapsto \text{Spec} \left(\frac{\mathbb{C}\langle\langle u, v, x, y \rangle\rangle}{u^2 + v^2 y - x(y^{2n+1} + (x + \varepsilon y^m)^2)} \right)$$

where $\varepsilon = 1$ if the lattice point is contained within the shaded region, and $\varepsilon = 0$ otherwise.

In particular, Type D flops do not admit moduli. Furthermore, the following hold.

- (1) *The quasi-homogeneous Type D flops are precisely those outside the shaded region, and these are the standard Laufer family.*
- (2) *The GV invariants n_1, n_2 of the flopping contraction associated to a point (n, m) are written in green. The ovals group together flops with the same GV invariants.*

Proof. It is immediate from 8.10 that the Type D normal forms in Table 1 classify Type D flops. In particular, once we exhibit one flop for each Type D normal form in Table 1, which has contraction algebra isomorphic to the prescribed Jacobi algebra, then the geometric classification is complete.

In the indexing of the diagram, for (n, m) with $n, m \geq 1$, consider the corresponding potential $xy^2 + x^{2n+2} + x^{2m+1}$. Under this assignment, the points (n, m) in the shaded region correspond to the normal forms $D_{n+1, m+1}$, which by definition have a restriction on m relative to n . The lattice points not in the shaded region, namely those $(n, 2n+1)$, correspond to $xy^2 + x^{2n+2} + x^{4n+3}$. By 6.16 this is isomorphic to $xy^2 + x^{2n+2}$, which is the normal form $D_{n+1, \infty}$ in Table 1. Thus, the lattice points stated are in bijection with the Type D normal forms.

Now by [vG, §2.2] and [Ka], given any potential $xy^2 + x^{2n+2} + x^{2m+1}$, the claimed commutative ring in the statement is the base of a flopping contraction, and further the corresponding contraction algebra is isomorphic to $\mathcal{J}ac(xy^2 + x^{2n+2} + x^{2m+1})$. The first statement regarding the bijection follows.

The fact that the quasi-homogeneous singularities correspond to those outside the shaded region follows from [vG2, §2.2.4], which computes the Milnor and Tjurina numbers. For the final statement regarding GV invariants, by 6.18(2) and Toda's dimension formula [T, 1.1], we can read off the GV invariants. Indeed, by [T] the pair n_1, n_2 where $n_1 = \dim_{\mathbb{C}} \mathcal{J}ac(f)^{ab}$ and $n_2 = \frac{1}{4}(\dim_{\mathbb{C}} \mathcal{J}ac(f) - \dim_{\mathbb{C}} \mathcal{J}ac(f)^{ab})$ are precisely the GV invariants for length two flops. Only those pairs illustrated in the diagram in 1.6 (or Figure 3) appear. \square

Remark 8.12. The map in 8.11 is in fact well defined on all points (n, m) with $m, n \geq 1$, not just those marked in the picture in 1.6. This follows since the commutative ring in 8.11 is always the base of a Type D flop, for every (n, m) [vG, §2.2]. The point is that the domain needs to be restricted in order to obtain a bijection. The previous result 6.16 shows that any lattice point (n, m) with $m \geq 2n+1$ gives an isomorphic flop to the lattice point $(n, 2n+1)$.

Simply inspecting 8.11(2) and the diagram in 1.6 gives the following corollary, which illustrates the significant gaps in the possible GV invariants that can arise.

Corollary 8.13. *Consider $(a, b) \in \mathbb{N}^2$. Then a, b are the GV invariants for a Type D flopping contraction if and only if either*

- (1) $(a, b) = (2m+3, m)$ for some $m \geq 1$, or
- (2) $(a, b) = (2n, b)$ for some $n \geq 2$, with $b \geq n-1$.

Further, when $a = 2m+3$ there are precisely $m+1$ distinct contraction algebras realising (a, b) , up to isomorphism, whilst for any given $(2n, b)$ the contraction algebra is unique.

Remark 8.14. It is possible to instead index the GV invariants to the classifying potentials, and this is done in Figure 3 below.

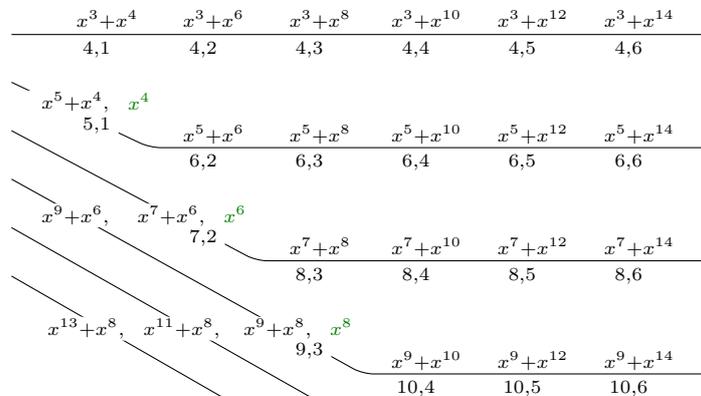


FIGURE 3. List of $p(x)$ for which $xy^2 + p(x)$ is one of the normal forms in $D_{n, m}$ or $D_{n, \infty}$. The pair n_1, n_2 associated to each $p(x)$ describes the GV invariants of any simple flop having isomorphic contraction algebra.

8.3. Constructing divisor to curve contractions. In the list of potentials in 6.18, the first appears as the contraction algebra of a divisor-to-curve contraction in [DW4, 2.18]. The second family, with $m = 1$, is isomorphic to $x^3 + y^3$, and so appears as a contraction algebra in [DW4, 2.25]. All the other three families are contraction algebras of D_4 flops by [vG, Ka], and the above subsection.

Motivated by Conjecture 1.11, this subsection will fill the last remaining gap, and show that the whole of the second family in 6.18, with arbitrary m , are realised as the contraction algebra of a divisor-to-curve contraction.

Remark 8.15. In the proof below we will first construct the contraction algebraically, before passing to the formal fibre to realise the contraction algebra. This algebraic construction is advantageous, since it conceptually distinguishes between the cases: in $\text{Spec } R_\infty$ below, which locally realises $D_{\infty, \infty}$, the origin is cD_4 whilst all other points on the singular locus are cA_2 . In contrast, in $\text{Spec } R_m$ below, which locally realises $D_{\infty, m}$, the origin is cD_4 whilst all other points on the singular locus are cA_1 . Compare the pictures in [DW4, 2.18] and [DW4, 2.25], and also [W].

Proposition 8.16. *For $m \in \mathbb{N} \cup \{\infty\}$, consider the element of $\mathbb{C}[[X, Y, Z, T]]$ defined by*

$$F_m := \begin{cases} Y(X^m + Y)^2 + XZ^2 - T^2 & \text{if } m \geq 1 \\ Y^3 + XZ^2 - T^2 & \text{if } m = \infty \end{cases}$$

and set $\mathcal{R}_m = \mathbb{C}[[X, Y, Z, T]]/F_m$. Then the following statements hold.

- (1) $\text{Sing}(\mathcal{R}_m)^{\text{red}} = (X^m + Y, Z, T)$ if $m \geq 1$, and (Y, Z, T) if $m = \infty$.
- (2) In either case, blowing up this locus gives rise to a crepant Type D divisorial contraction to a curve $\mathcal{X}_m \rightarrow \text{Spec } \mathcal{R}_m$ where \mathcal{X}_m is smooth.
- (3) The contraction algebra of $\mathcal{X}_m \rightarrow \text{Spec } \mathcal{R}_m$ is isomorphic to $\text{Jac}(xy^2 + x^{2m+1})$ when $m \geq 1$, respectively $\text{Jac}(xy^2)$ when $m = \infty$.
- (4) $\mathcal{R}_m \cong \mathcal{R}_n$ if and only if $m = n$, and so the F_m are all distinct up to isomorphism and so form an infinite family.

Proof. (1) is immediate.

(2) Working first on the case of finite $m \geq 1$, consider the affine algebra

$$R_m = \frac{\mathbb{C}[r, s, u, v]}{u^2 - r(r - s^m)^2 - sv^2}$$

whose completion at the origin is \mathcal{R}_m , in coordinates $(r, s, u, v) = (-Y, X, T, Z)$. The blowup along $(u, v, r - s^m)$ is covered by two affine patches: the first is $U = \text{Spec } \mathbb{C}[s, y_0, y_1]$, with $y_0 = u/(r - s^m)$ and $y_1 = v/(r - s^m)$, the second chart is a smooth hypersurface, and the map from U to the base is given by

$$(s, y_0, y_1) \mapsto (y_0^2 - sy_1^2, s, y_0(y_0^2 - sy_1^2 - s^m), y_1(y_0^2 - sy_1^2 - s^m)).$$

The exceptional locus in U is the divisor $y_0^2 - sy_1^2 - s^m = 0$. Pulling the canonical basis of differentials on $\text{Spec } R_m$ back to U gives

$$f_2^* \left(\frac{dr \wedge ds \wedge dv}{u} \right) = \frac{d(y_0^2 - sy_1^2) \wedge ds \wedge d(y_1(y_0^2 - sy_1^2 - s^m))}{y_0(y_0^2 - sy_1^2 - s^m)} = 2ds \wedge dy_0 \wedge dy_1$$

which is a regular differential on U , and in particular has no zero or pole along the exceptional divisor. Thus the map is crepant, as claimed. The case $m = \infty$ is similar, but easier, as both open charts are affine 3-space.

(3) The easiest way to establish the claim is to recognise F_m as a pullback from the universal D_4 flop and apply restriction theorems for contraction algebras. Consider the six-dimensional universal D_4 flop, given in [K, (1.1)] as

$$\mathcal{R} = \frac{\mathbb{C}[r, s, t, u, v, w, z]}{u^2 - rw^2 + 2zvw - sv^2 + (rs - z^2)t^2}$$

and its universal family $\mathcal{Y} \rightarrow \text{Spec } \mathcal{R}$, which is an isomorphism away from the singular locus in $\text{Spec } \mathcal{R}$. As observed by Van Garderen [vG2, §2.2.3], slicing by the sequence

$g_1 = z$, $g_2 = r - w - s^m$, and $g_3 = t$ yields a commutative diagram

$$\begin{array}{ccccccc} Y = Y_3 & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & \mathcal{Y} \\ f \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Spec } \mathcal{R}_3 & \longrightarrow & \text{Spec } \mathcal{R}_2 & \longrightarrow & \text{Spec } \mathcal{R}_1 & \longrightarrow & \text{Spec } \mathcal{R} \end{array}$$

where $\mathcal{R}_1 = \mathcal{R}/g_1$, $\mathcal{R}_2 = \mathcal{R}_1/g_2$ and $\mathcal{R}_3 = \mathcal{R}_2/g_3$. The result is the affine algebra

$$\mathcal{R}_3 = \frac{\mathbb{C}[r, s, u, v]}{u^2 - r(r - s^m)^2 - sv^2}$$

whose completion at the origin is \mathcal{R}_m . The pullback $f: Y \rightarrow \text{Spec } \mathcal{R}_3$ is visibly an isomorphism away from $\text{Sing}(\mathcal{R}_3)^{\text{red}} = (u, v, r - s^m)$, so in particular is birational.

Van Garderen observes that \mathcal{R}_3 is an integral domain [vG2, 2.12] and Y is smooth [vG2, 2.13], and that each g_i is a *slice*, in the terminology of [vG2, 2.9], which implies that f is projective and surjective with $\mathbf{R}f_*\mathcal{O} = \mathcal{O}$ [vG2, 2.10].

Furthermore, the tilting bundle yielding a derived equivalence between \mathcal{Y} and $\Lambda \in \text{CM } \mathcal{R}$ restricts to give a derived equivalence between Y and $\Lambda \otimes \mathcal{R}_3$ [vG2, (2.11)]. Since g_1, g_2, g_3 is a regular sequence, $\Lambda \otimes \mathcal{R}_3 \in \text{CM } \mathcal{R}_3$ and so in particular f is crepant [IW1, 4.14]. Since visibly both the blowup in (2) and f are crepant resolutions of the same variety, and both containing no flopping curves, they must be isomorphic (as varieties over the base $\text{Spec } \mathcal{R}_m$) since minimal models are unique up to flop. Thus the contraction algebra associated to (2) is isomorphic to the contraction algebra of the formal fibre of f . But by [vG2, 2.8] this is the claimed Jacobi algebra, namely $\mathcal{J}\text{ac}(xy^2 + x^{2m+1})$ when $m \geq 1$, respectively $\mathcal{J}\text{ac}(xy^2)$ when $m = \infty$.

(4) If $\mathcal{R}_m \cong \mathcal{R}_n$ are isomorphic, then the contraction algebras of $\mathcal{X}_m \rightarrow \text{Spec } \mathcal{R}_m$ and of $\mathcal{X}_n \rightarrow \text{Spec } \mathcal{R}_n$ must be isomorphic. But then their abelianisations must also be isomorphic, and so in particular must have the same dimension. But the abelianisations have dimension $2(m+1)$ and $2(n+1)$ respectively, and hence $m = n$. \square

The start of this subsection, combined with 8.16, then gives the following.

Corollary 8.17. *All the potentials in 6.18 are geometric.*

In turn, this establishes 1.12 in the introduction.

Corollary 8.18. *Conjecture 1.11 is true, except for the one remaining unresolved case when $f \cong x^3 + \mathcal{O}_4$, where some further analysis is required.*

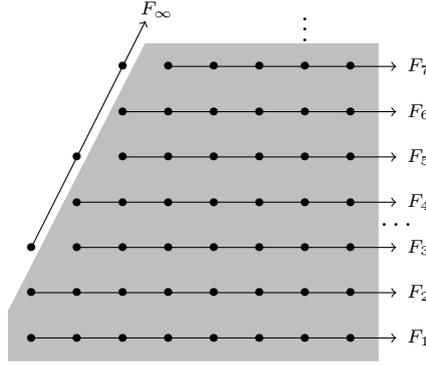
Proof. Every $A_{\text{con}} \cong \mathcal{J}\text{ac}(f)$ for some $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$. If $f_2 \neq 0$ the result is 5.2, so we can assume $f_2 = 0$. We need $f_3 \neq 0$ so that $\text{Jdim } \mathcal{J}\text{ac}(f) \leq 1$, and 4.13 splits into three cases. The first two cases are covered by 6.18, and 8.17 asserts that these are all geometric. The only remaining, unresolved, case from 4.13 is when $f_3 \cong x^3$. \square

Remark 8.19. It is possible to change variables to see that all type D normal forms can be realised in a uniform way. Indeed, the contraction algebra associated to the cD_4 singularity

$$\frac{\mathbb{C}[u, v, x, y]}{u^2 + v^2y - x(\varepsilon_2y^{2n+1} - (x - \varepsilon_3y^m)^2)}$$

realises the general Type D potential $f = xy^2 + \varepsilon_2x^{2n} + \varepsilon_3x^{2m-1}$, with the convention that each ε_i is either 0 or 1.

Remark 8.20. Much like in Pagoda [R2] for Type A , it is also possible to view each Type D divisor-to-curve contraction as an infinite limit of flops. The Type D situation is more delicate, since there are more possible directions in which to take such a limit. In relation to the classification of Type D flops in 8.11, the following are the limits which give rise to the divisor-to-curve contractions in 8.16.



Without the normal forms from noncommutative singularity theory, it is hard to either see or predict the above purely geometrically.

8.4. Classification for A and D divisor-to-curve contractions. This section is the divisor-to-curve analogue of §8.1.

Proposition 8.21. *If A_{con} is a contraction algebra of a Type A divisor-to-curve contraction, then $A_{\text{con}} \cong \mathcal{J}ac(x^2)$.*

Proof. For the same homological reason as in 8.21, in Type A the contraction algebra A_{con} is necessarily commutative. Since the contraction is divisor-to-curve, necessarily $\dim_{\mathbb{C}} A_{\text{con}} = \infty$ [DW3]. Combining 5.4 and 5.1, we see that $A_{\text{con}} \cong \mathcal{J}ac(x^2)$, since x^2 is the only infinite dimensional example in 5.1. \square

The following is the analogue of 8.9. However, it is slightly weaker, due to two key geometric facts having only been developed in the flops setting: (1) Katz–Morrison [KM] asserts that Type D is generically Type D_4 for flops, but this is open in the divisor-to-curve setting, and (2) Hua–Toda [HT] asserts that the isomorphism class of a contraction algebra determines the type, but again only for flops.

However, we can say the following, without using any geometric classifications.

Proposition 8.22. *If A_{con} is a contraction algebra of a Type D_4 divisor-to-curve contraction, then $A_{\text{con}} \cong \mathcal{J}ac(x^2y)$, or $A_{\text{con}} \cong \mathcal{J}ac(xy^2 + x^{2m+1})$ for some $m \geq 2$.*

Proof. The proof is very similar to 8.9. The algebra A_{con} is not commutative by 8.7, since A_{con} has a factor which is not commutative. As $A_{\text{con}} \cong \mathcal{J}ac(f)$ for some $f \in \mathbb{C}\langle\langle x, y \rangle\rangle_{\geq 2}$, appealing to 5.4 then gives $f_2 = 0$.

From this, 8.8 asserts that $f_3 \neq 0$, and $f_3 \not\cong \ell^3$. Hence by 6.18 $A_{\text{con}} \cong \mathcal{J}ac(f)$ for some f in the list stated there. Only the top two families are possible, since $\dim_{\mathbb{C}} A_{\text{con}} = \infty$ given the contraction is divisor-to-curve [DW3]. \square

APPENDIX A. \mathfrak{J} -DIMENSION RESTRICTIONS

The papers [ISm, IS2] introduce several new ideas that substantially strengthen the Golod–Shafarevich estimates [GS] for the growth of algebraic Jacobi rings, and prove that almost all have cubic or higher growth. In this appendix we extend the main results of [IS2] into the setting of formal noncommutative Jacobi algebras of 3.1, in a manner which should be viewed as the analogue of Vinberg’s [V3] extension of Golod–Shafarevich into the setting of topological rings.

A.1. Algebraic Notation. Throughout this appendix we let $d \geq 2$ and consider $\mathbb{C}\langle x \rangle = \mathbb{C}\langle x_1, \dots, x_d \rangle$. An element $F \in \mathbb{C}\langle x \rangle$ is called a *superpotential* if it is cyclically symmetric, in the sense of 3.3. For $m \geq k \geq 3$ write

$$SP_{k,m} = \{F \in \mathbb{C}\langle x \rangle \mid F \text{ is a superpotential with } F_j = 0 \text{ for } j < k \text{ and } j > m\},$$

where F_j is the homogeneous component of F of degree j , as in §2.1. In the special case $m = k$, write $SP_k := SP_{k,k}$, which consists of all homogeneous superpotentials of degree k , together with zero. Throughout, we will write elements of $\mathbb{C}\langle x \rangle$ and $\mathbb{C}\langle\langle x \rangle\rangle$ by small letters f and g , and superpotentials by capital letters F, G .

With the (left) strike-off derivatives ∂_i defined as in (3.A), the *algebraic Jacobi algebra* associated to a superpotential F is the algebra

$$A\mathcal{J}ac(F) := \frac{\mathbb{C}\langle x_1, \dots, x_d \rangle}{(\partial_1 F, \dots, \partial_d F)} = \frac{\mathbb{C}\langle x \rangle}{I_F}$$

where $I_F = (\partial_1 F, \dots, \partial_d F)$ is the two-sided ideal generated by $\partial_1 F, \dots, \partial_d F$. We write $\mathfrak{m} = (x_1, \dots, x_d) \subset \mathbb{C}\langle x \rangle$, a maximal two-sided ideal, and denote its image in $A\mathcal{J}ac(F)$ by $\mathfrak{R} = \mathfrak{m}/I_F$, the powers of which are $\mathfrak{R}^i = (\mathfrak{m}^i + I_F)/I_F$.

The use of strike-off derivatives ∂_i on superpotentials, as we use here to align with the statements and results of [ISm, IS2], or cyclic derivatives δ_i on any potential, as in 3.1, give equivalent theories but with minor differences in detail, which we address in §A.4.

A.2. Exact Potentials and Hilbert series. The differentiation package has two useful tools. The first is the following Euler relation.

Lemma A.1. [ISm, 3.5] *If F is a superpotential, then $\sum_{i=1}^d [x_i, \partial_i F] = 0$.*

The second is a sequence of right $A\mathcal{J}ac(F)$ -modules

$$0 \rightarrow A\mathcal{J}ac(F) \xrightarrow{d_3} A\mathcal{J}ac(F)^{\oplus d} \xrightarrow{d_2} A\mathcal{J}ac(F)^{\oplus d} \xrightarrow{d_1} A\mathcal{J}ac(F) \xrightarrow{d_0} \mathbb{C} \rightarrow 0 \quad (\text{A.A})$$

defined in e.g. [ISm, 3.4]. The precise form of the morphisms d_i will not concern us, as below we will only require the following two facts.

- (1) [ISm, 3.6] For any superpotential F , the sequence (A.A) is a complex, which is exact at the three right-most non-zero terms.
- (2) If further F is homogeneous, say $0 \neq F \in \text{SP}_k$, then the morphisms in the complex (A.A) satisfy $\deg(d_3) = 1$, $\deg(d_2) = k - 2$, $\deg(d_1) = 1$, and $\deg(d_0) = 0$.

Definition A.2. An element $F \in \text{SP}_{k,m}$ is called *exact* if (A.A) is exact.

If G is homogeneous, then the ideal $(\partial_1 G, \dots, \partial_d G)$ is a homogeneous ideal and so the graded decomposition of $\mathbb{C}\langle x \rangle$ induces a decomposition

$$A\mathcal{J}ac(G) = \bigoplus_{i \geq 0} A\mathcal{J}ac(G)_i. \quad (\text{A.B})$$

For $G \in \text{SP}_k$, the boundary maps d_i are homogeneous, and so furthermore the sequence (A.A) also decomposes into graded pieces, or *homogeneous slices*, each of which is a complex of finite-dimensional vector spaces, exact at the codomains of the restrictions of d_0 , d_1 and d_2 .

Definition A.3. For $G \in \text{SP}_k$, (A.B) determines the *Hilbert series* of $A\mathcal{J}ac(G)$

$$\mathcal{H}_G = \sum_{i \geq 0} \dim_{\mathbb{C}} (A\mathcal{J}ac(G)_i) t^i \in \mathbb{C}[[t]].$$

Throughout, recall that SP_k and $\text{SP}_{k,m}$ are defined only when $k \geq 3$. The following is an easy consequence of the degree of the morphisms in (2) above, see e.g. [ISm, §3].

Corollary A.4. *If $G \in \text{SP}_k$ is exact, then $\mathcal{H}_G = (1 - dt + dt^{k-1} - t^k)^{-1}$.*

It will be convenient to consider the following subset of homogeneous superpotentials

$$\text{ESP}_k := \{G \in \text{SP}_k \mid G \text{ is an exact potential, and } A\mathcal{J}ac(G) \xrightarrow{x_1} A\mathcal{J}ac(G) \text{ is injective}\}.$$

Recall that $d \geq 2$ is the number of variables in $\mathbb{C}\langle x \rangle$.

Lemma A.5 ([IS2, 2.1, 2.2]). *If $k \geq 3$ with $(d, k) \neq (2, 3)$, then $\text{ESP}_k \neq \emptyset$.*

Proof. Set

$$g = \begin{cases} \sum_{\sigma \in \mathfrak{S}_{d-1}} x_d^{k-d+1} x_{\sigma(1)} \dots x_{\sigma(d-1)} & \text{if } k \geq d \\ x_d x_{d-1} \dots x_{d-k+1} + \sum_{j \in \mathcal{S}} x_j x_d \mathfrak{m}_j & \text{if } k < d \end{cases}$$

where \mathfrak{S}_{d-1} is the symmetric group, $\mathcal{S} = \{j \mid 1 \leq j \leq d-1, j \neq d-k+1\}$, and the \mathfrak{m}_j are explicit monomials explained in [IS2, 2.2]. It is a reasonably elementary calculation to show that $G = \text{cyc}(g) \in \text{ESP}_k$; see [IS2, 2.1, 2.2]. \square

It turns out, although we do not need this, that if $d = 2$ then $\text{ESP}_3 = \emptyset$. This is why the argument in [A.13](#) below fails in the $(d, k) = (2, 3)$ case.

Recalling that $\mathfrak{R}^i = (\mathfrak{m}^i + I_F)/I_F$, the following is one of the main insights of [\[IS2\]](#).

Corollary A.6. *If $F \in \text{SP}_{k,m}$ with $F_k \in \text{ESP}_k$, then left multiplication*

$$\text{AJac}(F)/\mathfrak{R}^j \xrightarrow{x_1} \text{AJac}(F)/\mathfrak{R}^{j+1}$$

is injective for all $j \geq 1$.

Proof. This is [\[IS2, 3.1\]](#). A proof in the notation used here is in [arXiv:2111.05900v1](#), as Proposition A.12. \square

A.3. Very general elements. Fixing, once and for all, a basis f_1, f_2, \dots, f_r of SP_k , we treat SP_k as an irreducible algebraic family of superpotentials, and identify it with $\mathbb{C}^r = \text{Spec } \mathbb{C}[\mathfrak{t}]$, where $\mathfrak{t} = t_1, \dots, t_r$ are parameter variables and any element of SP_k is the specialisation $\mathfrak{t} = a$ at a point $a \in \mathbb{C}^r$ of the ‘generic’ superpotential $G_{\mathfrak{t}} = \sum t_i f_i$. In particular, SP_k inherits the Zariski topology from \mathbb{C}^r . Under this identification, it is natural to abbreviate $G_a \in \text{SP}_k$ by $a \in \text{SP}_k$.

Lemma A.7 (c.f. [\[ISm, 3.9\]](#)). *Fix (d, k) and consider SP_k with its Zariski topology as above. For any $i \geq 0$,*

- (1) *there is a non-empty Zariski open subset $U_i \subset \text{SP}_k$ on which $\dim \text{AJac}(G_a)_i$ is constant for all $a \in U_i$, and takes the minimum value of any $G \in \text{SP}_k$.*
- (2) *there is a largest Zariski open subset $V_i \subset U_i \cap U_{i+1}$ on which the rank of the restriction $\mathfrak{d}_3: \text{AJac}(G_a)_i \rightarrow (\text{AJac}(G_a)^{\oplus d})_{i+1}$ for any $a \in V_i$ is the maximum possible for a linear map between spaces of these dimensions (i.e. is injective).*
- (3) *there is a largest Zariski open subset $W_i \subset V_i \cap U_{i+k-1}$ on which the rank of the restriction $\mathfrak{d}_2: (\text{AJac}(G_a)^{\oplus d})_{i+1} \rightarrow (\text{AJac}(G_a)^{\oplus d})_{i+k-1}$ for any $a \in W_i$ is the maximum possible for a linear map whose kernel contains the image of \mathfrak{d}_3 .*

Thus if W_i not empty, then for $a \in W_i$, the homogeneous slice of [\(A.A\)](#) with the domain of \mathfrak{d}_3 in degree i is an exact sequence of finite-dimensional vector spaces.

Proof. (1) This is [\[IS1, 2.2\]](#), proved as in [\[IS1, 2.1\]](#). The point is this: let $a_0 \in \text{SP}_k$ be a point at which the minimum dimension is achieved, and choose a subset of the set Φ_i of all monomials of degree i for which the (square) coefficient matrix of a basis of the kernel of $\mathbb{C}\langle x \rangle_i \rightarrow \text{AJac}(G_{a_0})_i$ with respect to Φ_i is invertible (a section of the surjection). The entries of this square matrix are algebraic in a , so it remains invertible as a varies in a Zariski open subset $a_0 \in U_i \subset \text{SP}_k$. Thus the kernel cannot get smaller, so must have constant dimension, and thus the minimum is also achieved for all $a \in U_i$.

(2) Within $U_i \cap U_{i+1}$, the map $\mathfrak{d}_3: \text{AJac}(G_a)_i \rightarrow (\text{AJac}(G_a)^{\oplus d})_{i+1}$ is determined by a matrix of fixed size whose entries are functions in a . Maximising its rank is an open condition on a , since it occurs on the complement of a locus of vanishing minors. (It is possible that all relevant minors vanish identically, in which case the conclusion is simply that V_i is empty.)

(3) Similarly, maximising the rank of \mathfrak{d}_2 is an open condition prescribed by minors that are functions in a . The condition that the kernel contains the image of \mathfrak{d}_3 is already imposed by the entries of the matrix, since $a \in V_i$ and [\(A.A\)](#) is a complex. \square

Corollary A.8 ([\[ISm, 3.2\]](#)). *Let $k \geq 3$ and $(d, k) \neq (2, 3)$.*

- (1) *For each $i \geq 0$, the minimum value achieved by $\dim \text{AJac}(G)_i$ for any $G \in \text{SP}_k$, as in [A.7\(1\)](#), is the coefficient of t^i in the expansion of $(1 - dt + dt^{k-1} - t^k)^{-1}$.*
- (2) *If $G \in \text{SP}_k$ is exact, then \mathcal{H}_G attains the minimum coefficient for every term t^i .*

In particular, each W_i in [A.7\(2\)](#) is non-empty, and the minimum Hilbert series in the set $\{\mathcal{H}_F \mid F \in \text{SP}_k\}$ is $(1 - dt + dt^{k-1} - t^k)^{-1}$.

Proof. The proof follows [\[ISm, 3.2, 3.8\]](#), which is for $d = 2$ but generalises with no change, using the existence of the exact potentials in [A.5](#). \square

For formal power series $\varphi(t)$ and $\psi(t)$, write $\varphi \geq \psi$ to mean that the coefficients of $\varphi - \psi$ are all non-negative. If \mathcal{P} is a family of power series, then $\psi \in \mathcal{P}$ is called the *minimum* if $\varphi \geq \psi$ for all $\varphi \in \mathcal{P}$, noting that the minimum does not necessarily exist.

Proposition A.9. *If $k \geq 3$ and $(d, k) \neq (2, 3)$, then there exists a countable intersection \mathcal{U} of non-empty Zariski opens of \mathbf{SP}_k such that $G \in \mathbf{ESP}_k$ for all $G \in \mathcal{U}$.*

Proof. Let \mathcal{U}_1 be the countable intersection over all $i \geq 0$ of the Zariski open subsets W_i of A.7. By A.8, each W_i is non-empty, thus this intersection is non-empty. Similarly, there is another such countable intersection $\mathcal{U}_2 \subset \mathbf{SP}_k$ on which the left-multiplication map $x_1: \mathbf{AJac}(G) \rightarrow \mathbf{AJac}(G)$ is injective, as injectivity maximises rank in each degree. Since x_1 is injective on the free algebra (at $t = 0$), or again applying A.5, this intersection is also non-empty. Thus $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ is a non-empty countable intersection. \square

For a given $F \in \mathbf{SP}_{k,m}$, recall the notation \mathfrak{m} and I_F from §A.1. The *Poincaré series* of F is defined to be

$$\mathcal{P}_F = \sum_{i \geq 0} \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\langle x \rangle}{\mathfrak{m}^{i+1} + I_F} \right) t^i. \quad (\text{A.C})$$

This measures the growth of the quotients of $\mathbf{AJac}(F)$ by the ideals $(\mathfrak{m}^{i+1} + I_F)/I_F$. The general situation is more delicate than the homogeneous case of A.9, but still a minimum is achieved by a very general element.

Corollary A.10. *If $k \geq 3$ and $(d, k) \neq (2, 3)$, then for any $m \geq k$ there exists $F \in \mathbf{SP}_{k,m}$ such that the following statements hold.*

- (1) \mathcal{P}_F is the minimum in $\{\mathcal{P}_H \mid H \in \mathbf{SP}_{k,m}\}$.
- (2) $F_{\text{ord}(F)} \in \mathcal{U}$, where $\mathcal{U} \subset \mathbf{SP}_k$ is defined in A.9.

Proof. Similar to A.7(1) (cf. [IS1, 2.1]), minimising each coefficient of the Poincaré series is an open condition in a family, and so the minimum is realised on a countable intersection of non-empty Zariski open subsets $\mathcal{V} \subset \mathbf{SP}_{k,m}$.

Consider the map $\mathbf{SP}_{k,m} \rightarrow \mathbf{SP}_k$ given by $F \mapsto F_k$. Intersecting the preimage of $\mathcal{U} \subset \mathbf{SP}_k$ with \mathcal{V} determines a countable intersection of open subsets of $\mathbf{SP}_{k,m}$ on which the claims hold. Since \mathbb{C} is uncountable, this set contains a closed point. \square

A.4. Power Series. Given $f \in \mathbb{C}\langle x \rangle$, its Poincaré series is defined to be

$$\hat{\mathcal{P}}_f = \sum (\dim_{\mathbb{C}} \mathcal{J}ac(f)/\mathfrak{J}^i) t^i$$

where recall that \mathfrak{J} is the Jacobson radical of $\mathcal{J}ac(f)$.

When $F \in \mathbf{SP}_{k,m}$, we can view F as either a polynomial superpotential and form \mathcal{P}_F in (A.C), or we can view F as an element of $\mathbb{C}\langle x \rangle$ and form $\hat{\mathcal{P}}_F$. Since $\mathcal{J}ac(f)$ is defined with respect to the cyclic derivatives δ , and $\mathbf{AJac}(f)$ is defined with respect to strike-off derivatives ∂ , it is not quite true that $\hat{\mathcal{P}}_F = \mathcal{P}_F$.

Lemma A.11. *Given $G \in \mathbf{SP}_{k,m}$, then $\mathcal{P}_G = \hat{\mathcal{P}}_{\mathbb{G}}$, where $\mathbb{G} = \sum_{i=k}^m \frac{1}{i} G_i$. Thus, if $G \in \mathbf{SP}_k$, then $\hat{\mathcal{P}}_{\mathbb{G}} = \mathcal{P}_G = \frac{1}{1-t} \mathcal{H}_G$ where $\mathbb{G} = \frac{1}{k} G$.*

Now for $f \in \mathbb{C}\langle x \rangle$ and for any $i \geq 0$, set $\mathfrak{f}_i = f_{\leq i}$ and $\mathbf{F}_i = \text{cyc}(\mathfrak{f}_i)$. Then, continuing the notation of §2.2, and recalling that $\mathfrak{R}^i = (\mathfrak{m}^i + I_F)/I_F$,

$$\begin{aligned} \mathcal{J}ac(f)/\mathfrak{J}^i &\cong \frac{\mathbb{C}\langle x \rangle}{(\mathfrak{n}^i, \delta_1 f, \dots, \delta_d f)} \\ &\cong \frac{\mathbb{C}\langle x \rangle}{(\mathfrak{m}^i, \delta_1 \mathfrak{f}_i, \dots, \delta_d \mathfrak{f}_i)} \\ &\cong \frac{\mathbb{C}\langle x \rangle}{(\mathfrak{m}^i, \partial_1(\text{cyc } \mathfrak{f}_i), \dots, \partial_d(\text{cyc } \mathfrak{f}_i))} \cong \mathbf{AJac}(\mathbf{F}_i)/\mathfrak{R}^i \end{aligned} \quad (\text{A.D})$$

This gives a term-by-term algebraicisation of the Poincaré series, by

$$\hat{\mathcal{P}}_f = \sum_{i \geq 0} (\dim_{\mathbb{C}} \mathbf{AJac}(\mathbf{F}_i)/\mathfrak{R}^i) t^i.$$

A.5. Main Result. The main result, [A.13](#), requires the following elementary lemma.

Lemma A.12. For $d \in \mathbb{R}$ and $k \geq 2$ consider the formal power series

$$\frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)} = \sum_{i \geq 0} b_i t^i.$$

Setting $b_j = 0$ for $j < 0$, the following statements hold.

- (1) There is an equality $b_j = db_{j-1} - db_{j-k+1} + b_{j-k} + 1$ for all $j \geq 0$.
- (2) $b_0 = 1$, and further $b_j = 1 + d + \dots + d^j$ for all $1 \leq j \leq k-2$.

Proof. (1) Treating the series as a sum over $i \in \mathbb{Z}$, with $b_{<0} = 0$, and multiplying up shows at once that $b_0 = 1$ and that for any $j \in \mathbb{Z} \setminus \{0\}$

$$b_j - (d+1)b_{j-1} + db_{j-2} + db_{j-k+1} - (d+1)b_{j-k} + b_{j-k-1} = 0.$$

When $j = 0$, the claimed equality in (1) holds. For $j \geq 1$, splitting off a single b_{j-1} summand from the equation above we see by induction that

$$\begin{aligned} b_j &= db_{j-1} - db_{j-2} - db_{j-k+1} + (d+1)b_{j-k} - b_{j-k-1} \\ &\quad + (db_{j-2} - db_{j-k} + b_{j-k-1} + 1) \\ &= db_{j-1} - db_{j-k+1} + b_{j-k} + 1. \end{aligned}$$

(2) For $1 \leq j \leq k-2$, the equality in (1) reads $b_j = b_{j-1}d + 1$, and so the result follows since $b_0 = 1$. \square

The following is the main result of this appendix, and it asserts that, in almost all cases, the \mathfrak{J} -dimension of $\mathcal{J}ac(f)$ is ≥ 3 . In particular, in almost all cases the Jacobi algebra is infinite dimensional, as a vector space. Recall from [§2.1](#) that $\mathbb{C}\langle\mathbf{x}\rangle_{\geq k}$ consists of all those $f \in \mathbb{C}\langle\mathbf{x}\rangle$ for which $f_j = 0$ for all $j < k$, and note that $0 \in \mathbb{C}\langle\mathbf{x}\rangle_{\geq k}$.

Theorem A.13. Suppose that $d = 2$ and $k \geq 4$, or $d \geq 3$ and $k \geq 3$. If $f \in \mathbb{C}\langle\mathbf{x}\rangle$ has order k , then $\text{Jdim } \mathcal{J}ac(f) \geq 3$.

The proof will show that the coefficients of the Poincaré series of $\mathcal{J}ac(f)$ are no smaller than those of

$$\frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)}.$$

When $d = 2$ and $k = 4$ this lower bound is $1/((1-t)^3(1-t^2))$, and when $d = k = 3$ it is $1/(1-t)^4$, both of which have polynomial growth of degree 3. For all other d, k in the scope of the theorem, the growth is exponential and $\text{Jdim } \mathcal{J}ac(f) = \infty$.

Proof. Associated to the fixed $k = \text{ord}(f)$ and d is the power series

$$\frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)} = \sum_{i \geq 0} b_i t^i.$$

The b_i s are integers that depend only on k and d . Similarly, associated to k and d are the positive integers a_0, a_1, \dots defined to be

$$\begin{aligned} a_j &:= \min\{\dim_{\mathbb{C}} \left(\frac{\mathcal{J}ac(g)}{\mathfrak{J}^{j+1}} \right) \mid g \in \mathbb{C}\langle\mathbf{x}\rangle_{\geq k}\} \\ &= \min\{\dim_{\mathbb{C}} \left(\frac{\mathcal{J}ac(g)}{\mathfrak{J}^{j+1}} \right) \mid g \in \mathbb{C}\langle\mathbf{x}\rangle_{\geq k}\} \quad (\text{truncate terms mod } \mathfrak{J}^{j+1}) \\ &= \min\{\dim_{\mathbb{C}} \left(\frac{\mathcal{J}ac(g)}{\mathfrak{J}^{j+1}} \right) \mid g \in \mathbb{C}\langle\mathbf{x}\rangle_{\geq k}, g_t = 0 \text{ for } t > j+1\} \\ &= \begin{cases} \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\langle\mathbf{x}\rangle}{\mathfrak{n}^{j+1}} \right) & \text{if } j \leq k-2 \\ \min \left\{ \dim_{\mathbb{C}} \left(\frac{A\mathcal{J}ac(G)}{\mathfrak{A}^{j+1}} \right) \mid G \in \text{SP}_{k,j+1} \right\} & \text{if } j \geq k-1. \end{cases} \quad (\text{by (A.D)}) \end{aligned}$$

Certainly $\hat{\mathcal{P}}_f \geq \sum_{i \geq 0} a_i t^i$, by the minimality of the a_i . We claim that $a_i = b_i$ for all $i \geq 0$, since then

$$\hat{\mathcal{P}}_f \geq \sum_{i \geq 0} a_i t^i = \sum_{i \geq 0} b_i t^i = \frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)},$$

which has the prescribed growth as in the statement of the result.

So, from here on we discard the original f , and instead prove that $a_i = b_i$ for all $i \geq 0$. This is a statement which depends only on k and d .

Since by assumption $(d, k) \neq (2, 3)$, by A.5 $\text{ESP}_k \neq \emptyset$ and so choose $G \in \text{ESP}_k$. Since G is exact, by A.4 $\mathcal{H}_G = (1 - dt + dt^{k-1} - t^k)^{-1}$, and so by A.11 for $\mathbb{G} = \frac{1}{k}G$ we have

$$\hat{\mathcal{P}}_{\mathbb{G}} = \frac{1}{1-t} \cdot \mathcal{H}_G = \frac{1}{(1-t)(1-dt+dt^{k-1}-t^k)} = \sum_{i \geq 0} b_i t^i.$$

Since \mathbb{G} exists, it follows immediately by minimality of the a_i s that $a_i \leq b_i$ for all $i \geq 0$.

Now clearly $a_0 = b_0 = 1$, and further for all $1 \leq j \leq k-2$ we have $a_j = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\langle x \rangle}{\mathfrak{m}^{j+1}} \right)$, which equals b_j by A.12(2). Further, since d relations of degree k can cut down the dimension of $\mathbb{C}\langle x \rangle / \mathfrak{J}^{k+1}$ by at most d , it follows that $a_{k-1} \geq 1 + d + \dots + d^k - d$. This equals b_{k-1} by A.12(2), and so $a_{k-1} \geq b_{k-1}$, which in turn forces $a_{k-1} = b_{k-1}$.

Thus, by induction we can suppose that $a_j = b_j$ for all $0 \leq j \leq s$, for some $s \geq k-1$. The proof will be completed once we show that $a_{s+1} = b_{s+1}$.

Now by A.10 applied to $m = s+2$, there exists $F \in \text{SP}_{k, s+2}$ for which \mathcal{P}_F is the minimum in $\{\mathcal{P}_H \mid H \in \text{SP}_{k, s+2}\}$, and further $F_k \in \mathcal{U}$ where \mathcal{U} is from A.9. By the first of these facts, since for all $j \leq s+1$ by truncation

$$a_j = \min \left\{ \dim_{\mathbb{C}} \left(\frac{\text{AJac}(H)}{\mathfrak{R}^{j+1}} \right) \mid H \in \text{SP}_{k, j+1} \right\} = \min \left\{ \dim_{\mathbb{C}} \left(\frac{\text{AJac}(H)}{\mathfrak{R}^{j+1}} \right) \mid H \in \text{SP}_{k, s+2} \right\}$$

it follows that

$$\mathcal{P}_F = \sum_{j=0}^{s+1} a_j t^j + \mathcal{O}_{s+2}. \quad (\text{A.E})$$

On the other hand, since $F_k \in \mathcal{U}$, by definition $F_k \in \text{ESP}_k$. Thus, by A.6 for all $j \geq 1$ the left multiplication by x_1

$$\text{AJac}(F) / \mathfrak{R}^j \xrightarrow{x_1 \cdot} \text{AJac}(F) / \mathfrak{R}^{j+1}$$

is injective.

Set $I = (\partial_1 F, \dots, \partial_d F) \subset \mathbb{C}\langle x \rangle$. The above asserts that there is an injection

$$\frac{\mathbb{C}\langle x \rangle}{I + \mathfrak{m}^j} \xrightarrow{x_1 \cdot} \frac{\mathbb{C}\langle x \rangle}{I + \mathfrak{m}^{j+1}}$$

for all $j \geq 1$. This allows us to pick inductively sets of monomials M_j of $\mathbb{C}\langle x \rangle$ of degree j , starting with $M_0 = \{1\}$, such that the following two conditions are satisfied.

- (1) There is an inclusion $x_1 M_j \subseteq M_{j+1}$.
- (2) The necessarily disjoint union $N_j = M_0 \cup \dots \cup M_j$ projects down to give a basis of $\mathbb{C}\langle x \rangle / (I + \mathfrak{m}^{j+1})$.

To fix notation, set $B_j = \text{Span}_{\mathbb{C}}(N_j) \subset \mathbb{C}\langle x \rangle$, and define \mathbf{b}_j via the equality

$$\dim_{\mathbb{C}} B_j = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}\langle x \rangle}{I + \mathfrak{m}^{j+1}} \right) = \mathbf{b}_j.$$

Note that the second equality implies that $\mathcal{P}_F = \sum_{i \geq 0} \mathbf{b}_i t^i$.

Write $V = \text{Span}_{\mathbb{C}}\{x_1, \dots, x_d\} \subseteq \mathbb{C}\langle x \rangle$, $R = \text{Span}_{\mathbb{C}}\{\partial_1 F, \dots, \partial_d F\} \subseteq \mathbb{C}\langle x \rangle$, and for $j \geq 0$ consider the quotient map $\pi_j: \mathbb{C}\langle x \rangle \rightarrow \mathbb{C}\langle x \rangle / \mathfrak{m}^{j+1}$. By the definition of \mathbf{b}_j , for every $j \geq 0$

$$\dim_{\mathbb{C}} \pi_j(I) = 1 + d + \dots + d^j - \mathbf{b}_j. \quad (\text{A.F})$$

We now apply the adapted Vinberg argument: elements of the two-sided ideal I are sums of elements, each of which either starts with an x_i or starts with a derivative $\partial_i F$, so we can write

$$I = VI + R\mathbb{C}\langle x \rangle.$$

Applying π_{j+1} then gives an equality

$$\pi_{j+1}(I) = \pi_{j+1}(VI) + \pi_{j+1}(R\mathbb{C}\langle x \rangle).$$

Since B_j descends to span $\mathbb{C}\langle x \rangle / (I + \mathfrak{m}^{j+1})$, every element of $\mathbb{C}\langle x \rangle$ may be written as an element in B_j , plus an element in I , plus an element in \mathfrak{m}^{j+1} . Projecting down this sum via π_{j+1} , and noting that $RI \subset R\mathbb{C}\langle x \rangle$ gets absorbed into $\pi_{j+1}(I)$, and elements of R have degree $\geq k$, it follows that mod \mathfrak{m}^{j+1} there is an equality

$$\pi_{j+1}(I) = \pi_{j+1}(VI) + \pi_{j+1}(RB_{j+2-k}).$$

Write $R' = \text{Span}_{\mathbb{C}}\{\partial_2 F, \dots, \partial_d F\} \subseteq \mathbb{C}\langle x \rangle$, then using the Euler relation $\sum_{i=0}^d [x_i, \partial_i F] = 0$ of **A.1** we may get rid of any appearance of the product $(\partial_1 F)x_1$ at the cost of terms in the other summands. It follows that

$$\pi_{j+1}(I) = \pi_{j+1}(VI) + \pi_{j+1}(R'B_{j+2-k}) + \pi_{j+1}((\partial_1 F)B_{j+2-k}^+),$$

where $B_{j+2-k}^+ = \text{Span}_{\mathbb{C}}\{n \in N_{j+2-k} \mid n \neq x_1 m \text{ for any } m\}$.

The proof is completed by estimating the dimension of each of the three individual summands. Applying **(A.F)** for the first summand,

$$\dim_{\mathbb{C}} \pi_{j+1}(I) \leq d(1 + d + \dots + d^j - \mathbf{b}_j) + (d-1)\mathbf{b}_{j+2-k} + (\mathbf{b}_{j+2-k} - \mathbf{b}_{j+1-k})$$

since $\dim_{\mathbb{C}} B_{j+2-k}^+ \leq \mathbf{b}_{j+2-k} - \mathbf{b}_{j+1-k}$ holds by construction of the N_j in **(1)**. Plugging **(A.F)** for π_{j+1} into the above displayed equation, and then cancelling, it follows that

$$1 - \mathbf{b}_{j+1} \leq -d\mathbf{b}_j + d\mathbf{b}_{j+2-k} - \mathbf{b}_{j+1-k}.$$

which after re-arranging gives

$$\mathbf{b}_{j+1} \geq \mathbf{b}_j d - \mathbf{b}_{j+2-k} d + \mathbf{b}_{j+1-k} + 1. \quad (\text{A.G})$$

Since $\mathcal{P}_F = \sum_{i \geq 0} \mathbf{b}_j t^j$, by **(A.E)** we see that $\mathbf{b}_j = a_j$ for $0 \leq j \leq s+1$, and hence

$$\begin{aligned} a_{s+1} &\geq a_s d - a_{s+2-k} d + a_{s+1-k} + 1 && ((\text{A.G}) \text{ for } j = s) \\ &= b_s d - b_{s+2-k} d + b_{s+1-k} + 1 && (a_j = b_j \text{ for } j \leq s \text{ by induction}) \\ &= b_{s+1} && (\text{A.12(1)} \text{ for } j = s+1) \end{aligned}$$

Since we already know $a_{s+1} \leq b_{s+1}$ by minimality, the above forces $a_{s+1} = b_{s+1}$. Hence by induction $a_j = b_j$ for all $j \geq 0$, and the result follows. \square

A.6. Contractibility Consequence. The following is an immediate geometric consequence of the above.

Theorem A.14. *Let $C \subset \mathcal{X}$ be an irreducible rational curve in a smooth CY 3-fold, with NC deformation algebra Λ_{def} , such that $\mathcal{N}_{C|\mathcal{X}} \neq (-3, 1)$. Then $C \subset \mathcal{X}$ contracts to a point suitably locally, without contracting a divisor, if and only if $\dim_{\mathbb{C}} \Lambda_{\text{def}} < \infty$.*

Proof. The case of $(-1, -1)$ and $(-2, 0)$ -curves is of course well known [R2, DW1]. The point is that $(-4, 2), (-5, 3), \dots$ curves never contract [L2, 4.1], and further as a consequence of [V1] their noncommutative deformations are given as the Jacobi algebra quotient of a free power series ring in 3, 4, \dots variables. By A.13, these can never be finite dimensional. Thus in all these cases the statement is true, just since the curves never contract, and the deformation algebras are always infinite dimensional. \square

REFERENCES

- [A1] M. Artin, *Some numerical criteria for contractability of curves on algebraic surfaces*, Amer. J. Math. **84** (1962), 485–496.
- [AM] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.
- [AB] K. Ardakov and K. A. Brown, *Ring-theoretic properties of Iwasawa algebras: a survey*, Doc. Math. 2006, Extra Vol., 7–33.
- [A2] V. I. Arnold, *Local normal forms of functions*, Invent. Math. **35** (1976), 87–109.
- [A3] J. August, *The tilting theory of contraction algebras*, Adv. Math. **374** (2020), 107372, 56 pp.
- [BCP] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., **24** (1997), 235–265. <http://www.magma.usyd.edu>
- [BW1] G. Brown and M. Wemyss, *Gopakumar–Vafa invariants do not determine flops*, Comm. Math. Phys. **361** (2018), no. 1, 143–154.
- [BW2] G. Brown and M. Wemyss, *Derived deformation theory of crepant curves*, [arXiv:2310.06133](https://arxiv.org/abs/2310.06133).
- [BW4] G. Brown and M. Wemyss, *Exceptional noncommutative local normal forms*, in preparation.
- [BIRS] A. B. Buan, O. Iyama, I. Reiten, and D. Smith, *Mutation of cluster-tilting objects and potentials*, Amer. J. Math. **133** (2011), no. 4, 835–887.
- [BGS] R.-O. Buchweitz, G.-M. Greuel, and F.-O. Schreyer, *Cohen-Macaulay modules on hypersurface singularities. II*, Invent. Math. **88** (1987), no. 1, 165–182.
- [CB] W. Crawley-Boevey, *Noncommutative algebra 1*, Bielefeld, Summer Semester 2019 [pdf](https://www.math.uni-bielefeld.de/~crawley-boevey/).
- [CBH] W. Crawley-Boevey and M. P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no. 3, 605–635.
- [D] B. Davison, *Refined invariants of finite-dimensional Jacobi algebras*. [arXiv:1903.00659](https://arxiv.org/abs/1903.00659).

- [DGPS] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, *SINGULAR 4-1-0 — A computer algebra system for polynomial computations*, <http://www.singular.uni-kl.de> (2017).
- [DWZ] H. Derksen, J. Weyman, and A. Zelevinsky, *Quivers with potentials and their representations. I. Mutations*. *Selecta Math. (N.S.)* **14** (2008), no. 1, 59–119.
- [DW1] W. Donovan and M. Wemyss, *Noncommutative deformations and flops*, *Duke Math. J.* **165** (2016), no. 8, 1397–1474.
- [DW2] W. Donovan and M. Wemyss, *Contractions and deformations*, *Amer. J. Math.* **141** (2019), no. 3, 563–592.
- [DW3] W. Donovan and M. Wemyss, *Twists and braids for general 3-fold flops*, *J. Eur. Math. Soc. (JEMS)*, **21** (2019), no. 6, 1641–1701.
- [DW4] W. Donovan and M. Wemyss, *Noncommutative enhancements of contractions*, *Adv. Math.* **344** (2019), 99–136.
- [Du] T. Ducat, *Divisorial extractions from singular curves in smooth 3-folds*. *Internat. J. Math.* **27** (2016), no. 1, 1650005, 23 pp.
- [E1] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, *Graduate Texts in Mathematics*, **150**. Springer-Verlag, New York, 1995.
- [E2] M. Ershov, *Golod–Shafarevich groups: a survey*, *Internat. J. Algebra Comput.*, **22** (2012), no. 5, 1230001, 68 pp.
- [GS] E. S. Golod and I. R. Šafarevič, *On the class field tower*, *Izv. Akad. Nauk SSSR Ser. Mat.*, **28**, 261–272, 1964.
- [HT] Z. Hua and Y. Toda, *Contraction algebra and invariants of singularities*, *Int. Math. Res. Not. IMRN* 2018, no. 10, 3173–3198.
- [HZ] Z. Hua, and G. Zhou, *Quasi-homogeneity of potentials*, *J. Noncommut. Geom.* **15** (2021), no. 2, 399–422.
- [IW1] O. Iyama and M. Wemyss, *Singular Derived Categories of \mathbb{Q} -factorial terminalizations and Maximal Modification Algebras*, *Adv. Math.* **261** (2014), 85–121.
- [IW2] O. Iyama and M. Wemyss, *Reduction of triangulated categories and Maximal Modification Algebras for cA_n singularities*, *J. Reine Angew. Math.* **738** (2018), 149–202.
- [I] N. Iyudu, *Classification of contraction algebras and pre-Lie algebras associated to braces and trusses*, [arXiv:2008.06033](https://arxiv.org/abs/2008.06033).
- [IS1] N Iyudu and S. Shkarin, *Potential algebras with few generators*, [arXiv:1806.06829 v1](https://arxiv.org/abs/1806.06829)
- [IS2] N Iyudu and S. Shkarin, *Golod-Shafarevich-Vinberg type theorems and finiteness conditions for potential algebras*, [arXiv:2201.04479](https://arxiv.org/abs/2201.04479)
- [ISm] N Iyudu and A. Smoktunowicz, *Golod-Shafarevich-type theorems and potential algebras*, *Int. Math. Res. Not. IMRN* 2019, no. 15, 4822–4844.
- [JKM] G. Jasso, B. Keller and F. Muro, *The Derived Auslander-Iyama Correspondence*, [arXiv:2208.14413](https://arxiv.org/abs/2208.14413).
- [J] J. Jiménez, *Contraction of nonsingular curves*, *Duke Math. J.* **65** (1992), no. 2, 313–332.
- [KM] S. Katz and D. R. Morrison, *Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups*, *J. Algebraic Geom.* **1** (1992) 449–530.
- [K] J. Karmazyn, *The length classification of threefold flops via noncommutative algebras*, *Adv. Math.* **343** (2019), 393–447.
- [Ka] Y. Kawamata, *Non-commutative deformations of perverse coherent sheaves and rational curves*, [arXiv:2006.09547](https://arxiv.org/abs/2006.09547).
- [Ka2] Y. Kawamata, *On non-commutative formal deformations of coherent sheaves on an algebraic variety*, *EMS Surv. Math. Sci.* **8** (2021), no. 1-2, 237–263.
- [KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, *Algebraic geometry*, Sendai, 1985, 283–360, *Adv. Stud. Pure Math.*, **10**, North-Holland, Amsterdam, 1987.
- [KL] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, revised edition. *Graduate Studies in Mathematics*, **22**. American Mathematical Society, Providence, RI, 2000. x+212 pp. ISBN: 0-8218-0859-1.
- [L1] T. Y. Lam, *A first course in noncommutative rings*, Second edition. *Graduate Texts in Mathematics*, **131**. Springer-Verlag, New York, 2001. xx+385 pp.
- [L2] Henry B. Laufer, *On $\mathbb{C}P^1$ as an exceptional set*, In *Recent developments in several complex variables (Proc. Conf., Princeton Univ., Princeton, N. J., 1979)*, volume No. **100** of *Ann. of Math. Stud.*, pages 261–275. Princeton Univ. Press, Princeton, NJ, 1981.
- [M1] H. Matsumura, *Commutative ring theory*, Second edition. *Cambridge Studies in Advanced Mathematics*, **8**. Cambridge University Press, Cambridge, 1989. xiv+320 pp.
- [M2] A. Mellit, *Kleinian singularities and algebras generated by elements that have given spectra and satisfy a scalar sum relation*, *Algebra Discrete Math.* 2004, no. 3, 89–110.
- [M3] S. Mori, *On 3-dimensional terminal singularities*, *Nagoya Math. J.* **98** (1985), 43–66.
- [R1] M. Reid, *Canonical 3-folds*, *Journées de Géométrie Algébrique d’Angers, Juillet 1979*, pp. 273–310, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.
- [R2] M. Reid, *Minimal models of canonical 3-folds*, *Algebraic varieties and analytic varieties (Tokyo, 1981)*, 131–180, *Adv. Stud. Pure Math.*, **1**, North-Holland, Amsterdam, 1983.
- [T] Y. Toda, *Non-commutative width and Gopakumar–Vafa invariants*, *Manuscripta Math.* **148** (2015), no. 3-4, 521–533.

- [T2] N. Tziolas, *Terminal 3-fold divisorial contractions of a surface to a curve. I*, *Compositio Math.* **139** (2003), no. 3, 239–261.
- [T3] N. Tziolas, *Three-fold divisorial extremal neighborhoods over $cE7$ and $cE6$ compound DuVal singularities*, *Internat. J. Math.* **21** (2010), no. 1, 1–23.
- [V1] M. Van den Bergh, *Calabi-Yau algebras and superpotentials*, *Selecta Math. (N.S.)* **21** (2015), no. 2, 555–603.
- [V2] M. Van den Bergh, *Three-dimensional flops and noncommutative rings*, *Duke Math. J.* **122** (2004), no. 3, 423–455.
- [vG] O. van Garderen, *Donaldson–Thomas invariants of length 2 flops*, [arXiv:2008.02591](https://arxiv.org/abs/2008.02591).
- [vG2] O. van Garderen, *Donaldson–Thomas invariants of threefold flops*, PhD thesis, University of Glasgow, 2021.
- [V3] E. B. Vinberg, *On the theorem concerning the infinite-dimensionality of an associative algebra*, *Izv. Akad. Nauk SSSR Ser. Mat.*, **29**, 209–214, 1965.
- [Wa] S. Warner, *Topological rings*, North-Holland Mathematics Studies, **178**. North-Holland Publishing Co., Amsterdam, 1993. x+498 pp.
- [W] P. M. H. Wilson, *The Kähler cone on Calabi-Yau threefolds*, *Invent. Math.* **107** (1992), no. 3, 561–583. *Erratum: ‘The Kähler cone on Calabi-Yau threefolds’ [Invent. Math. 107 (1992), no. 3, 561–583]*, *Invent. Math.* **114** (1993), no. 1, 231–233.

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