NONCOMMUTATIVE RESOLUTIONS USING SYZYGIES

HAILONG DAO, OSAMU IYAMA, SRIKANTH B. IYENGAR
RYO TAKAHASHI, MICHAEL WEMYSS AND YUJI YOSHINO

Abstract. Given a noether algebra with a noncommutative resolution, a general construction of new noncommutative resolutions is given. As an application, it is proved that any finite length module over a regular local or polynomial ring gives rise, via suitable syzygies, to a noncommutative resolution.

The focus of this article is on constructing endomorphism rings with finite global dimension. This problem has arisen in various contexts, including Auslander's theory of representation dimension [1], Dlab and Ringel's approach to quasi-hereditary algebras in Lie theory [4, 6], Rouquier's dimension of triangulated categories [10], cluster tilting modules in Auslander–Reiten theory [8], and Van den Bergh's noncommutative crepant resolutions in birational geometry [12].

For a noetherian ring R which is not necessarily commutative, and a finitely generated faithful R-module M, the ring $\operatorname{End}_R(M)$ is a *noncommutative resolution* (abbreviated to NCR) if its global dimension is finite; see [5]. When this happens, M is said to *give an NCR of R*. We give a method for constructing new NCRs from a given one.

Theorem 1. *Let R be a noether algebra, and let $M, X \in \operatorname{mod} R$. If M is a d-torsionfree generator giving an NCR, and* $\operatorname{gldim}\operatorname{End}_R(X)$ *is finite, then for any integer* $0 \le c < \min\{d, \operatorname{grade}_R X\}$*, the following statements hold.*

(1) *The R-module $M \oplus \Omega^c X$ is a c-torsionfree generator.*
(2) *There is an inequality*

$$\operatorname{gldim}\operatorname{End}_R(M \oplus \Omega^c X) \le 2 \operatorname{gldim}\operatorname{End}_R(M) + \operatorname{gldim}\operatorname{End}_R(X) + 1.$$

In particular, $M \oplus \Omega^c X$ gives an NCR of R.

A commutative ring is *equicodimensional* if every maximal ideal has the same height. Typical examples of equicodimensional regular rings are polynomial rings over a field, and regular local rings.

Corollary 2. *Let R be an equicodimensional regular ring, and N a finite length R-module such that* $\operatorname{gldim}\operatorname{End}_R(N)$ *is finite. Given non-negative integers* $c_1, \dots, c_n$ *with $c_i < \dim R$ for each i, the R-module $M := R \oplus \Omega^{c_1} N \oplus \dots \oplus \Omega^{c_n} N$ satisfies*

$$\operatorname{gldim}\operatorname{End}_R(M) \le 2^n \dim R + (2^n - 1)(\operatorname{gldim}\operatorname{End}_R(N) + 1).$$

In particular, M gives an NCR of R.

For any finite length R-module X, there exists a finite length R-module Y such that $\operatorname{End}_R(X \oplus Y)$ has finite global dimension [7]. In the setting of the corollary, it follows that an NCR can be constructed using any finite length R-module.

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In the definition of noncommutative resolution, it is sometimes required that the module be reflexive [11]. If $\dim R \geq 3$ in the setting of the corollary, then for any finite length R-module, by taking all $c_i \geq 2$ it can be ensured that the module giving the NCR is reflexive, but is not free.

Proofs

Throughout, R will be a *noether algebra*, in the sense that it is finitely generated as a module over its centre, and the latter is a noetherian ring. Thus R is a noetherian ring, and for any M in $\operatorname{mod} R$, the category of finitely generated left R-modules, the ring $\operatorname{End}_R(M)$ is also a noether algebra, and hence noetherian.

The *grade* of $M \in \operatorname{mod} R$ is defined to be

$$\operatorname{grade}_R M = \inf\{n \mid \operatorname{Ext}^n_R(M,R) \neq 0\}.$$

When R is commutative, this is the length of a longest regular sequence in the annihilator of the R-module M; see, for instance, [9, Theorem 16.7].

A finitely generated R-module M is *d-torsionfree*, for some positive integer d, if

$$\operatorname{Ext}^i_R(\operatorname{Tr} M, R) = 0 \quad \text{for } 1 \leq i \leq d,$$

where $\operatorname{Tr} M$ be the Auslander transpose of M; see [2]. This is equivalent to the condition that M is the d-th syzygy of an R-module N satisfying $\operatorname{Ext}^i_R(N,R) = 0$ for $1 \leq i \leq d$; see [2].

Given R-modules X and Y we write $\underline{\operatorname{Hom}}_R(X,Y)$ for the quotient of $\operatorname{Hom}_R(X,Y)$ by the abelian subgroup of morphisms factoring through projective R-modules.

Lemma 3. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of R-modules. If an R-module W satisfies $\underline{\operatorname{Hom}}_R(W,Z) = 0$, then the following sequence is exact.*

$$0 \to \operatorname{Hom}_R(W,X) \to \operatorname{Hom}_R(W,Y) \to \operatorname{Hom}_R(W,Z) \to 0$$

Proof. By hypothesis any morphism $f\colon W \to Z$ factors as $W \to P \xrightarrow{f'} Z$, where P is a projective R-module, and since f' lifts to Y, so does f. □

As usual, we write ΩX for a syzygy of X.

Lemma 4. *Let X and Y be finitely generated R-modules.*

(1) *If $\operatorname{Ext}^1_R(X,R) = 0$, then there is an isomorphism*

$$\Omega\colon\ \underline{\operatorname{Hom}}_R(X,Y) \xrightarrow{\cong} \underline{\operatorname{Hom}}_R(\Omega X, \Omega Y).$$

(2) *If $0 \leq c < \operatorname{grade}_R X$ and $n \geq 1$, then $\underline{\operatorname{Hom}}_R(\Omega^c X, \Omega^{c+n} Y) = 0$.*

Proof. Part (1) is clear, and implies part (2) for its hypotheses yields

$$\underline{\operatorname{Hom}}_R(\Omega^c X, \Omega^{c+n} Y) \cong \underline{\operatorname{Hom}}_R(X, \Omega^n Y)$$

and the right-hand module is zero as $\operatorname{Hom}_R(X,R) = 0$ implies $\operatorname{Hom}_R(X, \Omega^n Y) = 0$, since $\Omega^n Y$ is a submodule of a projective R-module. □

Proof of Theorem 1. Part (1) is a direct verification.
For part (2), set $A := \operatorname{End}_R(M \oplus \Omega^c X)$ and let $e \in A$ be the idempotent corresponding to the direct summand M. Then $eAe = \operatorname{End}_R(M)$, so given the inequality

$$\operatorname{gldim} A \leq \operatorname{gldim}(eAe) + \operatorname{gldim} A/(e) + \operatorname{pd}_A(A/(e)) + 1$$

proved in [3, Theorem 5.4], it remains to prove the two claims below.

Claim. There is an isomorphism of rings $A/(e) \cong \operatorname{End}_R(X)$.

Indeed, first note that $A/(e) = \operatorname{End}_R(\Omega^c X)/[M]$, where $[M]$ denotes the two-sided ideal of morphisms factoring through $\operatorname{add} M$. This does not rely on any special properties of M or of X.

Since $\operatorname{Hom}_R(X, R) = 0$ one obtains the equality below

$$\operatorname{End}_R(X) = \underline{\operatorname{End}}_R(X) \cong \underline{\operatorname{End}}_R(\Omega^c X),$$

while the isomorphism is obtained by repeated application of Lemma 4(1), noting that $c < \operatorname{grade}_R X$. Therefore, to verify the claim, it is enough to prove $\operatorname{End}_R(\Omega^c X)/[M] = \underline{\operatorname{End}}_R(\Omega^c X)$, that is, any endomorphism of $\Omega^c X$ factoring through $\operatorname{add} M$ factors through $\operatorname{add} R$.

Given morphisms $\Omega^c X \xrightarrow{f} M \xrightarrow{g} \Omega^c X$, the morphism f factors through $\operatorname{add} R$ by Lemma 4(2), since M is a d-th syzygy module and $d > c$. This completes the proof of the claim.

Claim. There is an inequality $\operatorname{pd}_A(A/(e)) \leq \operatorname{gldim} \operatorname{End}_R(M)$.

Set $n := \operatorname{gldim} \operatorname{End}_R(M)$. Then, the $\operatorname{End}_R(M)$-module $\operatorname{Hom}_R(M, \Omega^c X)$ has a finite projective resolution

$$0 \to P_n \to \cdots \to P_0 \to \operatorname{Hom}_R(M, \Omega^c X) \to 0. \tag{A}$$

As $\operatorname{Hom}_R(M, -)\colon \operatorname{add}_R M \to \operatorname{proj} \operatorname{End}_R(M)$ is an equivalence, there is a sequence

$$0 \to M_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} M_0 \xrightarrow{f_0} \Omega^c X \to 0 \tag{B}$$

of R-modules, with $M_j \in \operatorname{add} M$ for all j, such that the induced sequence

$$0 \to \operatorname{Hom}_R(M, M_n) \to \cdots \to \operatorname{Hom}_R(M, M_0) \to \operatorname{Hom}_R(M, \Omega^c X) \to 0$$

is isomorphic to (A). Since $R \in \operatorname{add} M$, the sequence (B) is exact.

To justify the claim, it suffices to prove that the induced complex

$$0 \to \operatorname{Hom}_R(\Omega^c X, M_n) \to \cdots \to \operatorname{Hom}_R(\Omega^c X, M_0) \xrightarrow{g} \operatorname{Hom}_R(\Omega^c X, \Omega^c X) \tag{C}$$

obtained from (B) is exact, and $\operatorname{Cok}(g)$ is isomorphic to $\operatorname{End}_R(\Omega^c X)/[M] \cong A/(e)$. For, then there is a projective resolution

$$\begin{aligned} 0 \to \operatorname{Hom}_R(M \oplus \Omega^c X, M_n) \to \quad \cdots \quad &\to \operatorname{Hom}_R(M \oplus \Omega^c X, M_0) \\ &\to \operatorname{Hom}_R(M \oplus \Omega^c X, \Omega^c X) \to A/(e) \to 0 \end{aligned}$$

of the A-module $A/(e)$, as desired.

By construction, one obtains the exact sequence

$$\operatorname{Hom}_R(\Omega^c X, M_0) \xrightarrow{g} \operatorname{Hom}_R(\Omega^c X, \Omega^c X) \to \operatorname{End}_R(\Omega^c X)/[M] \to 0.$$

This justifies the assertion about $\operatorname{Cok}(g)$. As to the exactness, for each $0 \leq i \leq n$ set $K_i := \operatorname{Im}(f_i)$, where f_i are the maps in (B). Then there are exact sequences

$$0 \to K_{i+1} \to M_i \to K_i \to 0.$$

For each $i \geq 1$, using the fact that M_i is d-torsionfree, and $K_0 = \Omega^c X$, it follows by induction that K_i is a $(c+1)$-st syzygy. Lemma 4(2) then yields that $\underline{\operatorname{Hom}}_R(\Omega^c X, K_i) = 0$ for $i \geq 1$. By Lemma 3, one then obtains an exact sequence

$$0 \to \operatorname{Hom}_R(\Omega^c X, K_{i+1}) \to \operatorname{Hom}_R(\Omega^c X, M_i) \to \operatorname{Hom}_R(\Omega^c X, K_i) \to 0.$$

Thus the sequence (C) is exact, as desired. $\square$

Recall that a commutative ring R is *regular* if it is noetherian and every localization at a prime ideal has finite global dimension. When R is further equicodimensional, the global dimension of R is finite, since it equals $\dim R$.

Proof of Corollary 2. Up to Morita equivalence, we can assume that

$$c_1 > c_2 > \cdots > c_{n-1} > c_n.$$

Set $M_0 = R$ and for each integer $1 \le j \le n$, set

$$M_j := R \oplus \Omega^{c_1} N \oplus \cdots \oplus \Omega^{c_j} N.$$

We prove, by an induction on j, that M_j is c_j-torsionfree and that

$$\operatorname{gldim} \operatorname{End}_R(M_j) \le 2^j \dim R + (2^j - 1)(\operatorname{gldim} \operatorname{End}_R(N) + 1).$$

The base case $j = 0$ is a tautology, for R is regular and hence its global dimension equals $\dim R$. Assume the inequality holds for $j-1$ for some integer $j \ge 1$.

For the induction step, set $M = M_{j-1}$, so that

$$M_j = M_{j-1} \oplus \Omega^{c_j} N.$$

Since R is equicodimensional, $\operatorname{grade}_R N = \dim R$ and M_{j-1} is c_{j-1}-torsionfree, Theorem 1 applies to yield that M_j is c_j-torsionfree, and further that

$$\operatorname{gldim} \operatorname{End}_R(M_j) \le 2 \operatorname{gldim} \operatorname{End}_R(M_{j-1}) + \operatorname{gldim} \operatorname{End}_R(N) + 1.$$

Applying the induction hypothesis gives the desired upper bound for the global dimension of $\operatorname{End}_R(M_j)$. □

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Hailong Dao, Department of Mathematics, University of Kansas, Lawrence, KS 66045-7523, USA.
E-mail address: hdao@ku.edu

Osamu Iyama, Graduate School of Mathematics, Nagoya University, Chikusaku, Nagoya 464-8602, Japan.
E-mail address: iyama@math.nagoya-u.ac.jp

Srikanth B. Iyengar, Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA.
E-mail address: iyengar@math.utah.edu

Ryo Takahashi, Graduate School of Mathematics, Nagoya University, Chikusaku, Nagoya 464-8602, Japan.
E-mail address: takahashi@math.nagoya-u.ac.jp

Michael Wemyss: School of Mathematics and Statistics, University of Glasgow, 15 University Gardens, Glasgow, G12 8QW, UK.
E-mail address: michael.wemyss@glasgow.ac.uk

Yuji Yoshino, Department of Mathematics, Faculty of Science, Okayama University, Tsushima-Naka 3-1-1, Okayama, 700-8530, Japan.
E-mail address: yoshino@math.okayama-u.ac.jp