STABILITY ON CONTRACTION ALGEBRAS IMPLIES $K(\pi,1)$

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Abstract. This paper gives a description of the full space of Bridgeland stability conditions on the bounded derived category of a contraction algebra associated to a 3-fold flop. An immediate consequence is a homological proof of the $K(\pi,1)$-theorem in various finite settings, which includes ADE braid groups. The main result is that stability conditions provide the universal cover of a naturally associated hyperplane arrangement, which is known to be simplicial, and in special cases is an ADE root system. The key new insight is to study stability conditions not on the image of a spherical functor, but on the base, where contractibility can be approached via silting-discreteness. There are further geometric corollaries: a very short proof of faithfulness of pure braid group actions in various settings follows immediately, the first that avoids normal forms, as does a proof that a certain stability space $\operatorname{Stab}^{\circ}\mathcal{C}$ associated to a 3-CY category $\mathcal{C}$ is contractible.

1. Introduction

Let Λ_{con} denote the contraction algebra [DW1, DW3] associated to a 3-fold flopping contraction $f\colon X \to \operatorname{Spec} R$, where X has at worst Gorenstein terminal singularities. The purpose of this article is to give a description of stability conditions on the bounded derived category $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) = \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_{\mathrm{con}})$ of this finite dimensional algebra, motivated in part since this category plays a fundamental role in the proposed classification of flops, and in part from the purely algebraic viewpoint of cluster-tilting theory. Our main corollaries are topological and geometric.

1.1. **Change of Categories.** Associated to the flopping contraction f is the subcategory

$$\mathcal{C} := \{\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \mid \mathbf{R}f_*\mathcal{F} = 0\}$$

of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. With its finite dimensional Hom-spaces, and 3-CY property when X is smooth, the category $\mathcal{C}$ acts as a local model of a compact CY 3-fold. Furthermore, stability conditions on $\mathcal{C}$ are intimately related to the birational geometry of X. Indeed, it was recently established in [HW2] that there is a component of the space of stability conditions, $\operatorname{Stab}^{\circ}\mathcal{C}$, and a regular covering map

$$\operatorname{Stab}^{\circ}\mathcal{C} \to \mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}},$$

where $\mathcal{H}$ is the real simplicial hyperplane arrangement associated to f (see §2.2 for details), and $\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$ is its complexified complement. It is further proved that $\operatorname{Stab}^{\circ}\mathcal{C}$ is contractible, however this part *relies* on Deligne's proof of the $K(\pi,1)$-theorem for simplicial hyperplane arrangements [D1]. This is deeply unsatisfactory, as stability conditions are expected to be contractible in any reasonable geometric setting.

In this paper we solve this problem, by essentially mirroring the space of stability conditions on an associated singular 0-CY category, and proving contractibility there. This turns out to be much easier: we do not need to use *anything* about stability on the category $\mathcal{C}$, any known faithfulness results about actions on $\mathcal{C}$ or $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$, any normal forms, nor any previous work on surfaces [B3, BT]. Most of these fall out, for free, from our proof. In the process, we give an independent proof not only of Deligne's $K(\pi,1)$-theorem for ADE root systems, but also for their intersection arrangements.

The transfer between the algebra, in the form of the finite dimensional algebra Λ_{con}, and the geometry, in the form of X, is provided by deformation theory. Associated to the

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J.A. was supported by the Carnegie Trust for the Universities of Scotland, and M.W. was supported by EPSRC grants EP/R009325/1 and EP/R034826/1.

contraction f is a universal sheaf $\mathcal{E}$ [DW3] and a functor

$$-\otimes^{\mathbf{L}}_{\Lambda_{\mathrm{con}}}\mathcal{E}\colon \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})\to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

whose image equals $\mathcal{C}$. Our main new insight is to change categories, from $\mathcal{C}$ to $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$. Establishing results on $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is significantly easier. The above functor, which is known to intertwine well with tilting t-structures at simples, then allows us to transfer this information back to the geometry.

1.2. **Main result.** Our change in category brings one major advantage, namely that Λ_{con} is a silting-discrete algebra [A1, 3.10]. This fact has two happy consequences. First, we are able to describe the *full* space of stability conditions on $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$, not just a component as in the case of $\mathcal{C}$. Second, it is known that all silting-discrete algebras have contractible stability manifolds [PSZ], and thus $\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is contractible, before we even begin.

Alas, the change in category also brings one major disadvantage. Whilst contractibility comes for free, moving to finite dimensional algebras means that we lose the technology of Fourier–Mukai transforms, and so controlling standard equivalences becomes significantly more difficult. Indeed, after constructing in Lemma 4.7 a natural map

$$\operatorname{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})\to\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}},$$

all the hard work goes into proving that it is a regular cover. The age-old algebraic problem of knowing whether a given autoequivalence which is the identity on simples is globally the identity functor rears its head. Happily, in our setting we tame this problem by appealing to a commutative diagram that intertwines our algebraic equivalences with the geometric flop functors, where we can use a standard Fourier–Mukai argument (see Theorem 4.3).

Our main result is the following.

Theorem 1.1 (4.10)**.** *If Λ_{con} is the contraction algebra of f, then the natural map*

$$\operatorname{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})\to\mathcal{X}=\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$$

is a regular cover. Furthermore, $\operatorname{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ *is contractible and so this cover is universal.*

The simplicial hyperplane arrangement $\mathcal{H}$ can be described in many ways: intrinsically for Λ_{con}, it arises as the g-vector fan of its two-term silting complexes. Moreover, it should come as no surprise that $\mathcal{H}$ appears in the description of the stability manifold of the contraction algebra, as it is already known that $\mathcal{H}$ controls all of its tilting theory [A2]. The chambers of $\mathcal{H}$ are labelled by the contraction algebras of flopping contractions reached from f by iterated flops, and each path α in the skeleton graph of $\mathcal{H}$ is assigned a standard derived equivalence F_α. With this notation, the Galois group in Theorem 1.1 is the image of a *pure* braid group under the group homomorphism $\pi_1(\mathcal{X})\to\operatorname{Auteq}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ sending $\alpha\mapsto F_\alpha$.

1.3. **Corollaries.** The arrangement $\mathcal{H}$ associated to f is always an intersection arrangement inside the root system of some ADE Dynkin diagram. As such, Theorem 1.1 allows us to prove $K(\pi,1)$ for ADE braid groups without using [B3, BT], or normal forms. Indeed, if Λ_{con} is a contraction algebra with associated $\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}=\mathfrak{h}_{\mathrm{reg}}$, then it follows from Theorem 1.1 that the composition

$$\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})\to\mathfrak{h}_{\mathrm{reg}}\to\mathfrak{h}_{\mathrm{reg}}/W$$

is also a covering map.

Corollary 1.2 (5.1)**.** *The $K(\pi,1)$-conjecture holds for all ADE braid groups.*

We recall the $K(\pi,1)$-conjecture in Section 5. We remark that our techniques are slightly more general, and indeed we prove that given any intersection arrangement in any ADE root system, its complexified complement has contractible universal cover. This class of intersection arrangements is a bit eclectic: as a consequence, we prove $K(\pi,1)$ for the Coxeter groups I_n for $n=3,4,5,6,8$, but none of the other n. The upcoming memoir [IW2] describes in more detail the types of arrangements that can arise.

Another consequence of the silting-discrete contractibility theorem is the following, which gives a very short proof of [A2, 1.4]. Our approach here is also a conceptual improvement: faithful group actions should really be implied by the topology, and not from technical Ext group calculations and normal forms.

Corollary 1.3 (5.2). *The homomorphism* $\pi_1(\mathcal{X}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ *sending* $\alpha \mapsto F_\alpha$ *is injective.*

Combining the topological Corollary 1.3 with the algebraic-geometric Theorem 4.3 allows us to side-step algebra automorphism issues, and to fully classify one-sided tilting complexes in $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$. The following is a strengthening of [A2, 7.2].

Corollary 1.4 (5.3). *There is bijection between morphisms in the Deligne groupoid of* $\mathcal{H}$ *ending at the vertex associated to* Λ_{con} *and the set of isomorphism classes of basic one-sided tilting complexes of* Λ_{con}.

A more surprising consequence of the silting-discrete contractibility theorem is that it also implies both the faithfulness of the geometric action $\pi_1(\mathcal{X}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, and the contractibility of $\operatorname{Stab}^\circ \mathcal{C}$. The logic of the first statement is straightforward: given any element in the kernel of the geometric action, there is an element in the kernel of the algebraic action, and we can appeal directly to Corollary 1.3. As such, the following recovers the main result of the paper [HW1], but is a vast simplification. Relying on topological properties is both conceptually better, and shorter.

Corollary 1.5 (5.5). *The homomorphism* $\pi_1(\mathcal{X}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *sending* $\alpha \mapsto \mathsf{Flop}_\alpha$*, where* Flop_α *is the corresponding composition of Bridgeland-Chen flop functors, is injective.*

Given the faithful action from Corollary 1.5, which is implied from the silting-discrete contractibility theorem, we are also able to deduce that $\operatorname{Stab}^\circ \mathcal{C}$ is contractible. The logic again is straightforward: universal covers are unique. The proof of the following corollary is the only part of the paper where we use any prior results about $\operatorname{Stab}^\circ \mathcal{C}$.

Corollary 1.6 (5.6). $\operatorname{Stab}^\circ \mathcal{C}$ *is contractible.*

1.4. **Comparison to Literature.** It is possible to use surfaces to prove $K(\pi, 1)$ for ADE braid groups, by combining [B3], [BT] and [QW]. Indeed, the first describes the stability manifold of the surfaces analogue of $\mathcal{C}$ as a regular cover of an ADE root system, the second establishes universality, and using both of these results the third proves contractibility. Our approach via contraction algebras in the 3-fold setting is more general, but happily it is also easier and shorter. Our main point is that to extend $K(\pi, 1)$ to more general settings *requires* categories such as our $\mathcal{C}$ and $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ which no longer admit any easy combinatorial A_∞ description. However, this lack of combinatorial underpinning is more than made up for by the much nicer structural properties that the 3-fold setting affords.

The contraction algebra Λ_{con}, which is a symmetric finite dimensional algebra, allows us to construct the regular cover, control it, and prove contractibility, all very easily. As above, since Λ_{con} is silting-discrete [A1, 3.10], contractibility is known before we begin. The t-structures in $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ are all the tracking of module categories under the functors F_α, following [PSZ], and we furthermore argue in Theorem 4.3 that a certain autoequivalence of $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is the identity using Fourier–Mukai techniques. None of these arguments are topological in nature, although the latter is scheme-theoretic.

Topologically, we do rely on the following two facts.

1. That $\pi_1(\mathcal{X})$ is isomorphic to the vertex group of the Deligne groupoid of $\mathcal{H}$. This is a well-known and general fact (see e.g. [D2, p9]), not specific to finite simplicial hyperplane arrangements, and it is significantly weaker than $K(\pi, 1)$.
2. That the stability manifold for a silting-discrete algebra A is contractible [PSZ]. In turn, this relies on two results about the classifying spaces of posets: first, that the poset of silting pairs for A is a CW poset whose classifying space is contractible [BPP, 6.2, 7.1], and second, that $\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(A)$ has a regular, totally normal, CW-cellular stratification [QW] and hence it is homotopy equivalent to the classifying space of its poset of strata [FMT]. To relate these, [PSZ, 13] shows that the poset of strata is described algebraically as the poset of silting pairs. These techniques are not specific to finite simplicial hyperplane arrangements, as there are plenty of silting-discrete algebras whose g-vectors do not form a simplicial hyperplane arrangement.

Conventions. Throughout we work over the field of complex numbers. Given a noetherian ring A, modules will be right modules unless specified, and $\operatorname{mod} A$ denotes the category of finitely generated A-modules. We use the functional convention for composing arrows, so $f \circ g$ means g then f. With this convention, given a ring R, an R-module M is a left $\operatorname{End}_R(M)$-module. Furthermore, $\operatorname{Hom}_R(M,N)$ is a right $\operatorname{End}_R(M)$-module and a left $\operatorname{End}_R(N)$-module, in fact a bimodule.

2. Wall Crossing and Functorial Composition

Throughout this paper $f\colon X \to \operatorname{Spec} R$ is a fixed 3-fold flopping contraction, where X has at worst Gorenstein terminal singularities, and R is complete local. Necessarily R is an isolated cDV singularity [R]. Associated to f is a rigid Cohen–Macaulay (=CM) module N, an algebra $\Lambda := \operatorname{End}_R(N)$, and a contraction algebra Λ_{con}. We first briefly review these notions, mainly to set notation.

2.1. **Rigid Modules.** Since R is Gorenstein, recall that

$$\operatorname{CM} R := \{X \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(X,R) = 0 \text{ for all } i > 0\}.$$

For $X \in \operatorname{CM} R$, we say that X is *basic* if there are no repetitions in its Krull–Schmidt decomposition into indecomposables. We call X a *generator* if one of these indecomposable summands is R, and we call X *rigid* if $\operatorname{Ext}^1_R(X,X) = 0$. We say $X \in \operatorname{CM} R$ is *maximal rigid* if it is rigid, and furthermore it is maximal with respect to this property (see [IW1, 4.1]).

By the Auslander–McKay correspondence for cDV singularities, there is a one-to-one correspondence between flopping contractions $Y \to \operatorname{Spec} R$, up to R-isomorphism, and basic rigid generators in $\operatorname{CM} R$ [W, 4.13]. In particular, both sets are finite. For our fixed flopping contraction $f\colon X \to \operatorname{Spec} R$, the corresponding basic rigid CM generator across the bijection will throughout this paper be denoted N. The *contraction algebra* of f may then be defined as the stable endomorphism algebra $\Lambda_{\mathrm{con}} := \underline{\operatorname{End}}_R(N)$ [DW1, DW3].

The set of basic rigid CM generators carries an operation called *mutation.* Indeed, given such an L, with indecomposable summand $L_i \ncong R$, there is the so-called exchange sequence

$$0 \to L_i \to \bigoplus_{j \neq i} {L_j}^{\oplus b_{ij}} \to K_i \to 0 \tag{2.A}$$

and $\nu_i L := \frac{L}{L_i} \oplus K_i$ [W, (A.A)]. Given the summand K_i of $\nu_i L$ we can mutate again, and obtain $\nu_i\nu_i L$. It is a general fact for isolated cDV singularities [IW2, §7] that $\nu_i\nu_i L \cong L$, and moreover in the second exchange sequence

$$0 \to K_i \to \bigoplus_{j \neq i} {L_j}^{\oplus c_{ij}} \to L_i \to 0$$

we have $b_{ij} = c_{ij}$ for all i, j. Furthermore, there are equivalences of derived categories

$$\begin{aligned} \mathbf{R}\operatorname{Hom}(\operatorname{Hom}_R(L, \nu_i L), -)\colon\ & \mathrm{D^b}(\operatorname{mod} \operatorname{End}_R(L)) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod} \operatorname{End}_R(\nu_i L)) \\ \mathbf{R}\operatorname{Hom}(\operatorname{Hom}_R(\nu_i L, L), -)\colon\ & \mathrm{D^b}(\operatorname{mod} \operatorname{End}_R(\nu_i L)) \xrightarrow{\sim} \mathrm{D^b}(\operatorname{mod} \operatorname{End}_R(L)). \end{aligned}$$

We will abuse notation and notate both as Φ_i, and refer to them as the *mutation functors.*

Given our fixed basic rigid CM generator N corresponding to f, write $\operatorname{Mut}_0(N)$ for the set of basic rigid CM generators that can be obtained from N by iteratively mutating at non-free indecomposable summands. Geometrically, across the Auslander–McKay correspondence, this corresponds to all varieties that can be obtained from X by iteratively flopping curves.

2.2. **Hyperplanes and Labels.** For every $L \in \operatorname{Mut}_0(N)$, and each indecomposable summand L_i of L, there is an exact sequence

$$0 \to \bigoplus_{j=0}^{n} N_j^{\oplus a_{ij}} \to \bigoplus_{j=0}^{n} N_j^{\oplus b_{ij}} \to L_i \to 0$$

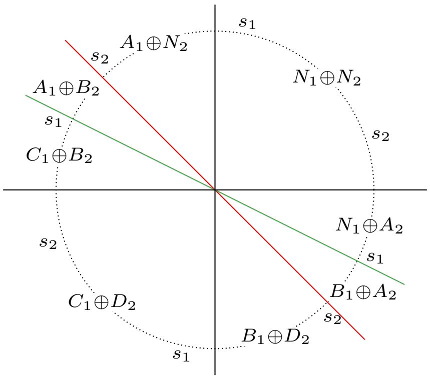
where the first and second terms do not share any indecomposable summands. In the setting where $N \in \operatorname{CM} R$ is maximal rigid, this fact is very-well known; in the setting here for rigid generators we rely instead on [IW2, §7]. In any case, consider the cone

$$C_L := \left\{ \sum_{i=1}^{n} \vartheta_i \left(\sum_{j=1}^{n} (b_{ij} - a_{ij})\, \mathbf{e}_i \right) \,\middle|\, \vartheta_i > 0 \text{ for all } i \right\} \subset \mathbb{R}^n.$$

It is clear that $C_N = \{\sum_{i=1}^{n} \vartheta_i \mathbf{e}_i \mid \vartheta_i > 0 \text{ for all } i\}$ which we denote throughout by C_+. Furthermore, the chambers C_L, as L varies over $\mathrm{Mut}_0(N)$, sweep out the chambers of a simplicial hyperplane arrangement $\mathcal{H}$ [HW1, IW2]. This $\mathcal{H}$ is always an intersection arrangement inside some ADE root system.

We fix a decomposition $N = R \oplus N_1 \oplus \ldots \oplus N_n$, and will often implicitly declare $N_0 = R$. As explained by the general Coxeter-style labelling of walls and chambers in [IW2, §1, §7], this fixed decomposition induces an ordering on the summands of all other elements L of $\mathrm{Mut}_0(N)$, such that crossing a wall locally labelled s_i always corresponds to replacing the ith summand. In this way, there is a global labelling on the edges of the skeleton of $\mathcal{H}$. We illustrate this in one example.

Example 2.1. There exists a two-curve flop over a cD_4 singularity with the following simplicial hyperplane arrangement. Each local wall is labelled by s_i for $i = 1, 2$, and under crossing wall s_i the ith summand gets replaced. For clarity, in each chamber we have not written the zeroth summand R.



Note that $C_N = C_+$ is the top right chamber.

Fixing an ordering of the summands of N not only fixes an ordering of the projective modules $\mathcal{P}_i = \mathrm{Hom}_R(N, N_i)$ of $\Lambda = \mathrm{End}_R(N)$, and via the pairing between simples and projectives, an ordering of its simples $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n$, but it also fixes an ordering on the projectives and simples of $\mathrm{End}_R(L)$ for all $L \in \mathrm{Mut}_0(N)$.

To fix notation, for $L \in \mathrm{Mut}_0(N)$ suppose that the induced ordering on the summands of L is $L = L_0 \oplus L_1 \oplus \ldots \oplus L_n$, where $L_0 \cong R$. There is an induced ordering on the projectives $\mathcal{Q}_i := \mathrm{Hom}_R(L, L_i)$ of $\mathrm{End}_R(L)$, and again via the pairing between simples and projectives, an ordering of its simples $\mathcal{S}'_0, \mathcal{S}'_1, \ldots, \mathcal{S}'_n$. The simples for the contraction algebra $\underline{\mathrm{End}}_R(L)$ are $\mathcal{S}'_1, \ldots, \mathcal{S}'_n$.

2.3. **Standard Equivalences for Simple Wall Crossings.** Let $L \in \mathrm{Mut}_0(N)$, with associated $\mathrm{A} := \mathrm{End}_R(L)$ and contraction algebra $\mathrm{A}_{\mathrm{con}} := \underline{\mathrm{End}}_R(L)$. To ease notation, suppose that $\mathrm{B} := \mathrm{End}_R(\nu_i L)$, with contraction algebra $\mathrm{B}_{\mathrm{con}}$, and consider the good truncation

$$\mathcal{T}_i := \tau_{\geq -1}(\mathrm{B}_{\mathrm{con}} \otimes^{\mathbf{L}}_{\mathrm{B}} \mathrm{Hom}_R(L, \nu_i L) \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}}), \tag{2.B}$$

which is a complex of $\mathrm{B}_{\mathrm{con}}$-$\mathrm{A}_{\mathrm{con}}$ bimodules. In this basic rigid setting, it is already known that the complex $\mathcal{T}_i$ is a two-sided tilting complex [A2, 4.3].

Theorem 2.2. *Suppose that $L, M \in \mathrm{Mut}_0(N)$, which correspond to the flopping contractions $X_L \to \operatorname{Spec} R$ and $X_M \to \operatorname{Spec} R$ say. Then the following are equivalent.*

1. *C_L and C_M share a codimension one wall in $\mathcal{H}$.*
2. *$M \cong \nu_i L$ for some $i \neq 0$.*
3. *X_L and X_M are related by a flop at a single irreducible curve.*

In this case, set $\mathrm{A} := \mathrm{End}_R(L)$, $\mathrm{A}_{\mathrm{con}} := \underline{\mathrm{End}}_R(L)$, *and* $\mathrm{B} := \mathrm{End}_R(\nu_i L)$, $\mathrm{B}_{\mathrm{con}} := \underline{\mathrm{End}}_R(\nu_i L)$. *Then the mutation operations intertwine via the following commutative diagram*

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\mathrm{A}_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] \arrow[d, "F_i := \mathbf{R}\mathrm{Hom}_{\mathrm{A}_{\mathrm{con}}}(\mathcal{T}_i, -)"', "\sim"] & \mathrm{D}^{\mathrm{b}}(\mathrm{A}) \arrow[r, "-\otimes^{\mathbf{L}}_{\mathrm{A}} \mathcal{V}", "\sim"'] \arrow[d, "\Phi_i"', "\sim"] & \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_L) \arrow[d, "\mathsf{Flop}_i"', "\sim"] \\
\mathrm{D}^{\mathrm{b}}(\mathrm{B}_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] & \mathrm{D}^{\mathrm{b}}(\mathrm{B}) \arrow[r, "-\otimes^{\mathbf{L}}_{\mathrm{B}} \mathcal{V}^{+}", "\sim"'] & \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X_{\nu_i L})
\end{tikzcd}$$

where Φ_i *is the mutation functor,* Flop_i *is the inverse of the Bridgeland–Chen flop functor* [B1, C], $\mathcal{V}$ *and* $\mathcal{V}^+$ *are the standard projective generators of zero perverse sheaves* [VdB], *and* res *is restriction of scalars induced from the ring homomorphisms* $\mathrm{A} \to \mathrm{A}_{\mathrm{con}}$ *and* $\mathrm{B} \to \mathrm{B}_{\mathrm{con}}$.

Proof. The statement (1)⇔(2) is clear; see also [HW2, 3.6]. The statement (2)⇔(3) is [W, 4.13], where since we are mutating only at indecomposable summands and R is isolated, [W, 4.20(1)] overrides the caveat in the latter part of [W, 4.13]. The left hand diagram commutes by [A2, 1.1], and the right hand diagram commutes by [W, 4.2]. □

2.4. **K-theory.** Since Λ_{con} is a finite dimensional algebra, consider the Grothendieck group

$$\mathsf{G}_0(\Lambda_{\mathrm{con}}) := K_0(\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})),$$

which is well-known to be a free abelian group based by the ordered simples $[\mathcal{S}_1], \dots, [\mathcal{S}_n]$. The notation G_0 is chosen since we are including in the Grothendieck group all modules, whereas K_0 often only deals with vector bundles, and thus projectives.

Continuing the notation from Theorem 2.2, for any $L \in \mathrm{Mut}_0(N)$ consider its associated contraction algebra $\mathrm{A}_{\mathrm{con}} = \underline{\mathrm{End}}_R(L)$ and its mutations $\nu_i \mathrm{A}_{\mathrm{con}} := \underline{\mathrm{End}}_R(\nu_i L)$. Abusing notation slightly, consider the standard equivalences

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\mathrm{A}_{\mathrm{con}}) \arrow[r, shift left, "F_i"] & \mathrm{D}^{\mathrm{b}}(\nu_i \mathrm{A}_{\mathrm{con}}) \arrow[l, shift left, "F_i"]
\end{tikzcd}$$

where the functor from right to left is induced by the mutation $\nu_i L \to \nu_i \nu_i L \cong L$. These, and their inverses, induce the following four isomorphisms on K-theory

$$\begin{tikzcd}
\mathsf{G}_0(\mathrm{A}_{\mathrm{con}}) \arrow[r, shift left, "\mathsf{F}_i"] & \mathsf{G}_0(\nu_i \mathrm{A}_{\mathrm{con}}) \arrow[l, shift left, "\mathsf{F}_i"] & \mathsf{G}_0(\mathrm{A}_{\mathrm{con}}) \arrow[r, shift right, "\mathsf{F}_i^{-1}"'] & \mathsf{G}_0(\nu_i \mathrm{A}_{\mathrm{con}}). \arrow[l, shift right, "\mathsf{F}_i^{-1}"']
\end{tikzcd} \tag{2.C}$$

As in Subsection 2.2, write $\{\mathcal{S}'_1, \dots, \mathcal{S}'_n\}$ for the ordered simples of $\mathrm{A}_{\mathrm{con}}$. For lack of suitable alternatives, also write $\{\mathcal{S}'_1, \dots, \mathcal{S}'_n\}$ for the correspondingly ordered simples in $\nu_i \mathrm{A}_{\mathrm{con}}$.

Lemma 2.3. *With the notation in* (2.A), $\mathsf{F}_i \colon \mathsf{G}_0(\mathrm{A}_{\mathrm{con}}) \to \mathsf{G}_0(\nu_i \mathrm{A}_{\mathrm{con}})$ *sends*

$$[\mathcal{S}'_t] \mapsto \left\{ \begin{array}{cl} -[\mathcal{S}'_i] & \text{if } t = i, \\ b_{it}[\mathcal{S}'_i] + [\mathcal{S}'_t] & \text{if } t \neq i. \end{array} \right. \tag{2.D}$$

Proof. The fact that $\Phi_i(\mathcal{S}'_i) = \mathcal{S}'_i[-1]$ is [W, 4.15]. Theorem 2.2 then implies that $F_i(\mathcal{S}'_i)$ maps, under restriction of scalars, to $\mathcal{S}'_i[-1]$. It follows that $F(\mathcal{S}'_i) \cong \mathcal{S}'_i[-1]$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{A}_{\mathrm{con}})$, see e.g. [A2, 6.6]. This establishes the top row.

For the second row, applying $\mathrm{Hom}_R(L, -)$ to (2.A) and using the rigidity of L gives an exact sequence

$$0 \to \mathcal{Q}_i \to \bigoplus_{j \neq i} \mathcal{Q}_j^{\oplus b_{ij}} \to \mathrm{Hom}_R(L, K_i) \to 0.$$

Applying $\mathrm{Hom}_{\mathrm{A}}(-, \mathcal{S}'_t)$ to this, with $t \neq i$, yields

$$\mathbf{R}\mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_R(L, K_i), \mathcal{S}'_t) = \mathbb{C}^{\oplus b_{it}}.$$

Further, it is clear that $\mathbf{R}\mathrm{Hom}_{\mathrm{A}}(\mathcal{Q}_j, \mathcal{S}'_t) \cong \mathrm{Hom}_{\mathrm{A}}(\mathcal{Q}_j, \mathcal{S}'_t)$ is zero if $j \neq t$, and equals $\mathbb{C}$ if $j = t$. Combining, we see that $\Phi_i(\mathcal{S}'_t) = \mathbf{R}\mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_R(L, \nu_i L), \mathcal{S}'_t)$ is a module, filtered by b_{it} copies of $\mathcal{S}'_i$, and one copy of $\mathcal{S}'_t$. Again, the left hand side of the commutative diagram in Theorem 2.2 then shows that $F_i(\mathcal{S}'_t)$ must also be a module, filtered by b_{it} copies of $\mathcal{S}'_i$, and one copy of $\mathcal{S}'_t$. The second row follows. □

Remark 2.4. Applying the above to $L = N$, basing $\mathsf{G}_0(\Lambda_{\mathrm{con}})$ by the ordered simples $[\mathcal{S}_1], \ldots, [\mathcal{S}_n]$, the above transformation (2.D) assembles into a $n \times n$ matrix, with coefficients in $\mathbb{Z}$, representing the map $\mathsf{F}_i \colon \mathbb{Z}^n \to \mathbb{Z}^n$. As is standard in linear algebra, the dual map

$$\mathsf{F}_i^* \colon \mathsf{G}_0(\nu_i\Lambda_{\mathrm{con}})^* \to \mathsf{G}_0(\Lambda_{\mathrm{con}})^*,$$

where $\mathsf{G}_0(\Lambda_{\mathrm{con}})^*$ has dual basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ say, is given by the transpose matrix. But the transpose is precisely the transformation

$$\mathbf{e}_t \mapsto \begin{cases} \mathbf{e}_t & \text{if } t \neq i, \\ -\mathbf{e}_i + \sum_{j\neq i} b_{ij}\mathbf{e}_j & \text{if } t = i. \end{cases}$$

which is precisely the transformation φ_i seen in moduli tracking [W, §5], or in K-theory of projectives in [HW2, 3.2]. In particular, it will be convenient to think of the dual basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ as being the basis $[\mathcal{P}_1], \ldots, [\mathcal{P}_n]$ of $\mathsf{K}_0(\operatorname{per}\Lambda)/[\mathcal{P}_0]$, where $\operatorname{per}\Lambda$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\Lambda)$ consisting of perfect complexes. Then the transformation F_i^* can be identified with these transformations elsewhere in the literature, and we can then use those results freely. Note that the hyperplane arrangement $\mathcal{H}$ from Subsection 2.2 is defined in terms of $\mathbf{e}_i$, and so naturally lives in $\mathsf{G}_0(\Lambda_{\mathrm{con}})^*$.

By the above remark, the proof of the following does follow as the dual of [HW2, 3.2]. It is however instructive to give a direct proof.

Lemma 2.5. *All four isomorphisms in* (2.C) *are given by the same matrix, namely the one from* (2.D)*, and this matrix squares to the identity.*

Proof. By (2.D), the matrices are controlled by the numbers b_{ij} appearing in the relevant exchange sequences. Say the top F_i is controlled by numbers b_{ij}, and the bottom F_i is controlled by numbers c_{ij}. That the two matrices labelled F_i are the same is simply the statement that $b_{ij} = c_{ij}$, which has already been explained in Subsection 2.1. Given this fact that $b_{ij} = c_{ij}$, we see that $\mathsf{F}_i\mathsf{F}_i = \mathrm{Id}$ by simply observing

$$[\mathcal{S}'_t] \overset{\mathsf{F}_i}{\longmapsto} \begin{cases} -[\mathcal{S}'_i] & \text{if } t = i, \\ b_{it}[\mathcal{S}'_i] + [\mathcal{S}'_t] & \text{if } t \neq i. \end{cases} \overset{\mathsf{F}_i}{\longmapsto} \begin{cases} -(-[\mathcal{S}'_i]) & \text{if } t = i \\ -b_{it}[\mathcal{S}'_i] + (b_{it}[\mathcal{S}'_i] + [\mathcal{S}'_t]) & \text{if } t \neq i, \end{cases}$$

which is clearly the identity. Applying F_i^{-1} to each side of the equation $\mathsf{F}_i\mathsf{F}_i = \mathrm{Id}$ gives $\mathsf{F}_i^{-1} = \mathsf{F}_i$, and all statements follow. □

2.5. **Groupoids.** As in Subsection 2.2, associated to every contraction algebra is a hyperplane arrangement $\mathcal{H}$. As is standard, there is an associated graph $\Gamma_{\mathcal{H}}$ defined as follows.

Definition 2.6. The vertices of $\Gamma_{\mathcal{H}}$ are the chambers, i.e. the connected components, of $\mathbb{R}^n \backslash \mathcal{H}$. There is a unique arrow $a \colon v_1 \to v_2$ from chamber v_1 to chamber v_2 if the chambers are adjacent, otherwise there is no arrow.

By definition, if there is an arrow $a \colon v_1 \to v_2$, then there is a unique arrow $b \colon v_2 \to v_1$ with the opposite direction of a. For an arrow $a \colon v_1 \to v_2$, set $s(a) := v_1$ and $t(a) := v_2$.

A *positive path of length* n in $\Gamma_{\mathcal{H}}$ is a formal symbol

$$p = a_n \circ \ldots \circ a_2 \circ a_1,$$

whenever there exists a sequence of vertices $v_0, \ldots, v_n$ of $\Gamma_{\mathcal{H}}$ and arrows $a_i \colon v_{i-1} \to v_i$ in $\Gamma_{\mathcal{H}}$. Set $s(p) := v_0$, $t(p) := v_n$, $\ell(p) := n$, and write $p \colon s(p) \to t(p)$. If $q = b_m \circ \ldots \circ b_2 \circ b_1$ is another positive path with $t(p) = s(q)$, we consider the formal symbol

$$q \circ p := b_m \circ \ldots \circ b_2 \circ b_1 \circ a_n \circ \ldots \circ a_2 \circ a_1,$$

and call it the *composition* of p and q.

Definition 2.7. A positive path α is called *minimal* if there is no shorter positive path in $\Gamma_{\mathcal{H}}$, with the same start and end points as α.

Following [D2, p7], let $\sim$ denote the smallest equivalence relation, compatible with morphism composition, that identifies all morphisms that arise as positive minimal paths with same source and target. Then consider the free category $\mathrm{Free}(\Gamma_{\mathcal{H}})$ on the graph $\Gamma_{\mathcal{H}}$, whose morphisms correspond to directed paths, and its quotient category

$$\mathbb{G}^+ := \mathrm{Free}(\Gamma_{\mathcal{H}})/\sim,$$

called the category of positive paths.

Definition 2.8. The *Deligne groupoid* $\mathbb{G}$ is the groupoid defined as the groupoid completion of $\mathbb{G}^+$, that is, a formal inverse is added to every morphism in $\mathbb{G}^+$.

It is a very well known fact [D1, P1, P3, S] (see also [P2, 2.1]) that for any vertex $v \in \mathbb{G}$ there is an isomorphism $\mathrm{End}_{\mathbb{G}}(v) \cong \pi_1(\mathcal{X})$ where $\mathcal{X} = \mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}$. This fact is general, and is significantly weaker and easier to establish than statements involving $K(\pi, 1)$. We will use this fact implicitly throughout.

2.6. Composition and K-theory.

For any $\alpha \in \mathrm{Free}(\Gamma_{\mathcal{H}})$, say $\alpha = s_{i_t} \circ \ldots \circ s_{i_1}$, consider

$$\begin{aligned} F_\alpha &:= F_{i_t} \circ \ldots \circ F_{i_1} \\ \Phi_\alpha &:= \Phi_{i_t} \circ \ldots \circ \Phi_{i_1}. \end{aligned}$$

The following is known, and is easy to establish just using the tilting order, in the case when the modules are maximal rigid (e.g. if X is smooth). In our more general situation of rigid generators, the same proof works, but it relies on some recent advances in [IW2].

Proposition 2.9. *Let $\alpha\colon C_L \to C_M$ be a positive minimal path. Set $\mathrm{A} := \mathrm{End}_R(L)$, $\mathrm{B} := \mathrm{End}_R(M)$, $\mathrm{A}_{\mathrm{con}} := \underline{\mathrm{End}}_R(L)$ and $\mathrm{B}_{\mathrm{con}} := \underline{\mathrm{End}}_R(M)$. Then the following hold.*

(1) *Φ_α is functorially isomorphic to $\mathbf{R}\mathrm{Hom}_{\mathrm{A}}(\mathrm{Hom}_R(L, M), -)$.*

(2) *F_α is functorially isomorphic to $\mathbf{R}\mathrm{Hom}_{\mathrm{A}_{\mathrm{con}}}(\mathcal{T}_{LM}, -)$ where $\mathcal{T}_{LM}$ is the two-sided tilting complex $\tau_{\geq -1}(\mathrm{B}_{\mathrm{con}} \otimes^{\mathbf{L}}_{\mathrm{B}} \mathrm{Hom}_R(L, M) \otimes^{\mathbf{L}}_{\mathrm{A}} \mathrm{A}_{\mathrm{con}})$.*

In particular, all positive minimal paths with the same start and end points are functorially isomorphic.

Proof. (1) When N is maximal rigid, this is precisely [A2, 4.9(1)], [HW1, 4.6]. In the more general setting here with $L, M \in \mathrm{Mut}_0(N)$, then certainly $L \in \mathrm{Mut}_0(M)$. Since R is isolated cDV, it follows from the combinatorial and geometric description of mutation of rigid modules in [IW2, §7] that $\mathrm{Hom}_R(L, M)$ is a tilting $\mathrm{End}_R(M)$-$\mathrm{End}_R(L)$-bimodule, of projective dimension one when viewed as a right $\mathrm{End}_R(L)$-module. This is the key technical condition, explained in [A2, 4.9(2)], and not available when [HW1] was written, that now allows us to use the main result [HW1, 4.6] freely in the more general setting here.
(2) When N is maximal rigid, this is [A2, 4.15]. The more general statement required here follows by (1), and again the Remark [A2, 4.9(2)] which asserts, given (1), we are able to apply the main result [A2, 4.15] to the setting of rigid objects. ☐

For any two positive minimal paths α and β with the same start and end points, the above proposition shows that there is a functorial isomorphism $F_\alpha \cong F_\beta$. Hence the association $\alpha \mapsto F_\alpha$ descends to a functor from $\mathbb{G}^+$. Since F_α is already an equivalence, this in turn formally descends to a functor from $\mathbb{G}$. Using the same logic, the assignment $\alpha \mapsto \Phi_\alpha$ also descends to a functor from $\mathbb{G}^+$, and then a functor from $\mathbb{G}$.

Furthermore, for every $\alpha \in \mathrm{Hom}_{\mathbb{G}}(C_L, C_M)$, recycling the notation from the second sentence in Proposition 2.9, the following diagram commutes

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(\mathrm{A}_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] \arrow[d, "F_\alpha"', "\sim"] & \mathrm{D}^{\mathrm{b}}(\mathrm{A}) \arrow[d, "\Phi_\alpha"', "\sim"] \\ \mathrm{D}^{\mathrm{b}}(\mathrm{B}_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] & \mathrm{D}^{\mathrm{b}}(\mathrm{B}) \end{tikzcd} \tag{2.E}$$

just by composition: by [A2, 1.1], respectively [A2, 3.2], both of the following commute.

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\mathrm{A}_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] \arrow[d, "F_i"', "\sim"] & \mathrm{D}^{\mathrm{b}}(\mathrm{A}) \arrow[d, "\Phi_i"', "\sim"] & \mathrm{D}^{\mathrm{b}}(\mathrm{A}_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] \arrow[d, "F_i^{-1}"', "\sim"] & \mathrm{D}^{\mathrm{b}}(\mathrm{A}) \arrow[d, "\Phi_i^{-1}"', "\sim"] \\
\mathrm{D}^{\mathrm{b}}(\nu_i\mathrm{A}_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] & \mathrm{D}^{\mathrm{b}}(\nu_i\mathrm{A}) & \mathrm{D}^{\mathrm{b}}(\nu_i\mathrm{A}_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] & \mathrm{D}^{\mathrm{b}}(\nu_i\mathrm{A})
\end{tikzcd}$$

Later, the following is one of the crucial ingredients in establishing that $\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is a covering space. Recall that for the standard derived equivalence F_α associated to a path α, the induced map on the K-theory is denoted F_α.

Proposition 2.10. *Suppose that $\beta\colon C \to D$ is a positive minimal path in* $\operatorname{Free}(\Gamma_{\mathcal{H}})$.

(1) *If $\alpha\colon C \to D$ is any positive path, then $\mathsf{F}_\alpha = \mathsf{F}_\beta$.*
(2) *If $\alpha \in \operatorname{End}_{\mathbb{G}}(C)$, then $\mathsf{F}_\alpha = \mathrm{Id}$.*
(3) *If $\alpha, \gamma \in \operatorname{Hom}_{\mathbb{G}}(C, D)$, then $\mathsf{F}_\alpha = \mathsf{F}_\gamma$.*

Proof. Having established Lemma 2.5, this is now word-for-word identical to [HW2, 4.8]. Note that this proof is elementary, and does not require Deligne normal form. □

2.7. **The Dual Composition.** For each $L \in \operatorname{Mut}_0(N)$, consider $\Lambda_L := \operatorname{End}_R(L)$ and recall that $\Lambda := \operatorname{End}_R(N)$. Choose a positive minimal path $\beta\colon C_L \to C_+$, which in turn gives rise to a derived equivalence

$$\Phi_L := \Phi_\beta \overset{2.9}{\cong} \mathbf{R}\mathrm{Hom}_{\Lambda_L}(\operatorname{Hom}_R(L,N), -)\colon\ \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_L) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda).$$

This derived equivalence is independent of choice of positive minimal path, by Proposition 2.9. It induces an isomorphism $\mathsf{K}_0(\operatorname{per}\Lambda_L) \to \mathsf{K}_0(\operatorname{per}\Lambda)$ on the K-theory of perfect complexes, so write

$$[\Phi_L(\mathcal{Q}_i)] = \sum_{j=0}^{n} (\varphi_L)_{ij}[\mathcal{P}_j]$$

in $\mathsf{K}_0(\operatorname{per}\Lambda) \cong \mathbb{Z}^{n+1}$, where $\mathcal{Q}_i = \operatorname{Hom}_R(L, L_i)$ and $\mathcal{P}_i = \operatorname{Hom}_R(N, N_i)$. Since α is a sequence of mutations that do not involve mutating the zeroth summand, at each stage the zeroth summand is fixed. Hence this isomorphism descends to an isomorphism

$$\varphi_L\colon \mathsf{K}_0(\operatorname{per}\Lambda_L)/[\mathcal{Q}_0] \xrightarrow{\sim} \mathsf{K}_0(\operatorname{per}\Lambda)/[\mathcal{P}_0].$$

Basing the first by $[\mathcal{Q}_1], \dots, [\mathcal{Q}_n]$ and the second by $[\mathcal{P}_1], \dots, [\mathcal{P}_n]$, the matrix representing the isomorphism is $(\varphi_L)_{ij}$ for $1 \leq i, j \leq n$. By Remark 2.4, later we will think of these bases as $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ of $\mathsf{G}_0(\underline{\operatorname{End}}_R(L))^*$ and $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $\mathsf{G}_0(\Lambda_{\mathrm{con}})^*$ respectively.

Remark 2.11. The above description of φ_L is in terms of projectives of the ambient $\operatorname{End}_R(L)$, since this is convenient later. There is however a much more intrinsic description of φ_L that does not rely on this larger algebra, via the two-term tilting complexes of the contraction algebra Λ_{con}. In particular, in the language of g-vectors, $\varphi_L(\mathbf{e}'_i) = g^{L_i}$, where g^{L_i} is the g-vector of the two-term complex of Λ_{con} associated to the rigid object L_i via the bijection [A1, 2.16]. We do not use this description below.

3. STABILITY AND T-STRUCTURES

3.1. **Stability Generalities.** Throughout this subsection, $\mathcal{T}$ denotes a triangulated category whose Grothendieck group $K_0(\mathcal{T})$ is a finitely generated free $\mathbb{Z}$-module.

Proposition 3.1 ([B2, 5.3])**.** *To give a stability condition on $\mathcal{T}$ is equivalent to giving a bounded t-structure $\mathcal{T}$ with heart $\mathcal{A}$, and a group homomorphism $Z\colon \mathsf{K}_0(\mathcal{A}) \to \mathbb{C}$, called the central charge, such that for all $0 \neq E \in \mathcal{A}$ the complex number $Z(E)$ lies in the semi-closed upper half-plane*

$$\mathbb{H} := \{re^{i\pi\varphi} \mid r > 0,\ 0 < \varphi \leq 1\},$$

and where furthermore Z must satisfy the Harder–Narasimhan property.

Write $\operatorname{Stab}\mathcal{T}$ for the set of *locally-finite* stability conditions on $\mathcal{T}$. We do not define these here, as below this condition is automatic for all stability conditions on $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$, since all hearts of bounded t-structures will be equivalent to finite dimensional modules on some finite dimensional algebra.

Theorem 3.2 ([B2, 1.2]). *The space* $\operatorname{Stab}\mathcal{T}$ *has the structure of a complex manifold, and the forgetful map*

$$\operatorname{Stab}\mathcal{T} \to \operatorname{Hom}_{\mathbb{Z}}(\mathsf{K}_0(\mathcal{T}), \mathbb{C})$$

is a local isomorphism onto an open subspace of $\operatorname{Hom}_{\mathbb{Z}}(\mathsf{K}_0(\mathcal{T}), \mathbb{C})$.

Any triangle equivalence $\Phi\colon \mathcal{T} \to \mathcal{T}'$ induces a natural map

$$\Phi_*\colon \operatorname{Stab}\mathcal{T} \to \operatorname{Stab}\mathcal{T}'$$

defined by $\Phi_*(Z, \mathcal{A}) := (Z \circ \phi^{-1}, \Phi(\mathcal{A}))$, where ϕ^{-1} is the corresponding isomorphism on K-theory $\mathsf{K}_0(\mathcal{T}') \xrightarrow{\sim} \mathsf{K}_0(\mathcal{T})$ induced by the functor Φ^{-1}. In this way, the group $\operatorname{Auteq}(\mathcal{T})$ of isomorphism classes of autoequivalences of $\mathcal{T}$ acts on $\operatorname{Stab}\mathcal{T}$.

3.2. t-structures for $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$.

The contraction algebra Λ_{con} is a *silting-discrete* symmetric algebra [A1, 3.3, 3.10]. Being symmetric, the technical condition of being silting-discrete is equivalent [AM, 2.11] to there being only finitely many basic tilting complexes between P and $P[1]$ (with respect to the silting order $\leq$), for every tilting complex P obtained by iterated irreducible left mutation from the free module Λ_{con}. Geometrically, for each such P, this set is finite since it is in bijection with R-schemes obtained by iterated flops of irreducible curves starting from X, which is well-known to be finite.

This fact has the following remarkable consequence.

Proposition 3.3. *Suppose that* $\mathcal{A}$ *is the heart of a bounded t-structure on* $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$. *Then* $\mathcal{A} = \mathcal{A}_\alpha$ *for some* $L \in \operatorname{Mut}_0(M)$ *and for some* $\alpha \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$, *where*

$$\mathcal{A}_\alpha := F_\alpha(\operatorname{mod} \underline{\operatorname{End}}_R(L))$$

and F_α *is the derived equivalence from Subsection 2.6 associated to* α.

Proof. Since Λ_{con} is silting-discrete, necessarily $\mathcal{A}$ has finite length [PSZ]. Furthermore, by the bijections in [KY, §5] there exists a silting complex T in $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ such that, in the notation of [KY, §5.4],

$$\mathcal{A} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0} = \{x \in \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \mid \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})}(T, x[i]) = 0 \text{ for all } i \neq 0\}. \tag{3.A}$$

Since Λ_{con} is symmetric, silting equals tilting, and so T is a tilting complex. It is already known (see [A1, 2.8]) that every tilting complex T in $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ can be obtained as a composition of mutations from Λ_{con}, so say $T \cong \mu_\beta \Lambda_{\mathrm{con}}$ for some $\beta \in \operatorname{Hom}_{\mathbb{G}}(C_+, C_L)$. Set $\mathrm{B}_{\mathrm{con}} := \underline{\operatorname{End}}_R(L)$, then [A1, 3.8(1)] gives $F_\beta(\mu_\beta \Lambda_{\mathrm{con}}) \cong \mathrm{B}_{\mathrm{con}}$, and hence $F_\beta(T) \cong \mathrm{B}_{\mathrm{con}}$.

Thus applying F_β to (3.A),

$$F_\beta(\mathcal{A}) = \{y \in \mathrm{D}^{\mathrm{b}}(\mathrm{B}_{\mathrm{con}}) \mid \operatorname{Hom}(\mathrm{B}_{\mathrm{con}}, y[i]) = 0 \text{ for all } i \neq 0\} = \operatorname{mod} \mathrm{B}_{\mathrm{con}},$$

and so applying $F_\beta^{-1} = F_{\beta^{-1}}$ shows that $\mathcal{A} = F_{\beta^{-1}}(\operatorname{mod} \mathrm{B}_{\mathrm{con}})$. Since $\beta^{-1} \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$, the result follows. ☐

Recall that inside $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ are the simples $\mathcal{S}_1, \ldots, \mathcal{S}_n$, which base the K-theory $\mathsf{G}_0(\Lambda_{\mathrm{con}})$. In a similar way, the simple modules $\mathcal{S}'_1, \ldots \mathcal{S}'_n$ of $\underline{\operatorname{End}}_R(L)$ base its Grothendieck group.

Corollary 3.4. *If* $\sigma \in \operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$, *then* $\sigma = (Z, \mathcal{A}_\alpha)$ *for some* $\alpha \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$ *and some* Z *satisfying* $Z(\mathsf{F}_\alpha[\mathcal{S}'_i]) \in \mathbb{H}$ *for all* $i = 1, \ldots, n$.

Proof. By Proposition 3.3 every abelian heart is of the form $\mathcal{A}_\alpha$ for some $\alpha \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$, and hence every point of $\operatorname{Stab}\mathcal{T}$ is of the form $(Z, \mathcal{A}_\alpha)$. To be a stability condition is equivalent to the map $Z\colon \mathsf{K}_0(\mathcal{A}_\alpha) \to \mathbb{C}$ sending all simples of $\mathcal{A}_\alpha$ to $\mathbb{H}$. Since the simples of $\mathcal{A}_\alpha$ are of the form $F_\alpha(\mathcal{S}'_i)$, it follows that $(Z, \mathcal{A}_\alpha)$ is a stability condition precisely when Z satisfies $Z([F_\alpha(\mathcal{S}'_i)]) = Z(\mathsf{F}_\alpha[\mathcal{S}'_i]) \in \mathbb{H}$ for all i. ☐

The general action of $\operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ on $\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ simplifies somewhat if we restrict to those standard equivalences given by $\operatorname{End}_{\mathbb{G}}(C_+)$. The functorial assignment $\alpha \to F_\alpha$ defines a group homomorphism

$$\pi_1(\mathcal{X}) \cong \operatorname{End}_{\mathbb{G}}(C_+) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$$

and we set PBr to be the image of this homomorphism. Then, using Corollary 3.4 to describe the points of $\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$, the action of $F_\beta \in \mathsf{PBr}$ on $\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is

$$F_\beta \cdot (Z, \mathcal{A}_\alpha) \overset{2.10(2)}{=} (Z, F_\beta(\mathcal{A}_\alpha)) = (Z, \mathcal{A}_{\beta\circ\alpha}). \tag{3.B}$$

4. Stability on Contraction Algebras as a Universal Cover

In order to realise $\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ as a universal cover, fix the isomorphism

$$\operatorname{Hom}_{\mathbb{C}}(\mathsf{G}_0(\Lambda_{\mathrm{con}}), \mathbb{C}) \xrightarrow{\sim} \mathbb{C}^n$$

given by $Z \mapsto (Z[\mathcal{S}_1], \dots, Z[\mathcal{S}_n]) = \sum_{i=1}^n Z[\mathcal{S}_i]\,\mathbf{e}_i$, where $\mathcal{S}_1, \dots, \mathcal{S}_n$ are the simples of Λ_{con}. Composing this with the forgetful map from Theorem 3.2, we thus obtain

$$p\colon \operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \to \operatorname{Hom}_{\mathbb{C}}(\mathsf{G}_0(\Lambda_{\mathrm{con}}), \mathbb{C}) \xrightarrow{\sim} \mathbb{C}^n. \tag{4.A}$$

Combining with Corollary 3.4, p sends an arbitrary point $(Z, \mathcal{A}_\alpha)$ to $(Z[\mathcal{S}_1], \dots, Z[\mathcal{S}_n])$. In this section we will show that p is a regular covering map onto its image. To do this, it will be convenient to also consider the stability manifolds of the other contraction algebras of R, and to track information between them.

Lemma 4.1. *For any $\alpha \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$, the following diagram commutes*

$$\begin{tikzcd}
\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\underline{\operatorname{End}}_R(L)) \arrow[r, "(F_\alpha)_*"] \arrow[d] & \operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \arrow[d] \\
\operatorname{Hom}(\mathsf{G}_0(\underline{\operatorname{End}}_R(L)), \mathbb{C}) \arrow[r, "-\circ \mathsf{F}_\alpha^{-1}"] \arrow[d, "\sim"] & \operatorname{Hom}(\mathsf{G}_0(\Lambda_{\mathrm{con}}), \mathbb{C}) \arrow[d, "\sim"] \\
\mathbb{C}^n \arrow[r, "\varphi_L"] & \mathbb{C}^n
\end{tikzcd}$$

where the topmost vertical arrows are the forgetful maps, F_α^{-1} is the image in K-theory of the inverse of the functor F_α defined in Subsection 2.6, and φ_L is defined in Subsection 2.7. The right hand vertical composition is p.

Proof. The top square commutes by definition of $(\mathsf{F}_\alpha)_*$. For the bottom square, by Proposition 2.10(3) applied to α^{-1}, we have $\mathsf{F}_\alpha^{-1} = \mathsf{F}_{\alpha^{-1}} = \mathsf{F}_\beta$, where β is a positive minimal path $C_+ \to C_L$. Writing $\beta = s_{i_t} \circ \dots \circ s_{i_1}$, then the middle map is the composition

$$\mathsf{G}_0(\underline{\operatorname{End}}_R(L))^* \xrightarrow{\mathsf{F}_{i_t}^*} \dots \xrightarrow{\mathsf{F}_{i_1}^*} \mathsf{G}_0(\Lambda_{\mathrm{con}})^*. \tag{4.B}$$

By Remark 2.4, each step is just the tracking of the projectives basing $\mathsf{K}_0(\mathrm{per})/[\mathcal{Q}_0]$ under the mutation functors. Hence (4.B) is precisely

$$\mathsf{G}_0(\underline{\operatorname{End}}_R(L))^* \xrightarrow{\varphi_{i_t}} \dots \xrightarrow{\varphi_{i_1}} \mathsf{G}_0(\Lambda_{\mathrm{con}})^*. \tag{4.C}$$

Consider the path $\overline{\beta} = s_{i_1} \circ \dots \circ s_{i_t} \colon C_L \to C_+$. Being the opposite path to β, it follows that $\overline{\beta}$ is also positive minimal. But then by Proposition 2.9 there is a functorial isomorphism

$$\Phi_L := \Phi_{\overline{\beta}} \cong \Phi_{i_1} \circ \dots \circ \Phi_{i_t}.$$

Hence φ_L, the image of this functor in $\mathsf{K}_0(\mathrm{per})/[\mathcal{P}_0]$, realises (4.C). $\square$

As is standard, consider the subset of $\mathbb{C}^n$

$$\mathbb{H}_+ := \left\{ \sum_{j=1}^n a_j \mathbf{e}_j \mid a_j \in \mathbb{H} \right\} \cong \mathbb{H}^n.$$

Corollary 4.2. *For any point of* $\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$*, which is necessarily of the form* $(Z, \mathcal{A}_\alpha)$ *for some* $\alpha \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$*,*

$$p(Z, \mathcal{A}_\alpha) = \varphi_L \left(\sum_{i=1}^{n} Z(\mathsf{F}_\alpha[\mathcal{S}'_i])\, \mathbf{e}'_i \right) \in \varphi_L(\mathbb{H}_+).$$

Proof. The first statement is Corollary 3.4. By definition $\mathcal{A}_\alpha = F_\alpha(\operatorname{mod}\underline{\operatorname{End}}_R(L))$, hence

$$\begin{aligned} p(Z, \mathcal{A}_\alpha) &= p\big(Z, F_\alpha(\operatorname{mod}\underline{\operatorname{End}}_R(L))\big) \\ &= p \circ (F_\alpha)_*(Z \circ \mathsf{F}_\alpha, \operatorname{mod}\underline{\operatorname{End}}_R(L)) && (\text{since } \mathsf{F}_\alpha \circ \mathsf{F}_\alpha^{-1} = \mathrm{Id}) \\ &= \varphi_L \left(\sum_{i=1}^{n} Z(\mathsf{F}_\alpha[\mathcal{S}'_i])\, \mathbf{e}'_i \right). && (\text{by Lemma 4.1}) \end{aligned}$$

The final statement that $p(Z, \mathcal{A}_\alpha) \in \varphi_L(\mathbb{H}_+)$ again follows from Corollary 3.4. ☐

4.1. **A Covering Map.** As in Section 3.2, let PBr be the image of the homomorphism

$$\pi_1(\mathcal{X}) \to \operatorname{Auteq}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$$

sending $\beta \mapsto F_\beta$. Then $F_\beta \in \mathsf{PBr}$ acts on the space of stability conditions via the action in (3.B). In this subsection we will establish that this action is free and properly discontinuous, so that $\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \to \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})/\,\mathsf{PBr}$ is a covering map.

The following is one of our main technical results. It establishes a condition when elements of PBr, inside the autoequivalence group of $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$, are the identity. The proof is via Fourier–Mukai techniques. Forgetting the ambient geometry is thus a bad idea: it seems extremely difficult to establish the following result in a purely algebraic manner.

Theorem 4.3. *Suppose that* $\alpha \in \operatorname{End}_{\mathbb{G}}(C_+)$ *satisfies* $F_\alpha(\Lambda_{\mathrm{con}}) \cong \Lambda_{\mathrm{con}}$*. Then there is a functorial isomorphism* $F_\alpha \cong \mathrm{Id}$*.*

Proof. By the assumption, the standard equivalence F_α is induced by the one-sided tilting complex Λ_{con}. By the usual lifting argument (see e.g. [RZ, 2.3]), the bimodule complex defining F_α must be isomorphic to ${}_1(\Lambda_{\mathrm{con}})_\zeta$ as bimodules, for some algebra automorphism $\zeta\colon \Lambda_{\mathrm{con}} \to \Lambda_{\mathrm{con}}$. Hence F_α is induced by this algebra automorphism.

Since F_α induces a Morita equivalence, it must take simples to simples. Furthermore, as F_α is the identity on K-theory $\mathsf{G}_0(\Lambda_{\mathrm{con}})$ by Proposition 2.10, F_α must fix all simples.

Now consider the commutative diagram

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] \arrow[d, "F_\alpha"', "\sim"] & \mathrm{D}^{\mathrm{b}}(\Lambda) \arrow[r, "-\otimes^{\mathbf{L}}_{\Lambda}\mathcal{V}", "\sim"'] \arrow[d, "\Phi_\alpha"', "\sim"] & \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \arrow[d, "\mathsf{Flop}_\alpha"', "\sim"] \\
\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \arrow[r, "\mathrm{res}"] & \mathrm{D}^{\mathrm{b}}(\Lambda) \arrow[r, "-\otimes^{\mathbf{L}}_{\Lambda}\mathcal{V}", "\sim"'] & \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)
\end{tikzcd}$$

where the left hand side is (2.E), and the right hand side can be obtained by iterating the right hand side of Theorem 2.2. Since F_α fixes simples, the left hand commutative diagram implies that Φ_α fixes the simples $\mathcal{S}_1, \ldots, \mathcal{S}_n$. By [VdB, 3.5.8], across the right hand commutative diagram, this in turn implies that Flop_α fixes the sheaves $\mathcal{O}_{\mathrm{C}_1}(-1), \ldots, \mathcal{O}_{\mathrm{C}_n}(-1)$, where each $\mathrm{C}_i \cong \mathbb{P}^1$. Since the flop functor and its inverse both map $\mathcal{O}_X$ to $\mathcal{O}_X$ [B1, (4.4)] and commute with the pushdown to $\operatorname{Spec} R$ (see e.g. [DW1, 7.16]), by the standard Fourier–Mukai argument (see [W, 4.3], which itself is based on [T]), $\mathsf{Flop}_\alpha \cong h_*$ for some isomorphism $h\colon X \to X$ that commutes with the pushdown. But this isomorphism is the identity on the dense open set obtained by removing the exceptional locus, and hence it must be the identity.

It follows that $\mathsf{Flop}_\alpha \cong \mathrm{Id}$ and hence $\Phi_\alpha \cong \mathrm{Id}$. Restricting the left hand commutative diagram to $\operatorname{mod}\Lambda_{\mathrm{con}}$, we obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{mod}\Lambda_{\mathrm{con}} \arrow[r, "\mathrm{res}", "\sim"'] \arrow[d, "F_\alpha"'] & \operatorname{mod}_{\{1,\dots,n\}}\Lambda \arrow[d, "\Phi_\alpha\cong\mathrm{Id}"] \\
\operatorname{mod}\Lambda_{\mathrm{con}} \arrow[r, "\mathrm{res}", "\sim"'] & \operatorname{mod}_{\{1,\dots,n\}}\Lambda
\end{tikzcd}$$

where $\operatorname{mod}_{\{1,\dots,n\}}\Lambda$ denotes those Λ-modules with a finite filtration by the simples $\mathcal{S}_1,\dots,\mathcal{S}_n$. In particular, F_α restricted to $\operatorname{mod}\Lambda_{\mathrm{con}}$ is functorially isomorphic to the identity. Hence ζ is an inner automorphism (see e.g. [L, 2.8.16]), which in turn implies that $F_\alpha \cong \mathrm{Id}$. ☐

Corollary 4.4. *For each $x \in \operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ there is an open neighbourhood $\mathcal{U}$ of x such that $\mathcal{U}\cap F_\beta(\mathcal{U}) = \emptyset$ for all $1 \neq F_\beta \in \mathsf{PBr}$.*

Proof. Consider the open neighbourhood $\mathcal{U}$ of x defined by

$$\mathcal{U} := \{y \in \operatorname{Stab}\ \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \mid d(x,y) < 1/4\},$$

where $d(-,-)$ is the metric on stability conditions introduced in [B2, Section 6]. Suppose that $\mathcal{U}\cap F_\beta(\mathcal{U}) \neq \emptyset$ for some $F_\beta \in \mathsf{PBr}$. We will show that $F_\beta \cong \mathrm{Id}$.

As the open balls intersect, every point $y \in \mathcal{U}$ must satisfy $d\big(y,(F_\beta)_*y\big) < 1$. Furthermore, the central charges of y and $(F_\beta)_*y$ are equal by Proposition 2.10(2) and the top commutative diagram in Lemma 4.1. Using [B2, Lemma 6.4] it follows immediately that $y = (F_\beta)_*y$, for every $y \in \mathcal{U}$.

In particular, by Corollary 3.4, say $x = (Z,\mathcal{A}_\alpha)$ for some $\alpha \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$. Since $x \in \mathcal{U}$, the property $(F_\beta)_*(x) = x$ implies that $\mathcal{A}_{\beta\circ\alpha} = \mathcal{A}_\alpha$, and so F_β restricts to an equivalence $\mathcal{A}_\alpha \to \mathcal{A}_\alpha$. In turn, this implies that the composition

$$F_{\alpha^{-1}\beta\alpha} = F_\alpha^{-1}\circ F_\beta\circ F_\alpha \colon\ \mathrm{D}^{\mathrm{b}}(\Gamma_{\mathrm{con}}) \to \mathrm{D}^{\mathrm{b}}(\Gamma_{\mathrm{con}}),$$

where $\Gamma_{\mathrm{con}} := \underline{\operatorname{End}}_R(L)$, restricts to an equivalence $\operatorname{mod}\Gamma_{\mathrm{con}} \to \operatorname{mod}\Gamma_{\mathrm{con}}$. It follows that $F_{\alpha^{-1}\beta\alpha}(\Gamma_{\mathrm{con}})$ must then be a basic tilting module, given that Γ_{con} is. Since Γ_{con} is symmetric, the only such module is Γ_{con}, and so $F_{\alpha^{-1}\beta\alpha}(\Gamma_{\mathrm{con}}) \cong \Gamma_{\mathrm{con}}$. By Proposition 4.3 applied to the contraction algebra Γ_{con}, we conclude that $F_{\alpha^{-1}\beta\alpha} \cong \mathrm{Id}$, and hence $F_\beta \cong \mathrm{Id}$. ☐

Corollary 4.5. *The map $\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \to \operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})/\mathsf{PBr}$ is the universal covering map, with Galois group PBr.*

Proof. $\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is contractible since contraction algebras are silting-discrete [A1, 4.12], and silting-discrete algebras have contractible stability manifolds [PSZ]. In particular, $\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is path connected and so the given map is a regular covering map by Corollary 4.4 together with the standard [H, 1.40(a)(b)]. The covering is clearly universal, since $\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is contractible and hence simply connected. ☐

4.2. The Regular Cover to the Complexified Complement.

In this subsection we will establish that p induces an isomorphism

$$\operatorname{Stab}\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})/\,\mathsf{PBr} \xrightarrow{\sim} \mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}. \tag{4.D}$$

Combining with Corollary 4.5 will then prove that (4.A) is the universal covering map onto its image, with Galois group PBr. We will establish (4.D) in two steps: first by showing that p has image $\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$, then second by establishing that (4.D) is well-defined and injective.

Our proof makes use of the following key combinatorial result, which is folklore when $\mathcal{H}$ is an ADE root system. In our mildly more general setting here, the proof, which is basically the same, can be found in [HW2, Appendix A].

Proposition 4.6. *With notation as above, the following hold.*

(1) *If α and β terminate at C_+, then*

$$\varphi_{s(\alpha)}(\mathbb{H}_+)\cap\varphi_{s(\beta)}(\mathbb{H}_+) \neq \emptyset \iff s(\alpha) = s(\beta).$$

(2) *There is a disjoint union*

$$\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}} = \bigsqcup_{L\in\mathrm{Mut}_0(N)} \varphi_L(\mathbb{H}_+).$$

The above combinatorics lead directly to the following. In the special case when $\mathcal{H}$ is an ADE root system, the following result is already implicit in [B3] and [T].

Corollary 4.7. *The image of p is $\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$.*

Proof. By Proposition 4.6(2), all the sets $\varphi_L(\mathbb{H}_+)$ avoid the complexified hyperplanes, so by Corollary 4.2 the image of p lies in $\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$. Further, given any $z \in \mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$, we may write $z = \varphi_L(h)$ for some $L \in \mathrm{Mut}_0(N)$ and some $h \in \mathbb{H}_+$, again by Proposition 4.6(2). Since the standard heart in $\mathrm{D}^{\mathrm{b}}(\underline{\mathrm{End}}_R(L))$ has finite length, we can find a stability condition $\sigma \in \mathrm{Stab}\,\mathrm{D}^{\mathrm{b}}(\underline{\mathrm{End}}_R(L))$ which maps to h via the left hand vertical composition in Lemma 4.1. Then for any $\alpha \in \mathrm{Hom}_{\mathbb{G}}(C_L, C_+)$, the commutative diagram in Lemma 4.1 shows that $(F_\alpha)_*(\sigma) \in \mathrm{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ maps, under p, to z. ☐

The following shows that (4.D) is both well-defined, and injective.

Lemma 4.8. *For $\sigma_1, \sigma_2 \in \mathrm{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$, then*

$$p(\sigma_1) = p(\sigma_2) \iff \sigma_1 = (F_\gamma)_*\sigma_2 \text{ for some } F_\gamma \in \mathsf{PBr}\,.$$

Proof. Note that ($\Leftarrow$) is clear since, by Proposition 2.10(2), the action in (3.B) of a pure braid does not effect the central charge of a stability condition.

For ($\Rightarrow$), recall that by Corollary 3.4 we can assume that $\sigma_1 = (Z_1, \mathcal{A}_\alpha)$ and $\sigma_2 = (Z_2, \mathcal{A}_\beta)$ where $\alpha \in \mathrm{Hom}_{\mathbb{G}}(C_{L_1}, C_+)$ and $\beta \in \mathrm{Hom}_{\mathbb{G}}(C_{L_2}, C_+)$. If $p(\sigma_1) = p(\sigma_2)$, then certainly $Z_1 = Z_2$ since p is simply the forgetful map followed by an isomorphism.

Furthermore, by Corollary 4.2, we see that $\varphi_{s(\alpha)}(\mathbb{H}_+) \cap \varphi_{s(\beta)}(\mathbb{H}_+) \neq \emptyset$, since the intersection contains $p(\sigma_1) = p(\sigma_2)$. Hence by Proposition 4.6(1) it follows that $s(\alpha) = s(\beta)$ and thus we can consider the composition $\gamma = \beta \circ \alpha^{-1} \in \mathrm{End}_{\mathbb{G}}(C_+)$. Then $F_\gamma \in \mathsf{PBr}$ and

$$\begin{aligned}
(F_\gamma)_*(Z_1, \mathcal{A}_\alpha) &= (Z_1, \mathcal{A}_{\gamma\circ\alpha}) && \text{(by (3.B) since } F_\gamma \in \mathsf{PBr}) \\
&= (Z_1, \mathcal{A}_{\beta\circ\alpha^{-1}\circ\alpha}) && \text{(using } \gamma = \beta\circ\alpha^{-1}) \\
&= (Z_2, \mathcal{A}_\beta), && \text{(since } Z_1 = Z_2 \text{ and } \alpha^{-1}\circ\alpha = \mathrm{Id})
\end{aligned}$$

proving the statement. ☐

Corollary 4.9. *The map $p\colon \mathrm{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \to \mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$ induces a homeomorphism*

$$\mathrm{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})/\,\mathsf{PBr} \to \mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}.$$

Proof. The map p is surjective by Corollary 4.7, and by Lemma 4.8 it induces the bijection in the statement. The induced map is itself a homeomorphism by the definition of the quotient topology, and the fact that p is a local homeomorphism. ☐

The following is our main result.

Corollary 4.10. *The map $p\colon \mathrm{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \to \mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$ is the universal cover, with Galois group* PBr. *Furthermore,* $\mathrm{Stab}\,\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ *is contractible.*

Proof. The first statement is obtained by composing the universal cover from Corollary 4.5 with the homeomorphism from Corollary 4.9. As already stated, the second part follows from [PSZ] and [A1, 4.12]. ☐

5. COROLLARIES

In this section we prove the five main corollaries stated in the introduction. For an ADE root system, it is well-known [B4] that $\pi_1(\mathfrak{h}_{\mathrm{reg}}/W)$ is isomorphic to the associated ADE braid group. Recall that the $K(\pi, 1)$-conjecture for ADE braid groups, which is already a theorem in this setting, asserts that the universal cover of $\mathfrak{h}_{\mathrm{reg}}/W$ is contractible.

Corollary 5.1. *The $K(\pi, 1)$-conjecture holds for all ADE braid groups.*

Proof. As in [T, §3] or [KM, §4.3], we may choose a flopping contraction for which the hyperplane arrangement $\mathcal{H}$ is an ADE root system $\mathfrak{h}$. It is well known that the complexified complement $\mathcal{X} = \mathfrak{h}_{\mathrm{reg}}$. Write W for the Weyl group, which is finite, thus clearly the covering map $\mathfrak{h}_{\mathrm{reg}} \to \mathfrak{h}_{\mathrm{reg}}/W$ has finite fibres. It follows that the composition

$$\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}}) \to \mathcal{X} = \mathfrak{h}_{\mathrm{reg}} \to \mathfrak{h}_{\mathrm{reg}}/W$$

is then also a covering map. As above, it is well-known that $\pi_1(\mathfrak{h}_{\mathrm{reg}}/W)$ is the braid group. By Corollary 4.10, the fact that $\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is contractible implies that it is simply connected. Hence the composition is the universal cover, and furthermore the universal cover is contractible. □

Corollary 5.2. *The homomorphism* $\pi_1(\mathcal{X}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ *sending* $\alpha \mapsto F_\alpha$ *is injective.*

Proof. As in Sections 3 and 4, by definition PBr is the image of this homomorphism. Since p is a regular covering map, as is standard [H, 1.40(c)] there is a short exact sequence of groups

$$1 \to \pi_1(\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})) \to \pi_1(\mathcal{X}) \to \mathsf{PBr} \to 1 \tag{5.A}$$

where $\pi_1(\mathcal{X}) \to \mathsf{PBr}$ as before takes $\alpha \mapsto F_\alpha$. However, since by Corollary 4.10 $\operatorname{Stab} \mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is contractible, its fundamental group is trivial. □

For $\alpha \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$, set $T_\alpha := F_\alpha(\underline{\operatorname{End}}_R(L))$, which is necessarily a tilting complex for Λ_{con} since equivalences map tilting complexes to tilting complexes.

Corollary 5.3. *The map* $\alpha \mapsto T_\alpha$ *is a bijection from morphisms in the Deligne groupoid ending at* C_+ *to the set of basic one-sided tilting complexes of* Λ_{con}*, up to isomorphism.*

Proof. Surjectivity is already known. Indeed, since Λ_{con} is silting-discrete, by [A1, 3.16(2)] every standard derived equivalence from $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$ is, up to algebra automorphism, isomorphic to F_β for some $\beta \in \operatorname{Hom}_{\mathbb{G}}(C_+, C_L)$. In particular every one-sided tilting complex for Λ_{con} is isomorphic to $F_\beta^{-1}(\underline{\operatorname{End}}_R(L))$ for some $\beta \in \operatorname{Hom}_{\mathbb{G}}(C_+, C_L)$. Since $\beta^{-1} \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$ and $F_\beta^{-1} = F_{\beta^{-1}}$, surjectivity of the map follows.

The content is that the map is also injective. Suppose that $\alpha \in \operatorname{Hom}_{\mathbb{G}}(C_L, C_+)$ and $\beta \in \operatorname{Hom}_{\mathbb{G}}(C_M, C_+)$ are such that $T_\alpha \cong T_\beta$, where $L, M \in \operatorname{Mut}_0(N)$. Then, by definition,

$$F_\alpha(\underline{\operatorname{End}}_R(L)) \cong F_\beta(\underline{\operatorname{End}}_R(M)) \tag{5.B}$$

and hence $\mathcal{A}_\alpha = \mathcal{A}_\beta$. In particular, choosing any central charge Z, we have $(Z, \mathcal{A}_\alpha) = (Z, \mathcal{A}_\beta)$ and hence $p(Z, \mathcal{A}_\alpha) = p(Z, \mathcal{A}_\beta)$. But Corollary 4.2 then implies that $\varphi_L(\mathbb{H}_+)$ and $\varphi_M(\mathbb{H}_+)$ intersect, which by Proposition 4.6 implies that $L \cong M$.

Set $\mathrm{B}_{\mathrm{con}} := \underline{\operatorname{End}}_R(L)$. Applying $F_\beta^{-1} = F_{\beta^{-1}}$ to (5.B) gives $F_{\beta^{-1} \circ \alpha}(\mathrm{B}_{\mathrm{con}}) \cong \mathrm{B}_{\mathrm{con}}$. Thus by applying Theorem 4.3 to the composition $\beta^{-1} \circ \alpha \in \operatorname{End}_{\mathbb{G}}(C_L)$ we deduce that there is an isomorphism $F_{\beta^{-1} \circ \alpha} \cong \mathrm{Id}$. Corollary 5.2 applied to the contraction algebra $\mathrm{B}_{\mathrm{con}}$ then shows that $\beta^{-1} \circ \alpha$ must be the identity, and hence $\alpha = \beta$ in the Deligne groupoid. □

Remark 5.4. If we instead assign to a path $\alpha \colon C_+ \to C_L$ the tilting complex $F_\alpha^{-1}(\underline{\operatorname{End}}_R(L))$, we equivalently obtain a bijection between the paths in the Deligne groupoid that *start* at C_+ and basic one-sided tilting complexes of Λ_{con}.

Corollary 5.5. *The homomorphism* $\pi_1(\mathcal{X}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *sending* $\alpha \mapsto \mathsf{Flop}_\alpha$ *is injective.*

Proof. Suppose that α belongs to the kernel, so $\mathsf{Flop}_\alpha = \mathrm{Id}$. Since Φ_α is functorially isomorphic to Flop_α after tilting by Theorem 2.2, necessarily $\Phi_\alpha \cong \mathrm{Id}$. The left-hand part of the commutative digram in Theorem 2.2 then implies that $F_\alpha(\Lambda_{\mathrm{con}})$ maps, under restriction of scalars, to Λ_{con}. It follows that $F_\alpha(\Lambda_{\mathrm{con}}) \cong \Lambda_{\mathrm{con}}$ in $\mathrm{D}^{\mathrm{b}}(\Lambda_{\mathrm{con}})$, see e.g. [A2, 6.6]. By Lemma 4.3, there is a functorial isomorphism $F_\alpha \cong \mathrm{Id}$. By Corollary 5.2 we see that $\alpha = 1$, proving the statement. □

Corollary 5.6. *With the notation as in the introduction,* $\operatorname{Stab}^\circ \mathcal{C}$ *is contractible.*

Proof. By [HW2] there is a regular covering map $\mathrm{Stab}^{\circ}\mathcal{C} \to \mathcal{X}$ with Galois group G equal to the image of $\pi_1(\mathcal{X}) \to \mathrm{Auteq}\,\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X)$. But by Corollary 5.5, the map $\pi_1(\mathcal{X}) \to G$ is an isomorphism. By the corresponding version of (5.A), $\pi_1(\mathrm{Stab}^{\circ}\mathcal{C})$ is trivial and so the cover is universal. Universal covers are unique, so by Corollary 4.10 it follows that $\mathrm{Stab}^{\circ}\mathcal{C}$ is contractible. □

References

[AM] T. Aihara and Y. Mizuno, *Classifying Tilting Complexes Over Preprojective Algebras of Dynkin Type*, Algebra Number Theory **11** (2017), no. 6, 1287–1315.

[A1] J. August, *On the Finiteness of the Derived Equivalence Classes of some Stable Endomorphism Rings*, arXiv:1801.05687.

[A2] J. August, *The Tilting Theory of Contraction Algebras*, arXiv:1802.10366.

[BT] C. Brav and H. Thomas, *Braid groups and Kleinian singularities*, Math. Ann. **351** (2011), no. 4, 1005–1017.

[B1] T. Bridgeland, *Flops and derived categories.* Invent. Math. **147** (2002), no. 3, 613–632.

[B2] T. Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) **166** (2007), no. 2, 317–345.

[B3] T. Bridgeland, *Stability conditions and Kleinian singularities*, Int. Math. Res. Not. IMRN (2009), no. 21, 4142–4157.

[B4] E. Brieskorn, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, Invent. Math. **12** (1971), 57–61.

[BPP] N. Broomhead, D. Pauksztello and D. Ploog, *Discrete derived categories II: the silting pairs CW complex and the stability manifold*, J. Lond. Math. Soc. (2) **93** (2016), no. 2, 273–300.

[C] J-C. Chen, *Flops and equivalences of derived categories for threefolds with only terminal Gorenstein singularities*, J. Differential Geom. **61** (2002), no. 2, 227–261.

[D1] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302.

[D2] E. Delucchi, *Combinatorics of covers of complexified hyperplane arrangements*, Arrangements, local systems and singularities, 1–38, Progr. Math., **283**, Birkhäuser Verlag, Basel, 2010.

[DW1] W. Donovan and M. Wemyss, *Noncommutative deformations and flops*, Duke Math. J. **165** (2016), no. 8, 1397–1474.

[DW3] W. Donovan and M. Wemyss, *Twists and braids for general 3-fold flops*, J. Eur. Math. Soc. (JEMS), **21** (2019), no. 6, 1641–1701.

[FMT] M. Furuse, T. Mukouyama, and D. Tamaki, *Totally normal cellular stratified spaces and applications to the configuration space of graphs*, Topol. Methods Nonlinear Anal. **45** (2015), no. 1, 169–214.

[H] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge (2002), xii+544pp.

[HW1] Y. Hirano and M. Wemyss, *Faithful actions from hyperplane arrangements*, Geom. & Topol. **22** (2018), no. 6, 3395–3433.

[HW2] Y. Hirano and M. Wemyss, *Stability conditions for* 3*-fold flops*, arXiv:1907.09742.

[IW1] O. Iyama and M. Wemyss, *Maximal modifications and Auslander–Reiten duality for non-isolated singularities*, Invent. Math. **197** (2014), no. 3, 521–586.

[IW2] O. Iyama and M. Wemyss, *Tits cones intersections, contracted preprojective algebras, and affine actions on* 3*-fold flops*, in preparation.

[KY] S. Koenig and D. Yang, *Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras*, Doc. Math. **19** (2014), 403–438.

[KM] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Math., **134**, Cambridge Univ. Press, Cambridge, 1998.

[L] M. Linckelmann, *The block theory of finite group algebras. Vol. I.*, London Mathematical Society Student Texts, **91**, CUP, Cambridge, (2018), x+515pp.

[P1] L. Paris, *Universal cover of Salvetti's complex and topology of simplicial arrangements of hyperplanes*, Trans. Amer. Math. Soc. **340** (1993), no. 1, 149–178.

[P2] L. Paris, *On the fundamental group of the complement of a complex hyperplane arrangement.* Arrangements—Tokyo 1998, 257–272, Adv. Stud. Pure Math., **27**, Kinokuniya, Tokyo, 2000.

[P3] L. Paris, *The covers of a complexified real arrangement of hyperplanes and their fundamental groups*, Topology Appl. **53** (1993), no. 1, 75–103.

[PSZ] D. Pauksztello, M. Saorín and A. Zvonareva, *Contractibility of the stability manifold for silting-discrete algebras*, Forum Math. **30** (2018), no. 5, 1255–1263.

[QW] Y. Qiu and J. Woolf, *Contractible stability spaces and faithful braid group actions*, Geom. Topol. **22** (2018), no. 6, 3701–3760.

[R] M. Reid, *Minimal models of canonical 3-folds*, Algebraic varieties and analytic varieties (Tokyo, 1981), 131–180, Adv. Stud. Pure Math., 1, North-Holland, Amsterdam, 1983.

[RZ] R. Rouquier and A. Zimmermann, *Picard groups for derived module categories*, Proc. London Math. Soc. (3) **87** (2003), no. 1, 197–225.

[S] M. Salvetti, *Topology of the complement of real hyperplanes in* $\mathbb{C}^N$, Invent. Math. **88** (1987), no. 3, 603–618.

[T] Y. Toda, *Stability conditions and crepant small resolutions*, Trans. Amer. Math. Soc. **360** (2008), no. 11, 6149–6178.

[VdB] M. Van den Bergh, *Three-dimensional flops and noncommutative rings*, Duke Math. J. **122** (2004), no. 3, 423–455.

[W] M. Wemyss, *Flops and Clusters in the Homological Minimal Model Program*, Invent. Math. **211** (2018), no. 2, 435–521.

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