NONCOMMUTATIVE RESOLUTIONS USING SYZYGIES

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Abstract. Given a noether algebra with a noncommutative resolution, a general construction of new noncommutative resolutions is given. As an application, it is proved that any finite length module over a regular local or polynomial ring gives rise, via suitable syzygies, to a noncommutative resolution.

The focus of this article is on constructing endomorphism rings with finite global dimension. This problem has arisen in various contexts, including Auslander's theory of representation dimension [1], Dlab and Ringel's approach to quasi-hereditary algebras in Lie theory [5, 8], Rouquier's dimension of triangulated categories [12], cluster tilting modules in Auslander–Reiten theory [10], and Van den Bergh's noncommutative crepant resolutions in birational geometry [14].

For a noetherian ring R which is not necessarily commutative, and a finitely generated faithful R-module M, the ring $\operatorname{End}_R(M)$ is a *noncommutative resolution* (abbreviated to NCR) if its global dimension is finite; see [7]. When this happens, M is said to *give an NCR of R*. In the result below, we give a method for constructing new NCRs from a given one. As usual $\Omega^c X$ denotes a cth syzygy modue of X. We defer introducing other terminology and notation to the next section.

Theorem 1. *Let R be a noether algebra, and let $M, X \in \operatorname{mod} R$. If M is a d-torsionfree generator giving an NCR, and $\operatorname{gldim} \operatorname{End}_R(X)$ is finite, then for any integer $0 \le c < \min\{d, \operatorname{grade}_R X\}$, the following statements hold.*

(1) *The R-module $M \oplus \Omega^c X$ is a c-torsionfree generator.*
(2) *There is an inequality*

$$\operatorname{gldim} \operatorname{End}_R(M \oplus \Omega^c X) \le 2 \operatorname{gldim} \operatorname{End}_R(M) + \operatorname{gldim} \operatorname{End}_R(X) + 1.$$

In particular, $M \oplus \Omega^c X$ gives an NCR of R.

The statement of Theorem 1 is inspried by a recollement type inequality (1.1) that yields that the finiteness of global dimensions of eAe and $A/(e)$ implies that of A, provided $\operatorname{pd}_A(A/(e))$ is finite. The hypotheses in Theorem 1 enable us to apply this fact to $A = \operatorname{End}_R(M \oplus \Omega^c X)$ and the idempotent $e \in A$ corresponding to the direct summand M.

A commutative ring is *equicodimensional* if every maximal ideal has the same height. Typical examples are polynomial rings over a field, and regular local rings. The following corollary generalises, and is inspired by, a result of Buchweitz and Pham [4], who considered the case $N = k$; see also [6, Corollary 5.2].

Corollary 2. *Let R be an equicodimensional regular ring, and N a finite length R-module such that $\operatorname{gldim} \operatorname{End}_R(N)$ is finite. Given non-negative integers $c_1, \dots, c_n$*

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with $c_i < \dim R$ for each i, the R-module $M := R \oplus \Omega^{c_1} N \oplus \ldots \oplus \Omega^{c_n} N$ satisfies

$$\operatorname{gldim} \operatorname{End}_R(M) \leq 2^n \dim R + (2^n - 1)(\operatorname{gldim} \operatorname{End}_R(N) + 1).$$

In particular, M gives an NCR of R.

For any finite length R-module X, there exists a finite length R-module Y such that $\operatorname{End}_R(X \oplus Y)$ has finite global dimension [9]. In the setting of the corollary, it follows that an NCR can be constructed using any finite length R-module.

In the definition of noncommutative resolution, it is sometimes required that the module be reflexive [13]. If $\dim R \geq 3$ in the setting of the corollary, then for any finite length R-module, by taking all $c_i \geq 2$ it can be ensured that the module giving the NCR is reflexive, but is not free.

Terminology and Proofs

Throughout, R will be a *noether algebra*, in the sense that it is finitely generated as a module over its centre, and the latter is a noetherian ring. Thus R is a noetherian ring, and for any M in $\operatorname{mod} R$, the category of finitely generated left R-modules, the ring $\operatorname{End}_R(M)$ is also a noether algebra, and hence noetherian.

The *grade* of $M \in \operatorname{mod} R$ is defined to be

$$\operatorname{grade}_R M = \inf\{n \mid \operatorname{Ext}^n_R(M, R) \neq 0\}.$$

When R is commutative, this is the length of a longest regular sequence in the annihilator of the R-module M; see, for instance, [11, Theorem 16.7].

A finitely generated R-module M is *d-torsionfree*, for some positive integer d, if

$$\operatorname{Ext}^i_R(\operatorname{Tr} M, R) = 0 \quad \text{for } 1 \leq i \leq d,$$

where $\operatorname{Tr} M$ be the Auslander transpose of M; see [2]. This is equivalent to the condition that M is the d-th syzygy of an R-module N satisfying $\operatorname{Ext}^i_R(N, R) = 0$ for $1 \leq i \leq d$; see [2]. For example, if R is commutative and Gorenstein, any $(\dim R)$th syzygy module of a finitely generated module is d-torsion free, for any d.

Given R-modules X and Y we write $\underline{\operatorname{Hom}}_R(X, Y)$ for the quotient of $\operatorname{Hom}_R(X, Y)$ by the abelian subgroup of morphisms factoring through projective R-modules.

Lemma 3. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of R-modules. If an R-module W satisfies $\underline{\operatorname{Hom}}_R(W, Z) = 0$, then the following sequence is exact.*

$$0 \to \operatorname{Hom}_R(W, X) \to \operatorname{Hom}_R(W, Y) \to \operatorname{Hom}_R(W, Z) \to 0$$

Proof. By hypothesis any morphism $f \colon W \to Z$ factors as $W \to P \xrightarrow{f'} Z$, where P is a projective R-module, and since f' lifts to Y, so does f. $\square$

As usual, we write ΩX for a syzygy of X.

Lemma 4. *Let X and Y be finitely generated R-modules.*

(1) *If $\operatorname{Ext}^1_R(X, R) = 0$, then there is an isomorphism*

$$\Omega \colon \underline{\operatorname{Hom}}_R(X, Y) \xrightarrow{\cong} \underline{\operatorname{Hom}}_R(\Omega X, \Omega Y).$$

(2) *If $0 \leq c < \operatorname{grade}_R X$ and $n \geq 1$, then $\underline{\operatorname{Hom}}_R(\Omega^c X, \Omega^{c+n} Y) = 0$.*

Proof. Part (1) is clear and standard, for instance both sides are isomorphic to $\operatorname{Ext}^1_R(X, \Omega Y)$ using short exact sequences $0 \to \Omega X \to F \to X \to 0$ and $0 \to \Omega Y \to G \to Y \to 0$. Part (2) follows for its hypotheses yield

$$\underline{\operatorname{Hom}}_R(\Omega^c X, \Omega^{c+n} Y) \cong \underline{\operatorname{Hom}}_R(X, \Omega^n Y)$$

and the right-hand module is zero as $\operatorname{Hom}_R(X, R) = 0$ implies $\operatorname{Hom}_R(X, \Omega^n Y) = 0$, since $\Omega^n Y$ is a submodule of a projective R-module. $\square$

Proof of Theorem 1. Part (1): since $c < \operatorname{grade}_R X$, $\Omega^c X$ is c-torsionfree, see [2]. For part (2), set $A := \operatorname{End}_R(M \oplus \Omega^c X)$ and let $e \in A$ be the idempotent corresponding to the direct summand M. Then $eAe = \operatorname{End}_R(M)$, so given the inequality

$$\operatorname{gldim} A \le \operatorname{gldim}(eAe) + \operatorname{gldim} A/(e) + \operatorname{pd}_A(A/(e)) + 1 \tag{1.1}$$

proved in [3, Theorem 5.4], it remains to prove the two claims below.

Claim. There is an isomorphism of rings $A/(e) \cong \operatorname{End}_R(X)$.

Indeed, first note that $A/(e) = \operatorname{End}_R(\Omega^c X)/[M]$, where $[M]$ denotes the two-sided ideal of morphisms factoring through $\operatorname{add} M$. This does not rely on any special properties of M or of X.

Since $\operatorname{grade}_R X \ge 1$, one has $\operatorname{Hom}_R(X, R) = 0$ and this yields the equality below

$$\operatorname{End}_R(X) = \underline{\operatorname{End}}_R(X) \cong \underline{\operatorname{End}}_R(\Omega^c X),$$

while the isomorphism is obtained by repeated application of Lemma 4(1), noting that $c < \operatorname{grade}_R X$. Therefore, to verify the claim, it is enough to prove $\operatorname{End}_R(\Omega^c X)/[M] = \underline{\operatorname{End}}_R(\Omega^c X)$, that is, any endomorphism of $\Omega^c X$ factoring through $\operatorname{add} M$ factors through $\operatorname{add} R$.

Given morphisms $\Omega^c X \xrightarrow{f} M \xrightarrow{g} \Omega^c X$, the morphism f factors through $\operatorname{add} R$ by Lemma 4(2), since M is a d-th syzygy module and $d > c$. This completes the proof of the claim.

Claim. There is an inequality $\operatorname{pd}_A(A/(e)) \le \operatorname{gldim} \operatorname{End}_R(M)$.

Set $n := \operatorname{gldim} \operatorname{End}_R(M)$. Then, the $\operatorname{End}_R(M)$-module $\operatorname{Hom}_R(M, \Omega^c X)$ has a finite projective resolution

$$0 \to P_n \to \cdots \to P_0 \to \operatorname{Hom}_R(M, \Omega^c X) \to 0. \tag{1.2}$$

As $\operatorname{Hom}_R(M, -)\colon \operatorname{add}_R M \to \operatorname{proj} \operatorname{End}_R(M)$ is an equivalence, there is a sequence

$$0 \to M_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} M_0 \xrightarrow{f_0} \Omega^c X \to 0 \tag{1.3}$$

of R-modules, with $M_j \in \operatorname{add} M$ for all j, such that the induced sequence

$$0 \to \operatorname{Hom}_R(M, M_n) \to \cdots \to \operatorname{Hom}_R(M, M_0) \to \operatorname{Hom}_R(M, \Omega^c X) \to 0$$

is isomorphic to (1.2). Since $R \in \operatorname{add} M$, the sequence (1.3) is exact.

To justify the claim, it suffices to prove that the induced complex

$$0 \to \operatorname{Hom}_R(\Omega^c X, M_n) \to \cdots \to \operatorname{Hom}_R(\Omega^c X, M_0) \xrightarrow{g} \operatorname{Hom}_R(\Omega^c X, \Omega^c X) \tag{1.4}$$

obtained from (1.3) is exact, and $\operatorname{Cok}(g)$ is isomorphic to $\operatorname{End}_R(\Omega^c X)/[M] \cong A/(e)$. For, then there is a projective resolution

$$\begin{aligned} 0 \to \operatorname{Hom}_R(M \oplus \Omega^c X, M_n) \to \quad \cdots \quad & \to \operatorname{Hom}_R(M \oplus \Omega^c X, M_0) \\ & \to \operatorname{Hom}_R(M \oplus \Omega^c X, \Omega^c X) \to A/(e) \to 0 \end{aligned}$$

of the A-module $A/(e)$, as desired.

By construction, one obtains the exact sequence

$$\operatorname{Hom}_R(\Omega^c X, M_0) \xrightarrow{g} \operatorname{Hom}_R(\Omega^c X, \Omega^c X) \to \operatorname{End}_R(\Omega^c X)/[M] \to 0.$$

This justifies the assertion about $\operatorname{Cok}(g)$. As to the exactness, for each $0 \le i \le n$ set $K_i := \operatorname{Im}(f_i)$, where f_i are the maps in (1.3). Then there are exact sequences

$$0 \to K_{i+1} \to M_i \to K_i \to 0.$$

For each $i \geq 1$, using the fact that M_i is d-torsionfree, and $K_0 = \Omega^c X$, it follows by induction that K_i is a $(c+i)$-th syzygy. Lemma 4(2) then yields that $\underline{\operatorname{Hom}}_R(\Omega^c X, K_i) = 0$ for $i \geq 1$. By Lemma 3, one then obtains an exact sequence

$$0 \to \operatorname{Hom}_R(\Omega^c X, K_{i+1}) \to \operatorname{Hom}_R(\Omega^c X, M_i) \to \operatorname{Hom}_R(\Omega^c X, K_i) \to 0.$$

Thus the sequence (1.4) is exact, as desired. □

Recall that a commutative ring R is *regular* if it is noetherian and every localization at a prime ideal has finite global dimension. When R is further equicodimensional, the global dimension of R is finite, since it equals $\dim R$.

Proof of Corollary 2. Up to Morita equivalence, we can assume that

$$c_1 > c_2 > \cdots > c_{n-1} > c_n.$$

Set $M_0 = R$ and for each integer $1 \le j \le n$, set

$$M_j := R \oplus \Omega^{c_1} N \oplus \cdots \oplus \Omega^{c_j} N.$$

We prove, by an induction on j, that M_j is c_j-torsionfree and that

$$\operatorname{gldim} \operatorname{End}_R(M_j) \le 2^j \dim R + (2^j - 1)(\operatorname{gldim} \operatorname{End}_R(N) + 1).$$

The base case $j = 0$ is a tautology, for R is regular and hence its global dimension equals $\dim R$. Assume the inequality holds for $j-1$ for some integer $j \ge 1$.

For the induction step, set $M = M_{j-1}$, so that

$$M_j = M_{j-1} \oplus \Omega^{c_j} N.$$

Since R is equicodimensional, $\operatorname{grade}_R N = \dim R$ and M_{j-1} is c_{j-1}-torsionfree, Theorem 1 applies to yield that M_j is c_j-torsionfree, and further that

$$\operatorname{gldim} \operatorname{End}_R(M_j) \le 2 \operatorname{gldim} \operatorname{End}_R(M_{j-1}) + \operatorname{gldim} \operatorname{End}_R(N) + 1.$$

Applying the induction hypothesis gives the desired upper bound for the global dimension of $\operatorname{End}_R(M_j)$. □

The following examples illustrate that, without additional inequality on c in Theorem 1, $M \oplus \Omega^c X$ need not give an NCR.

Example 5. Let Q be the cycle of length two and R the quotient of $\mathbb{C}Q$ by all paths of length three. Let $M = R \oplus S_1$ and $X = S_2$, where S_1 and S_2 are simple R-modules. Then by a direct calculation one gets $\operatorname{gldim} \operatorname{End}_R(M) = 3$ and $\operatorname{gldim} \operatorname{End}_R(X) = 0$, whilst $\operatorname{gldim} \operatorname{End}_R(M \oplus X) = \infty$.

Here is an example from commutative algebra: Let $R = \mathbb{C}[x,y]/(x^{2n} - y^2)$ with x, y commuting indeterminates, and $n \ge 2$ a positive integer. Then R is a curve singularity of type A_{2n-1}. For $M = R \oplus (R/(x^n - y))$ and $X = R/(x^n + y)$, one gets $\operatorname{gldim} \operatorname{End}_R(M) = 3$ (in fact M is 2-cluster tilting) and $\operatorname{gldim} \operatorname{End}_R(X) = 1$, whilst $\operatorname{gldim} \operatorname{End}_R(M \oplus X) = \infty$.

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