

DOUBLE BUBBLE PLUMBINGS AND TWO-CURVE FLOPS

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ABSTRACT. We discuss the symplectic topology of the Stein manifolds obtained by plumbing two 3-dimensional spheres along a circle. These spaces are related, at a derived level and working in a characteristic determined by the specific geometry, to local threefolds which contain two floppable $(-1, -1)$ -curves meeting at a point. Using contraction algebras we classify spherical objects on the B-side, and derive topological consequences including a complete description of the homology classes realised by graded exact Lagrangians.

1. INTRODUCTION

1.1. Context. Let (Q, \mathbb{W}) be a quiver with potential, i.e. a directed graph Q with a cyclic word \mathbb{W} on the oriented edges of Q . Ginzburg [20] associates to the pair (Q, \mathbb{W}) a 3-dimensional Calabi-Yau category $\mathcal{C}(Q, \mathbb{W})$ over a field \mathbb{K} . A basic question is to find geometric models for these categories, for instance as derived categories of sheaves on local Calabi-Yau 3-folds or Fukaya categories of convex symplectic six-manifolds.

One remarkable aspect of this question is that extremely elementary quivers already lead to, and in many respects control, rich geometry. The famous conifold example, which corresponds to the very simplest case when the quiver has one vertex and no arrows, compares on the B-side the local geometry near a floppable $(-1, -1)$ -curve in a Calabi-Yau 3-fold, with on the A-side the cotangent bundle T^*S^3 .

This paper treats the simplest case of a loopless quiver which admits non-trivial potentials, namely the oriented 2-cycle. We construct geometric models, on both the A- and B-sides of the mirror, which through necessity leave the world of toric geometry and cotangent bundles. Our main results relate our two constructions through a range of categorical equivalences and intertwining of autoequivalence groups. We then deploy various techniques on the B-side to establish the classification of spherical objects, one of the first in a 3-fold setting and as far as we know the first such when the underlying A_∞ -algebras are not formal, which has a number of symplectic-topological corollaries.

1.2. Constructions. On the A-side, our models are plumblings of two cotangent bundles T^*S^3 along a circle, namely ‘double bubble plumblings’. Consider the Stein manifold W_η obtained by plumbing two 3-spheres Q_0 and Q_1 along an unknotted circle $Z \subset Q_i$ with respect to an identification $\eta: \nu_{Z/Q_0} \xrightarrow{\sim} \nu_{Z/Q_1}$ (see Section 2.1 for details). The choice of η forms a torsor for \mathbb{Z} ; we normalise so that $\eta = 0 \in \mathbb{Z}$ defines the plumbing in which the Morse-Bott surgery of the Q_i along Z is diffeomorphic to $S^1 \times S^2$, whilst the Lens space $L(k, 1)$ is the Morse-Bott surgery when $|\eta| = k$. We may assume $k \geq 0$, since the

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spaces W_k and W_{-k} are symplectomorphic. Fix a field \mathbb{K} , and write \mathcal{Q}_k for the subcategory of the \mathbb{Z} -graded compact \mathbb{K} -linear Fukaya category $D^\pi \mathcal{F}(W_k)$ split-generated by the core components $\{Q_0, Q_1\}$.

Let Br_3 denote the braid group on 3 strings, and $\rho: \mathrm{Br}_3 \rightarrow \mathrm{Sym}_3$ the natural homomorphism to the symmetric group. Recall that the mixed braid group MBr_3 is $\rho^{-1}(1, 3)$ and the pure braid group PBr_3 is $\ker(\rho)$. By considering parallel transport in appropriate families, there are actions of MBr_3 on W_0 and of PBr_3 on W_k for $k > 0$.

On the B-side, our models are neighbourhoods of pairs of floppable curves meeting at a point. Set $R_0 = \mathbb{K}[u, v, x, y]/(uv - xy^2)$ and $R_k = \mathbb{K}[u, v, x, y]/(uv - xy(x^k + y))$ for all $k \geq 1$. It turns out in all cases that there is a specific crepant resolution $f: Y_k \rightarrow \mathrm{Spec} R_k$ with $f^{-1}(0)$ comprising a pair of $(-1, -1)$ -rational curves C_1 and C_2 meeting at a single point. Consider the subcategory

$$\mathcal{C}_k := \langle \mathcal{O}_{C_1}(-1), \mathcal{O}_{C_2}(-1) \rangle \subset D^b(\mathrm{coh} Y_k).$$

It is easy to see that $\mathcal{C}_k \subseteq \{a \in D^b(\mathrm{coh} Y_k) \mid \mathbf{R}f_* a = 0\}$, with equality holding if and only if $k \geq 1$. Either way, all objects of \mathcal{C}_k are necessarily supported on the exceptional locus. Results of [7] show that \mathcal{C}_k is independent of the choice of small resolution in characteristic zero, and we extend this to characteristic p below. Similarly, in characteristic zero the papers [14, 51] show that \mathcal{C}_0 admits a MBr_3 action, whereas \mathcal{C}_k admits a PBr_3 action for all $k > 0$. Happily, these too generalise to all characteristics.

1.3. Categorical Results. The following is our first main result.

Theorem 1.1 (3.13). *With notation as above, the following hold.*

- (1) *If $\mathbb{K} = \mathbb{C}$, then there are equivalences $\mathcal{Q}_0 \simeq \mathcal{C}_0$ and $\mathcal{Q}_k \simeq \mathcal{C}_1$ for all $k \geq 1$.*
- (2) *If $p > 2$ is prime and $\mathrm{char} \mathbb{K} = p$, then there is an equivalence $\mathcal{Q}_p \simeq \mathcal{C}_p$.*

The subcategory \mathcal{Q}_p split-generates $D^\pi \mathcal{F}(W_p)$ over \mathbb{C} if $p = 1$, or over \mathbb{K} when $\mathrm{char} \mathbb{K} = p > 2$, so symplectically \mathcal{Q}_p is a natural object. The comparison in Theorem 1.1 should be viewed as an example of mirror symmetry in the non-toric case, where we need to ‘turn on’ the characteristic of the ground field in order to access certain fat-spherical / Lens space objects in part (2). This positive characteristic case is not simply a curiosity; we *need* part (2) later in order to deduce symplectic-topological properties of W_p (see 1.5 and 1.7).

Addendum 1.2 (4.27). In (1), when $k = 1$, the equivalence entwines the actions of the pure braid group. In (1) when $k = 0$, and in (2) generally, a similar statement holds at the level of the actions on objects; we refer the reader to the discussion after (4.2).

Writing e and f for the arrows in the oriented two-cycle, implicit in the above is the assertion that in characteristic zero $\mathcal{Q}_0 \simeq \mathcal{C}_0$ are geometric models for the zero potential, whilst $\mathcal{Q}_k \simeq \mathcal{C}_1$ are geometric models for the potential $(ef)^2$. In characteristic p , $\mathcal{Q}_p \simeq \mathcal{C}_p$ should be viewed as geometric models for a fabled potential $(ef)^p$. Even although such an object does not exist in a technical sense (cyclically permuting gives $p(ef)^p + p(fe)^p = 0$), our models still exhibit the higher multiplications that we would expect from $(ef)^p$.

Thus, although potentials are not well adapted to the case $\mathrm{char} \mathbb{K} = p$, mirror symmetry is: our A- and B-side models match directly, regardless of the characteristic, and still exhibit the ‘expected’ higher multiplications from the potentials framework.

By appealing to known results concerning A_∞ -Koszul duality [15, 37, 27], we then extend Theorem 1.1 to a statement relating wrapped Fukaya categories and relative singularity categories. This should be viewed as a natural lifting of the equivalence in Theorem 1.1 to include non-compact objects.

Corollary 1.3 (5.6). *When $k = 1$ and $\mathbb{K} = \mathbb{C}$, or when $k > 2$ is prime and $\text{char } \mathbb{K} = k$, then there is an equivalence of categories $\mathcal{W}(W_k; \mathbb{K}) \simeq \text{D}^b(\text{coh } Y_k) / \langle \mathcal{O}_{Y_k} \rangle$.*

Both Theorem 1.1 and Corollary 1.3 can be seen as a direct if *ad hoc* instance of mirror symmetry. There are SYZ-type prescriptions for constructing mirrors of conic fibrations [2]: when applied to the X_k , as explained in Remark 3.14, these yield rather singular spaces to which the plumbings W_k are fair approximations.

Our attention then shifts to classifying fat-spherical objects in \mathcal{C}_k , over any field. It should be compared with [22], which discusses two-dimensional A_k -Milnor fibres, and [3], which discusses the case of the A_2 -Milnor fibre in any dimension.

Theorem 1.4 (6.12). *Fix $k \geq 1$, and suppose that $a \in \mathcal{C}_k$ satisfies $\text{Hom}_{\mathcal{C}_k}(a, a[i]) = 0$ for all $i < 0$, and further $\dim_k \text{Hom}_{\mathcal{C}_k}(a, a) = 1$. Then the following statements hold.*

- (1) *There exists a composition of mutation functors $F: \text{D}^b(\Lambda) \rightarrow \text{D}^b(\Lambda_\beta)$ such that Fa is either \mathcal{S}_1 or \mathcal{S}_2 .*
- (2) *The complex a is fat-spherical, and up to the action of the pure braid group, is isomorphic to either $\mathcal{S}_1, \mathcal{S}_2, \text{Cone}(\mathcal{S}_1[-1] \rightarrow \mathcal{S}_2)$, or their shifts by [1].*

Our techniques are constructive: given a (fat) spherical object, there is an algorithm which expresses it as a pure braid image of a core component or of the surgery. It is however reasonable to object to the lack of symmetry in the statement of Theorem 1.4(2), since $\text{Cone}(\mathcal{S}_1[-1] \rightarrow \mathcal{S}_2)$ is preferred over $\text{Cone}(\mathcal{S}_2[-1] \rightarrow \mathcal{S}_1)$, and both are fat-spherical. However, this is just a manifestation of the rampant choice involved in determining generators for the pure braid group; different choices lead to different presentations.

Our method to prove Theorem 1.4 iteratively determines how to ‘improve’ an arbitrary fat-spherical object by applying mutation functors and their inverses, and our induction is in fact much simpler than the surfaces case [22]. The method of proof, which may be of independent interest, uses the representation theory of the associated contraction algebras in a crucial way; we refer the reader to §6.2 for more details. A key property is that our particular contraction algebras have only finitely many indecomposable modules, something which is not true generally.

1.4. Corollaries. We show in Lemma 4.5 that W_1 is an affine flag 3-fold, and there is a natural action of the pure braid group on $\pi_0 \text{Symp}_{ct}(W_1)$ coming from parallel transport in the universal family of flag 3-folds. Even although we do not know if W_k is affine in general, in Lemma 4.15 we use Morse-Bott-Lefschetz descriptions of the plumbings to show that for all $k \geq 1$ there is a natural representation $\rho_k: \text{PBr}_3 \rightarrow \pi_0 \text{Symp}(W_k)$ whose image is generated by (spherical and Lens space) Dehn twists in the core components and their surgery.

Known results of faithfulness of the pure braid action on the B-side [21], together with Addendum 1.2, then establishes the following.

Corollary 1.5 (4.29). *If $p = 1$ or $p > 2$ is prime, then the natural representation $\rho_p: \text{PBr}_3 \rightarrow \pi_0 \text{Symp}_{gr}(W_p)$ is faithful.*

Remark 1.6. We consider the representation to graded symplectomorphisms, which act on the \mathbb{Z} -graded Fukaya category. For $p = 1$, one can check that the representation lifts to a faithful representation $\text{PBr}_3 \rightarrow \pi_0 \text{Symp}_{ct}(W_1)$; for $p > 1$ whilst the generators of the pure braid group are represented by compactly supported maps, our methods do not show that the whole representation lifts. The centre of the pure braid group acts by a symplectomorphism which is isotopic to the identity through non-compactly-supported maps (and which acts by a non-trivial shift on the wrapped category), cf. Lemma 4.21, so certainly $\text{PBr}_3 \rightarrow \pi_0 \text{Symp}(W_k)$ is not injective.

Presumably it is possible to establish Corollary 1.5 more directly, using Floer cohomology computations à la [31, 28]. Of course, we suspect that faithfulness of the action holds for all $k \geq 1$, not just for primes, but our current methods are hampered by our inability to work over \mathbb{Z} instead of fields of positive characteristic.

On the other hand, our categorical classification of fat-spherical objects – more generally, simple objects with no negative-degree self-Ext’s – in Theorem 1.4 leads to our main, purely topological corollary.

Corollary 1.7 (6.13). *Let $p = 1$ or $p > 2$ be prime. Let $L \subset W_p$ be a closed connected exact Lagrangian submanifold with vanishing Maslov class. Then L is quasi-isomorphic to a pure braid image of one of the two core components or the Morse-Bott surgery of the core components.*

In particular, identifying $H_3(W_p; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators the core components Q_i , with either choice of orientation, we obtain the homological analogue of the ‘nearby Lagrangian conjecture’ in this case:

Corollary 1.8 (6.16). *Let $p = 1$ or $p > 2$ be prime. Let $L \subset W_p$ be an exact Lagrangian submanifold with vanishing Maslov class. Then $\pm[L] \in \{(1, 0), (0, 1), (1, \pm 1)\} \subset \mathbb{Z} \oplus \mathbb{Z}$.*

The Corollaries are illustrative but not exhaustive. For instance, the proof of Corollary 6.12 pins down precisely which classes are represented by spherical objects, and which by fat spherical objects. It follows that when $p > 2$ is prime, the classes $\pm(1, \pm 1) \in H_3(W_p; \mathbb{Z})$ can be represented by Lagrangian Lens spaces $L(p, 1)$ but *not* by Lagrangian spheres.

In the same vein, we obtain the following.

Corollary 1.9 (6.18). *If $p = 1$ or $p > 2$ is prime, there is no exact Lagrangian $S^1 \times S^2 \subset W_p$ with vanishing Maslov class.*

Using the existence of a dilation in symplectic cohomology over a field of characteristic three, Ganatra and Pomerleano [19] show that an exact graded $L \subset W_1$ has prime summands certain spherical space forms, or copies of $S^1 \times S^2$.

In contrast to the faithful action in Corollary 1.5, it is more challenging to see how Corollary 1.7 or Corollary 1.8 could be proved directly. Our proof heavily relies on the existence of a square root of the twist functors (which exist on the B-side), and an analysis of the representation theory of various contraction algebras. Techniques based on symplectic cohomology tend to yield primitivity results for homology classes of exact Lagrangians, rather than a complete classification.

1.5. Future Work. One might hope that all the results above should fit into a much larger conjectural framework of flopping contractions, and there should be a vast sea of mirror symmetry statements, each controlled by the contraction algebra that arises. We briefly outline, in Section 7, this ‘realisation problem’ for symplectic models of threefold flops, and we explain some obstructions and subtleties that occur when we naively add more spheres and Lens spaces (respectively, more flopping curves) to the spaces considered in this paper.

Conventions. Throughout, all fields \mathbb{K} are algebraically closed. This is mostly required for the birational geometry in Section 3.1.

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2. PLUMBINGS AND THE TWO-CYCLE QUIVER

2.1. Plumbings. Fix closed smooth n -dimensional manifolds Q_1 and Q_2 , a smooth k -dimensional manifold Z with $0 \leq k < n$, and embeddings $\iota_j: Z \hookrightarrow Q_j$ with a fixed identification

$$(2.1) \quad \eta: \nu_{Z/Q_1} \xrightarrow{\cong} \nu_{Z/Q_2}$$

between the normal bundles of the two embeddings. This data determines a Stein manifold

$$W_\eta = W_\eta(\iota_1, \iota_2) = T^*Q_1 \#_{Z, \eta} T^*Q_2$$

which is the *plumbing* of the cotangent bundles of the Q_i along the submanifold Z with respect to the identification η . The plumbing is the completion of a Stein domain obtained as follows. Let D^*Q_i denote the disc cotangent bundle (for some chosen Riemannian metric). Weinstein’s neighbourhood theorem implies that there is an open neighbourhood $U(Z)_i$ of $D^*Z \subset D^*Q_i$ which is symplectomorphic to a disc sub-bundle of the symplectic vector bundle $\nu_{Z/Q_i} \otimes \mathbb{C}$ over Z . W_η is obtained by gluing $U(Z)_1 \xrightarrow{\cong} U(Z)_2$ by the map $\eta \otimes (\cdot\sqrt{-1})$ obtained from composing the given isomorphism of normal bundles with multiplication by $\sqrt{-1}$ in the complexification. W_η is an exact symplectic manifold which contains both Q_i as exact Lagrangian submanifolds; it retracts to the union $Q_1 \cup_Z Q_2$ (its ‘skeleton’).

Conversely, if (X, ω) is any symplectic manifold which contains a pair of closed Lagrangian submanifolds Q_i meeting cleanly along a closed submanifold $Z \subset Q_i$, then ω induces a perfect pairing between ν_{Z/Q_1} and ν_{Z/Q_2} , hence an isomorphism η of the underlying unoriented bundles. An open neighbourhood of $Q_1 \cup_Z Q_2 \subset X$ is symplectically equivalent to a Stein subdomain of the corresponding W_η . Plumbings therefore describe the local geometry of symplectic manifolds near unions of cleanly intersecting Lagrangian submanifolds.

Example 2.1. The A_n -Milnor fibre $\{x^2 + y^2 + z^{n+1} = 1\} \subset (\mathbb{C}^3, \omega_{\text{st}})$ is the iterated plumbing of n copies of T^*S^2 along points. Because S^2 admits an orientation-reversing diffeomorphism, the choice of identification of normal bundles is not important.

Example 2.2. The self-plumbing of T^*S^2 at two points $\{p \sqcup q\} \subset S^2$ depends on whether $T_p(S^2) \xrightarrow{\cong} T_q(S^2)$ preserves or reverses orientation. One obtains a neighbourhood of an immersed Lagrangian sphere of self-intersection 0 respectively -4 in the two cases.

2.2. Double-bubble plumbings. Let $Q_1 = S^3 = Q_2$ and let $Z = S^1$. The embedding $Z \hookrightarrow Q_i$ defines a knot, which we will denote κ_i but usually assume is trivial. The normal bundle $\nu_{Z/Q_i} \cong S^1 \times D^2$ is trivial, so the possible isomorphisms $\eta: \nu_{Z/Q_1} \rightarrow \nu_{Z/Q_2}$ covering an orientation-preserving diffeomorphism of Z form a torsor for the integers.

Lemma 2.3. *The plumbing $W_\eta(\kappa_1, \kappa_2)$ contains two exact Maslov zero Lagrangian submanifolds K, K' obtained from the two possible Morse-Bott surgeries of the Q_i along Z .*

Proof. There are two Lagrange surgeries of the planes $\mathbb{R}^2 \amalg i\mathbb{R}^2 \subset \mathbb{C}^2$, which are distinct up to compactly supported Hamiltonian isotopy. The intersection locus $Z \subset W_\eta$ is an isotropic submanifold, and an open neighbourhood $U(Z) \subset W_\eta$ is symplectomorphic to a disc bundle in the total space of a symplectic vector bundle $E \rightarrow T^*Z$. The bundle E admits two transverse Lagrangian subbundles coming from the normal bundles ν_{Z/Q_i} , which means that E is globally symplectically trivial. The surgeries of two transverse planes in \mathbb{C}^2 may then be performed pointwise. The 1-handles in each fibre of E are exact and graded, from which the corresponding properties for the Bott surgeries follow. \square

The Bott surgeries K and K' are not in general Hamiltonian isotopic, but they are diffeomorphic because the Lagrange handles in the local model are smoothly isotopic.

Fix a field \mathbb{K} . The Floer differential vanishes for action reasons, and $HF(Q_1, Q_2) \cong H^*(Z; \mathbb{K})$. Choose gradings on the Q_i so that

$$HF(Q_1, Q_2) \cong H^*(S^1)[-1] \cong HF(Q_2, Q_1)$$

are symmetrically graded in degrees 1 and 2. Let e and f denote the degree 1 generators in $HF^1(Q_1, Q_2)$ respectively $HF^1(Q_2, Q_1)$, and $e^\vee \in HF^2(Q_2, Q_1)$ and $f^\vee \in HF^2(Q_1, Q_2)$ denote the Poincaré dual degree 2 generators. In a minimal model, the simplest higher-order operations in the A_∞ -structure on the Q_i are Massey products

$$(2.2) \quad \mu^3(e, f, e) = \lambda \cdot f^\vee; \quad \mu^3(f, e, f) = \lambda' \cdot e^\vee, \quad \lambda, \lambda' \in \mathbb{K}.$$

The values λ, λ' are well-defined symplectic invariants, independent of the choice of almost complex structure.

Lemma 2.4. *The values λ and λ' co-incide. If $\lambda = 0$ then $H^*(K; \mathbb{Z}) \cong H^*(S^1 \times S^2; \mathbb{Z})$. If $|\lambda| > 0$, then K is a homology Lens space with $H^1(K; \mathbb{Z}) \cong \mathbb{Z}/|\lambda|$.*

Proof. Equality of λ and λ' follows from K and K' being diffeomorphic. There is a Lagrangian cobordism between K and $Q_1 \amalg Q_2$, which shows that the surgery is quasi-isomorphic to a mapping cone $Q_1 \xrightarrow{\kappa} Q_2$ in $\mathcal{F}(W_\eta)$, where $\kappa \in \text{Hom}_{\mathcal{F}}(Q_1, Q_2) = HF(Q_1, Q_2)$.

(The other surgery K' is quasi-isomorphic to a mapping cone $Q_2 \xrightarrow{\kappa'} Q_1$.) The class κ is a primitive multiple $\kappa = \pm e$ of the unique degree one morphism, corresponding to the fundamental class of the clean-intersection locus (the multiple must be primitive since the mapping cone is quasi-isomorphic to the surgery, and hence indecomposable, over any field). The endomorphisms of the mapping cone depend on $\mu_{\text{Tw}(\mathcal{F})}^1(f)$ which has non-trivial term $\mu_{\mathcal{F}}^3(e, f, e)$; since K is exact, $\text{Hom}_{\mathcal{F}}(K, K) = H^*(K; \mathbb{K})$. The result follows. \square

Example 2.5. If κ_2 is an unknot, then K and K' are obtained by Dehn surgery on κ_1 with integer slopes $\pm\lambda$.

Via Lemma 2.4, we can fix a base-point in the \mathbb{Z} -torsor of normal bundle identifications. (The same base-point arises if one uses a metric on S^3 to identify the normal and conormal bundles along Z .) We will write

$$W_n(\kappa_1, \kappa_2)$$

for the plumbing in which the surgeries have $H^1(K; \mathbb{Z}) = \mathbb{Z}/n$, with $n = 0$ corresponding to the case of a homology $S^1 \times S^2$. When the knots κ_i are both the unknot, for instance plumbing along a linearly embedded circle, we simply write W_n .

2.3. Encoding by a quiver with potential. Let $\mathcal{F}(W_n)$ denote the Fukaya category of W_n as constructed in [44], whose objects are closed exact spin Lagrangian branes with vanishing Maslov class; this is a \mathbb{Z} -graded A_∞ -category over \mathbb{K} . Let $\mathcal{Q}_n \subset \mathcal{F}(W_n)$ denote the subcategory generated by the two core components Q_i . The categories \mathcal{Q}_n and $\mathcal{F}(W_n)$ are 3-Calabi-Yau categories.

It will turn out that when $\text{char } \mathbb{K} = 0$ the categories \mathcal{Q}_n can be encoded by a quiver with potential; besides being an efficient packaging of the information, it makes it easy to compare to categories arising elsewhere.

Let \mathcal{C} be a minimal \mathbb{K} -linear A_∞ category which is 3-Calabi-Yau and has a finite set of objects $\{S_i\}$. Suppose further that

- (1) Each S_i is spherical, and
- (2) $\text{Hom}_{\mathcal{C}}(S_i, S_j)$ is supported in degrees 1 and 2 for $i \neq j$.

Then there are \mathbb{K} -vector spaces V_{ij} with

$$\text{Hom}_{\mathcal{C}}(S_i, S_j) = \mathbb{K}^{\delta_{ij}} \oplus V_{ij}^*[-1] \oplus V_{ji}[-2] \oplus \mathbb{K}^{\delta_{ij}}[-3].$$

If the A_∞ -structure on \mathcal{C} is *both* cyclic, in the sense that

$$\langle \mu^n(f_n \cdots, f_1), f_0 \rangle = \langle \mu^n(f_{n-1}, \dots, f_1, f_0), f_n \rangle,$$

and strictly unital, then the only possible non-trivial A_∞ -products

$$\mu^n: \text{Hom}_{\mathcal{C}}(S_{j_{n-1}}, S_{j_n}) \otimes \cdots \otimes \text{Hom}_{\mathcal{C}}(S_{j_1}, S_{j_2}) \longrightarrow \text{Hom}_{\mathcal{C}}(S_{j_1}, S_{j_n})[2-n]$$

are those where all inputs f_i have degree 1. There is a manifest non-degenerate pairing

$$\langle -, - \rangle: \text{Hom}_{\mathcal{C}}(S_i, S_j) \times \text{Hom}_{\mathcal{C}}(S_j, S_i) \rightarrow \mathbb{K}[-3],$$

and the non-trivial A_∞ -products are encoded by the values

$$c_{n+1}(f_{n+1}, \dots, f_1) = \langle f_{n+1}, \mu^n(f_n, \dots, f_1) \rangle,$$

hence by linear maps

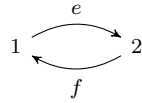
$$(2.3) \quad c_{n+1}: \text{Hom}_{\mathcal{C}}^1(S_{j_{n+1}}, S_{j_1}) \otimes \text{Hom}_{\mathcal{C}}^1(S_{j_n}, S_{j_{n+1}}) \otimes \cdots \otimes \text{Hom}_{\mathcal{C}}^1(S_{j_1}, S_{j_2}) \longrightarrow \mathbb{K}[n+1].$$

We may therefore encode \mathcal{C} entirely by a quiver Q , which has a vertex for each S_i and $\text{rk}_{\mathbb{K}}(V_{ij})$ arrows from vertex i to vertex j . The coefficients c_{n+1} determine a potential on Q , i.e. an element $\mathbb{W} \in \widehat{\mathbb{K}Q}$ of the closure of the subspace of the path algebra of Q (completed with respect to path length) spanned by cyclic paths of length at least two. We will write $\mathcal{C}(Q, \mathbb{W})$ for the A_∞ -category associated to a quiver Q with potential \mathbb{W} ; the zero-potential $\mathbb{W} = 0$ defines the formal A_∞ -structure.

Remark 2.6. Any A_∞ -structure over a field is quasi-isomorphic to a strictly unital structure. Cyclicity can always be achieved in characteristic zero [32], but not in general in finite characteristic: $QH^*(S^2; \mathbb{Z}/2)$ has a non-trivial A_∞ -product μ^4 with output in the fundamental class [16], and this violates cyclicity in any strictly unital model.

Two potentials \mathbb{W} and \mathbb{W}' are *weakly right-equivalent* if there is an automorphism $\phi: \widehat{\mathbb{K}Q} \rightarrow \widehat{\mathbb{K}Q}$ of the completed path algebra which fixes the zero-length paths and a scalar $t \in \mathbb{K}^*$ such that $\phi(\mathbb{W}) - t\mathbb{W}'$ lies in the closure of the subspace generated by differences $a_1 \dots a_s - a_2 \dots a_s a_1$, where $a_1 \dots a_s$ is a cycle in $\mathbb{K}Q$. [30, Lemma 2.9] shows weakly right-equivalent potentials yield quasi-equivalent A_∞ -categories $\mathcal{C}(Q, \mathbb{W}) \simeq \mathcal{C}(Q, \mathbb{W}')$.

Let Cyc_2 denote the quiver which is a single oriented 2-cycle, with arrows e and f .



Lemma 2.7. If $\text{char } \mathbb{K} = 0$, then any potential on Cyc_2 is weakly equivalent to some $(ef)^n$.

Proof. The only cyclic words in $\widehat{\mathbb{C}Q}$ have the form $\mathbb{W} = \sum_j a_j (ef)^j$ for scalars $a_j \in \mathbb{K}$. Since we are working formally, a power series $1 + \sum_j b_j (ef)^j$ is invertible, so \mathbb{W} is weakly equivalent to $(ef)^n$ where $n = \min\{j \mid a_j \neq 0\}$. \square

Proposition 2.8. Over \mathbb{C} , there are equivalences

$$\mathcal{Q}_0 \simeq (\text{Cyc}_2, \mathbb{W} = 0) \quad \text{and} \quad \mathcal{Q}_n \simeq (\text{Cyc}_2, \mathbb{W} = (ef)^2) \text{ for } n \geq 1.$$

Proof. This follows from Lemmas 2.4 and 2.7: the former says that once $n > 0$ the potential must have a non-trivial quartic term to yield the non-trivial μ^3 in the A_∞ -structure. \square

In characteristic zero, the potentials $(ef)^n$ with $n \geq 3$ cannot arise from any $W_\eta(\kappa_1, \kappa_2)$, even if one allows the embedding $Z \hookrightarrow Q_i$ to be knotted. Indeed, for $n \geq 3$, the surgery K is an exact Lagrangian homology $S^1 \times S^2$, and triviality of the classical A_∞ -structure on $H^*(S^1 \times S^2)$ rules out potentials of higher degree.

2.4. Finite characteristic. Proposition 2.8 shows that, over \mathbb{C} , the categories \mathcal{Q}_n are all equivalent, for any $n \geq 1$. To see the more general phenomena, for example the Lens space twist, requires us to work over a different field, and to turn on the characteristic. Because of this, due to Remark 2.6, from this section onwards we will work directly with the higher A_∞ products.

Let $p > 2$ be prime, and write W_p for the plumbing in which the surgery of the two cores is a Lens space $L(p, 1)$. Then, since $\text{char } \mathbb{K} = p$,

$$\mu^3(e, f, e) = p \cdot f^\vee \equiv 0.$$

It is possible to pin down another non-trivial higher product. For $\mathbb{F}_p = \mathbb{Z}/p$, and $L(p, 1)$ the 3-dimensional Lens space, the cohomology $H^*(L(p, 1); \mathbb{F}_p) = \Lambda(y) \otimes \mathbb{F}_p[z]/\langle z^2 \rangle$ with $|y| = 1$ and $|z| = 2$.

Lemma 2.9. There is a non-trivial A_∞ -product $\mu^p(y, \dots, y) = z$ in $H^*(L(p, 1); \mathbb{F}_p)$.

Proof. The infinite-dimensional Lens space $S^\infty/(\mathbb{Z}/p)$ is also the Eilenberg-MacLane space $K(\mathbb{Z}/p, 1)$, so its based loop space $\Omega(K(\mathbb{Z}/p, 1)) = \mathbb{Z}/p\mathbb{Z}$ is just a finite set up to homotopy (with an addition structure since it is an h -space). A classical group cohomology computation asserts that, when the field $\mathbb{F}_p = \mathbb{Z}/p$,

$$(2.4) \quad H^*(S^\infty/\mathbb{Z}/p; \mathbb{F}_p) = \text{Ext}_{\mathbb{F}_p[\mathbb{Z}/p]}^*(\mathbb{F}_p, \mathbb{F}_p) = \Lambda(y) \otimes \mathbb{F}_p[z] \quad \text{with } |y| = 1, |z| = 2.$$

Moreover, this carries a non-trivial A_∞ -product

$$(2.5) \quad \mu^p(y, y, \dots, y) = z.$$

This is the first non-trivial product, see [39, Example 6.3], so $\mu^i = 0$ for $2 < i < p$.

Consider an A_∞ -minimal model $H^*(L(p, 1); \mathbb{F}_p)$ for the cohomology of the 3-dimensional Lens space $L(p, 1)$, and a minimal A_∞ -functor $F: H^*(L(p, 1); \mathbb{F}_p) \dashrightarrow H^*(K(\mathbb{Z}/p, 1); \mathbb{F}_p)$ for the map which classifies the degree one generator y . This map is represented by the inclusion $L(p, 1) \subset K(\mathbb{Z}/p, 1)$ for suitable cellular cochain models, which shows that it is an isomorphism on $H^{<4}$. The A_∞ -functor equations yield

$$\mu^k(F^1(y), \dots, F^1(y)) + \sum_{j < k} \mu^j(\dots) = F^1(\mu^k(y, \dots, y)) + \sum_{1 < j} F^j(\dots)$$

If $k < p$, then the A_∞ -operations μ^j for $j \leq k$ all vanish on the target, so the left hand side vanishes identically, which inductively shows that the A_∞ -product $\mu^k(y, \dots, y) = 0$ on $L(p, 1)$. However, for $k = p$, one obtains the identity

$$z = \mu^p(y, \dots, y) = \mu^p(F^1(y), \dots, F^1(y)) = F^1(\mu^p(y, \dots, y))$$

showing that μ^p is non-trivial for the finite-dimensional $L(p, 1)$, as well as for $K(\mathbb{Z}/p, 1)$. \square

Corollary 2.10. *If $\text{char } \mathbb{K} = p$, then in the category \mathcal{Q}_p there is a non-trivial product $\mu^{2p-1}(e, f, \dots, e, f, e) = f^\vee$.*

Proof. As in Lemma 2.4, we know from the existence of the surgery cobordism that the Lens space $K = L(p, 1)$ quasi-represents the mapping cone $Q_0 \xrightarrow{e} Q_1$; then its degree one cohomological generator is represented by the cocycle f . The product in Lemma 2.9 is then the twisted complex differential $\mu_{\text{Tw}}^p(f, \dots, f)$, which has non-trivial contribution exactly from the term appearing in the statement. \square

Recall from Lemma 2.7 that over \mathbb{C} , every A_∞ -structure on the cohomological algebra underlying the two-cycle quiver is quasi-isomorphic to that given by the potential $(ef)^n$ for some n . Over a field \mathbb{K} of finite characteristic p , we will in mild abuse of notation write $(\text{Cyc}_2, (ef)^p)$ for the unique quasi-isomorphism class of A_∞ -structures on the algebra underlying $\text{Cyc}(2)$ in which the only non-vanishing higher products are $\mu^{2p-1}(e, f, \dots, e) = \pm f^\vee$ and $\mu^{2p-1}(f, e, \dots, f) = \pm e^\vee$.

Corollary 2.11. *If $\text{char } \mathbb{K} = p$, there is an equivalence $\mathcal{Q}_p \simeq (\text{Cyc}_2, (ef)^p)$.*

Proof. *A priori* we cannot make the A_∞ -structure strictly cyclic over \mathbb{K} , but it can be made strictly unital, since the reduced Hochschild complex is quasi-isomorphic to the full Hochschild complex over any field. For degree reasons there are then only two kinds of possible non-zero A_∞ -product: those with only degree one inputs, and products with a single degree two input, say

$$\mu^k(e, e^\vee, e, f, e, f, \dots, e, f) = ?[Q_0] \quad \text{or} \quad \mu^k(e, f, \dots, e, e^\vee) = ?[Q_0]$$

and the corresponding products with output $[Q_1]$. The latter products have the feature that, for at least one of e, f , if one removes all occurrences of that symbol, one is left only with copies of the other symbol and its dual (e.g. removing f 's from the first expression leaves all inputs being e, e^\vee). Such A_∞ -products arise in the A_∞ -structure on the twisted complex of $Q_1 \xrightarrow{f} Q_0$ or $Q_0 \xrightarrow{e} Q_1$. These represent the exact Morse-Bott surgery $L(p, 1)$, for which the A_∞ -structure on its endomorphism algebra recovers the classical A_∞ -structure on cohomology over any field. Arguing as in Lemma 2.9 and Corollary 2.10, if these A_∞ -products were non-trivial they would be visible in the group cohomology $\text{Ext}_{\mathbb{F}_p[\mathbb{Z}/p]}(\mathbb{F}_p, \mathbb{F}_p)$. The vanishing of the products with degree two output follows by Koszul duality as in [38, Example 8.2] (or from an explicit resolution of the diagonal). Indeed, [38] shows that the only non-trivial A_∞ -products in $H^*(S^\infty/(\mathbb{Z}/p); \mathbb{F}_p)$ are of the form

$$\mu^p(yz^{j_1}, \dots, yz^{j_p}) = z^{1+\sum j_i}.$$

None of the products with $j_i > 0$ can survive to $L(p, 1)$ for degree reasons. It follows that the displayed product (2.5) is the only non-trivial product up to the symmetry between the degree one generators in the algebra, or geometrically the two Morse-Bott Lagrange surgeries; compare to the argument of Proposition 3.11. \square

Remark 2.12. Lazaroiu [35] has given a sufficient criterion for a DG-algebra with a pairing to be strictly cyclic over fields of characteristic $\neq 2$, in terms of a homotopy operator with suitable properties. It would be interesting to establish cyclicity directly for double bubbles from Abouzaid's topological model [1].

3. MIRRORS FROM FLOPS

3.1. Neighbourhoods of two-curve flops. The quivers with potential appearing in Section 2.3 also arise when studying neighbourhoods of a pair of intersecting floppable curves in a threefold. We focus on the affine 3-folds $X_n = \text{Spec } R_n$ where

$$(3.1) \quad R_n = \frac{\mathbb{K}[u, v, x, y]}{(uv - xy(x^n + y))} \quad \text{for } n \geq 1; \quad R_0 = \frac{\mathbb{K}[u, v, x, y]}{(uv - xy^2)}.$$

For $n \neq 0$ it is clear that R_n localises to a regular local ring away from the origin, so this affine variety has a unique singular point $0 \in X_n$. Being an isolated hypersurface, it follows immediately from Serre's criterion (in any characteristic) that R_n is normal. On the other hand, X_0 is singular precisely along the x -axis. However, it too is normal.

From these varieties, over any \mathbb{K} , we will geometrically construct 3-CY categories \mathcal{C} . This construction is standard when $\text{char } \mathbb{K} = 0$; the main issue in this subsection is determining that the characteristic does not effect the birational and derived geometry.

Lemma 3.1. *For each $n \geq 0$ the variety X_n admits a specific resolution $f: Y_n \rightarrow X_n$ for which $f^{-1}(0)$ comprises two $(-1, -1)$ -curves meeting at one point.*

Proof. See cf. [25, §5.1]. When \mathbb{K} has characteristic zero, this is standard. Below, the main fact that we will use is that

$$\varphi: \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \rightarrow \text{Spec } \mathbb{K}[u, v, x, y]/(uv - xy),$$

which is the blowup of either (u, x) or (u, y) , results in a projective birational morphism with a single $(-1, -1)$ -curve as the exceptional locus, regardless of the characteristic of the

field. Given the base is normal, Zariski's Main Theorem implies that $\varphi_*\mathcal{O} = \mathcal{O}$, and an easy Čech calculation shows that $\mathbf{R}^1\varphi_*\mathcal{O} = 0$. Given the fibres are at most one dimensional, this in turn implies that $\mathbf{R}\varphi_*\mathcal{O} = \mathcal{O}$.

Write $R_n = \mathbb{K}[u, v, x, y]/(uv - g)$, with $g = g_1g_2g_3$ a factorization into primes in $\mathbb{K}[x, y]$, then small resolutions can be obtained by blowing up ideals locally of the form (u, g_i) . Different sequences can lead to different resolutions.

To construct our specific resolution, consider the ordering

$$g_1g_2g_3 = \begin{cases} yx(x^n + y) & \text{if } n > 0, \\ yxy & \text{if } n = 0. \end{cases}$$

Then for all X_n with $n \geq 0$, we first blowup the ideal $(u, g_1) = (u, y)$ to obtain

$$g: Z \rightarrow \text{Spec } R_n,$$

where the variety Z is covered by the two open charts

$$\mathbb{K}[u, x, S] \quad \text{and} \quad \mathbb{K}[v, x, y, T]/(vT - g_2g_3).$$

The co-ordinates $S = g_1/u$ and $T = u/g_1$ patch to define a copy of \mathbb{P}^1 in Z . Since the base is normal, again Zariski's Main Theorem shows that $g_*\mathcal{O} = \mathcal{O}$, and an explicit Čech cohomology calculation gives $\mathbf{R}^1g_*\mathcal{O} = 0$. Given the dimension of the fibres is at most one, we conclude that $\mathbf{R}\varphi_*\mathcal{O} = \mathcal{O}$.

When $n = 0$ the second chart is given by $vT = xy$, whereas when $n \geq 1$ the second chart is $vT = x(x^n + y)$. However, after the polynomial change in co-ordinates $Y \mapsto x^n + y$, the second chart is isomorphic to $\mathbb{K}[v, x, Y, T]/(vT - xY)$. Hence in all cases, it is clear that the second chart localises to a regular local ring away from the origin, and we can resolve the second chart by blowing up $(T, g_2) = (T, x)$, whose zero set does not intersect the first chart. This results in a morphism

$$h: Y_n \rightarrow Z$$

Using the fact at the beginning of the proof, the second blowup replaces the unique singular point in the second chart of Z with a $(-1, -1)$ -curve. Since Z is normal necessarily $h_*\mathcal{O} = \mathcal{O}$, and also we have that $\mathbf{R}^1h_*\mathcal{O} = 0$ since this can be checked locally. Again as the dimension of the fibres are at most one, it follows that $\mathbf{R}h_*\mathcal{O} = \mathcal{O}$.

Composing, we obtain a resolution $f = g \circ h: Y_n \rightarrow \text{Spec } R_n$, such that $\mathbf{R}f_*\mathcal{O} = \mathcal{O}$. Since the co-ordinates $S = g_1/u$ and $T = u/g_1$ patch to define a copy of \mathbb{P}^1 in the partial resolution Z , and this \mathbb{P}^1 passes through the node $\{vT = g_2g_3\}$ in the second chart, in the full resolution Y we thus have two intersecting \mathbb{P}^1 s, say C_1 and C_2 , where by construction the second of which is a $(-1, -1)$ curve. It can be seen explicitly that contracting C_1 results in $uv = g_1g_2$, which in all cases is $uv = xy$. Hence C_1 is also a $(-1, -1)$ -curve. \square

Remark 3.2. Over any field \mathbb{K} , the following hold.

- (1) The variety X_n admits three distinct crepant resolutions if $n = 0$ and six if $n > 0$. This is since we can also blowup using different orderings of $g_1g_2g_3$; there are $3! = 6$ options if the factors are distinct (which they are when $n > 0$), but when $n = 0$ the repetition of one of the factors means we only obtain three. All are related by flops, as can explicitly be seen.

- (2) When $n = 0$ the two curves belong to a surface; one can flop either curve to obtain a resolution (now containing a $(-2, 0)$ -curve) whose exceptional set is not pure-dimensional, see [13, Figure on p.24], but the curves cannot flop together, as then the whole surface is contracted.
- (3) When $n > 0$ the picture is similar to the above, but the surface containing the two curves is now infinitesimal. In the case $n = 1$, after all possible flops, all curves remain $(-1, -1)$ -curves. However, in the case $n > 1$, floppable $(-2, 0)$ -curves arise.

As in the introduction, for $n \geq 0$ consider the category

$$\mathcal{C}_n := \langle \mathcal{O}_{C_1}(-1), \mathcal{O}_{C_2}(-1) \rangle \subset D^b(\text{coh } Y_n).$$

where $f: Y_n \rightarrow X_n$ is the specific resolution constructed in Lemma 3.1. To relate this to the categories \mathcal{Q}_k , we pass to noncommutative resolutions, which give a third realisation of the same category.

3.2. Noncommutative Resolutions. For each $n \geq 0$, consider the maximal Cohen–Macaulay R_n -module $M := R_n \oplus (u, y) \oplus (u, xy)$.

Proposition 3.3. *For each $n \geq 0$, the following statements hold.*

- (1) Y_n admits a tilting bundle \mathcal{V} , which is a direct sum of line bundles.
- (2) There is an isomorphism $f_*\mathcal{V} \cong M$.

Consequently, Y_n is derived equivalent to $\Lambda_n := \text{End}_{R_n}(M)$, the resolution f is crepant, and Λ_n is a NCCR.

Proof. By the explicit construction of Y_n , we can find a divisor D_1 such that $D_1 \cdot C_1$ is a point, and $D_1 \cdot C_2$ is empty (e.g. consider $S = 0$ in the first chart). Similarly, we can find a divisor D_2 such that $D_2 \cdot C_2$ is a point, and $D_2 \cdot C_1$ is empty. Set $\mathcal{L}_i = \mathcal{O}(D_i)$, and $\mathcal{V} = \mathcal{O} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$.

The fact that \mathcal{V} is tilting in characteristic zero is [49]. In characteristic p here, we can instead follow the more explicit method of [50]. Indeed, exactly as in [50, 5.7] the fact that $\mathbf{R}f_*\mathcal{O} = \mathcal{O}$, together with all tensor shifts of the short exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{L}_i^* \rightarrow \mathcal{O}^{\oplus 2} \rightarrow \mathcal{L}_i \rightarrow 0 \\ 0 \rightarrow \mathcal{O} \rightarrow \mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathcal{L}_1 \otimes \mathcal{L}_2 \rightarrow 0 \end{aligned}$$

is enough to establish that $\text{Ext}^t(\mathcal{V}, \mathcal{V}) = 0$ for all $t > 0$. Generation is then word-for-word identical to [50, 5.8], since here $\mathcal{L}_1 \otimes \mathcal{L}_2$ is ample. Hence \mathcal{V} is a tilting bundle.

It is easy to see that $h_*\mathcal{V} = \mathcal{O} \oplus \mathcal{L}'_1 \oplus (T, g_2)$, where \mathcal{L}'_1 is also a line bundle on Z . Pushing this down once more, since $T = u/g_1$ we obtain

$$f_*\mathcal{V} = \mathcal{O} \oplus (u, g_1) \oplus (u, g_1g_2),$$

proving the second statement. The fact that the natural map

$$\text{End}_{Y_n}(\mathcal{V}) \rightarrow \text{End}_{R_n}(f_*\mathcal{V}) = \text{End}_{R_n}(M) := \Lambda_n$$

is an isomorphism follows immediately from [12, 4.3], since f is a proper birational morphism between varieties, \mathcal{V} is generated by global sections, and $\text{Ext}^1(\mathcal{V}, \mathcal{V}) = 0$.

It is already known, regardless of the characteristic, that the property of being an NCCR can be checked complete locally [23, 2.17], and complete locally in the situation here the result

is [25, §5.1]. Hence Λ_n is a NCCR, so Y_n , being derived equivalent to Λ_n , is automatically a crepant resolution [24]. \square

The algebra Λ_n has a simple module corresponding to each summand of M , say $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ corresponding to $R, (u, y), (u, xy)$ respectively. In characteristic zero the following is standard; here we just need to directly verify it.

Corollary 3.4. *For any $n \geq 0$, under the derived equivalence induced by the tilting bundle \mathcal{V} , the simple \mathcal{S}_i corresponds to $\mathcal{O}_{C_i}(-1)[1]$.*

Proof. We prove $i = 1$. It is clear that $\mathbf{RHom}_{Y_n}(\mathcal{O}, \mathcal{O}_{C_1}(-1)[1]) = 0$. Further, since $D_2 \cdot C_1$ is empty, it is also clear that $\mathbf{RHom}_{Y_n}(\mathcal{L}_2, \mathcal{O}_{C_1}(-1)[1]) = 0$. Lastly, observe that $\mathbf{RHom}_{Y_n}(\mathcal{L}_1, \mathcal{O}_{C_1}(-1)[1]) = \mathbf{RHom}_{Y_n}(\mathcal{O}, \mathcal{O}_{C_1}(-2)[1]) = \mathbb{K}$ in homological degree zero. The result follows. \square

It follows that the category \mathcal{C}_n is equivalent to all those complexes in $D^b(\text{mod } \Lambda_n)$ whose cohomology modules are filtered by $\mathcal{S}_1, \mathcal{S}_2$. Being generated by $\mathcal{S}_1 \oplus \mathcal{S}_2$, the category \mathcal{C}_n is Morita equivalent to perfect modules over the A_∞ -algebra $\text{Ext}_{\Lambda_n}^*(\mathcal{S}_1 \oplus \mathcal{S}_2, \mathcal{S}_1 \oplus \mathcal{S}_2)$. We will compute this A_∞ -algebra using the following presentation of Λ_n .

Lemma 3.5. *For $n \geq 0$, the ring Λ_n can be presented as the following quiver with relations:*

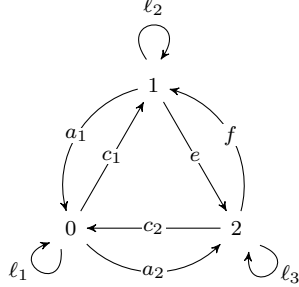
$$\begin{aligned} efa_1 &= a_1\ell \\ \ell c_1 &= c_1ef \\ fec_2 &= c_2\ell \\ \ell a_2 &= a_2fe \end{aligned}$$

$$\begin{aligned} a_1c_1e &= e(c_2a_2 - (fe)^n) \\ fa_1c_1 &= (c_2a_2 - (fe)^n)f \\ \ell^n &= a_2c_2 - c_1a_1 \end{aligned}$$

where if $n = 0$, any monomial involving a power of n should be interpreted as being zero.

Proof. Consider first the key variety $K = \mathbb{K}[u, v, x, y, z]/(uv - xyz)$. It is easier to first present the NCCR $\Delta = \text{End}_K(K \oplus (u, y) \oplus (u, xy))$ of K , by considering the following morphisms:

Indeed, it is easy to see that the stated maps generate the algebra Δ , and furthermore the following relations hold:



$$\begin{array}{lll}
 \ell_3 f = f a_1 c_1 & \ell_1 a_2 = a_2 f e & \ell_2 a_1 = a_1 a_2 c_2 \\
 \ell_3 c_2 = c_2 c_1 a_1 & \ell_1 c_1 = c_1 e f & \ell_2 e = e c_2 a_2 \\
 e \ell_3 = a_1 c_1 e & c_2 \ell_1 = f e c_2 & c_1 \ell_2 = a_2 c_2 c_1 \\
 a_2 \ell_3 = c_1 a_1 a_2 & a_1 \ell_1 = e f a_1 & f \ell_2 = c_2 a_2 f
 \end{array}$$

Thus there is an induced surjection $\mathbb{K}Q/\mathcal{R} \twoheadrightarrow \Delta$. Since both algebras are graded, by comparing the dimension of the graded pieces on both sides, this is in fact an isomorphism.

Then, exactly as in [26, §6.2], the algebra Λ_n can be obtained from Δ by slicing by the central element corresponding to $z - (x^n + y) \in K$. Hence, a presentation of Λ_n can be obtained from a presentation of Δ by adding in the following three additional relations:

$$\begin{aligned}
 a_2 c_2 &= \ell_1^n + c_1 a_1 \\
 \ell_2 &= (e f)^n + a_1 c_1 \\
 c_2 a_2 &= (f e)^n + \ell_3
 \end{aligned}$$

The second allows us to eliminate ℓ_2 , and the third allows us to eliminate ℓ_3 . This leaves $15 - 2 = 13$ relations, but it is easy to check that six come for free from the rest, and so we are left with the seven relations in the statement of the lemma. \square

Remark 3.6. Note that Lemma 3.5 gives a non-minimal presentation when $n = 1$, since in that case the loop at 0 is not strictly required. This does not effect the computation of the A_∞ structure below, which takes place at the other vertices.

3.3. Computing the A_∞ structure. Set $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$. We now compute $A_n = \text{Ext}_{\Lambda_n}^*(\mathcal{S}, \mathcal{S})$, for all $n \geq 0$, over an arbitrary field \mathbb{K} .

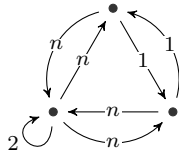
Lemma 3.7. *The sequences*

$$\begin{aligned}
 0 \rightarrow \mathcal{P}_1 \xrightarrow{\begin{pmatrix} c_1 \\ f \end{pmatrix}} \mathcal{P}_0 \oplus \mathcal{P}_2 \xrightarrow{\begin{pmatrix} \ell & -c_1 e \\ -f a_1 & c_2 a_2 - (f e)^n \end{pmatrix}} \mathcal{P}_0 \oplus \mathcal{P}_2 \xrightarrow{(a_1 \ e)} \mathcal{P}_1 \rightarrow \mathcal{S}_1 \rightarrow 0 \\
 0 \rightarrow \mathcal{P}_2 \xrightarrow{\begin{pmatrix} a_2 \\ e \end{pmatrix}} \mathcal{P}_0 \oplus \mathcal{P}_1 \xrightarrow{\begin{pmatrix} \ell & -a_2 f \\ -e c_2 & a_1 c_1 + (e f)^n \end{pmatrix}} \mathcal{P}_0 \oplus \mathcal{P}_1 \xrightarrow{(c_2 \ f)} \mathcal{P}_2 \rightarrow \mathcal{S}_2 \rightarrow 0
 \end{aligned}$$

are projective resolutions of the simples \mathcal{S}_1 and \mathcal{S}_2 .

Proof. This is just projectivisation applied to the mutation sequences [25, 5.10, 5.11]. \square

Lemma 3.8. *For all $n \geq 0$ the algebra Λ_n is \mathbb{N} -graded, with arrows graded as follows:*



Proof. It is a direct verification that the seven relations in Lemma 3.5 are all homogeneous with respect to the stated grading. \square

To avoid confusion, we will refer to the above grading as the Adams grade, to distinguish it from the homological degree. We can shift the projective modules via the Adams grade to make the differentials in Lemma 3.7 have grade zero, namely

$$\mathcal{P}_1(-2n-2) \longrightarrow \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) \longrightarrow \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) \longrightarrow \mathcal{P}_1$$

$$\mathcal{P}_2(-2n-2) \longrightarrow \mathcal{P}_0(-n-2) \oplus \mathcal{P}_1(-2n-1) \longrightarrow \mathcal{P}_0(-n) \oplus \mathcal{P}_1(-1) \longrightarrow \mathcal{P}_2$$

Write \mathcal{P} for the direct sum of the above two projective resolutions, and consider the DGA $\text{End}_{\Lambda_n}(\mathcal{P})$. This inherits a graded structure, as all morphisms between the projectives are Adams graded.

From Lemma 3.7 it follows that the cohomology of $\text{End}_{\Lambda_n}(\mathcal{P})$, namely the Ext algebra $\text{Ext}_{\Lambda_n}^*(\mathcal{S}, \mathcal{S})$, has Hilbert series $2 + 2t + 2t^2 + 2t^3$, and so we next seek generators. For degree one, there are obvious maps representing $\text{Ext}^1(\mathcal{S}_1, \mathcal{S}_2) = \mathbb{K}$ and $\text{Ext}^1(\mathcal{S}_2, \mathcal{S}_1) = \mathbb{K}$, namely

$$\begin{array}{ccccccc} & & \mathcal{P}_1(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) \longrightarrow \mathcal{P}_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ \xi_{12} = & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \begin{pmatrix} 0 & a_2 \\ -a_1 & -(ef)^{n-1}e \end{pmatrix} & & \begin{pmatrix} 0 & -1 \end{pmatrix} \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_2(-2n-2) & \rightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_1(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_1(-1) & \longrightarrow & \mathcal{P}_2 \\ & & \mathcal{P}_2(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_1(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_1(-1) \longrightarrow \mathcal{P}_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ \xi_{21} = & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \begin{pmatrix} 0 & c_1 \\ -c_2 & (fe)^{n-1}f \end{pmatrix} & & \begin{pmatrix} 0 & -1 \end{pmatrix} \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_1(-2n-2) & \rightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) & \longrightarrow & \mathcal{P}_1 \end{array}$$

All diagrams above commute, and both chain maps give non-zero elements of Ext because the 1's on the vertical maps cannot be obtained by any homotopy; otherwise the trivial path is in the arrow ideal, which would contradict the grading. For dimension reasons, these generate Ext^1 . Both ξ_{12}, ξ_{21} have Adams grade +1.

It is easy to check that both $\xi_{21} \circ \xi_{12}$ and $\xi_{12} \circ \xi_{21}$ are homotopic to zero. We next seek generators of $\text{Ext}^2(\mathcal{S}_1, \mathcal{S}_2)$ and $\text{Ext}^2(\mathcal{S}_2, \mathcal{S}_1)$, so consider the following:

$$\begin{array}{ccccccc} & & & & \mathcal{P}_1 & \xrightarrow{\begin{pmatrix} c_1 \\ f \end{pmatrix}} & \mathcal{P}_0 \oplus \mathcal{P}_2 \longrightarrow \mathcal{P}_0 \oplus \mathcal{P}_2 \longrightarrow \mathcal{P}_1 \\ & & & & \downarrow & & \downarrow \\ \zeta_{12} = & & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & & & \begin{pmatrix} 0 & 1 \end{pmatrix} \\ & & \downarrow & & & & \downarrow \\ \mathcal{P}_2 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_1 & \xrightarrow{\begin{pmatrix} c_2 & f \end{pmatrix}} & \mathcal{P}_2 \end{array}$$

$$\zeta_{21} = \begin{array}{ccccccc} & & & \mathcal{P}_2 & \xrightarrow{\begin{pmatrix} a_2 \\ e \end{pmatrix}} & \mathcal{P}_0 \oplus \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_1 & \longrightarrow & \mathcal{P}_2 \\ & & & \downarrow & & \downarrow & & & & \\ \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_2 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_2 & \xrightarrow{\begin{pmatrix} a_1 & e \end{pmatrix}} & \mathcal{P}_1 & & & \end{array}$$

These give non-zero elements in Ext^2 for the same reason as before: the trivial paths don't belong to the arrow ideal. Both ζ_{12}, ζ_{21} have Adams grade $2n + 1$.

It is very easy to check that $\zeta_{21} \circ \xi_{12} = -\xi_{21} \circ \zeta_{12}$, as the left hand side is the identity on \mathcal{P}_1 (and is zero elsewhere), whilst the right hand side is the negative of the identity on \mathcal{P}_1 . Consequently both are non-zero elements of $\text{Ext}^3(\mathcal{S}_1, \mathcal{S}_1)$, again since the trivial path is not in the arrow ideal. A similar thing happens for the indices the other way round. Thus we can realise all non-zero elements in $\text{Ext}^*(\mathcal{S}, \mathcal{S})$ via combinations of the elements $\xi_{12}, \xi_{21}, \zeta_{12}, \zeta_{21}$, and so these generate $\text{Ext}^*(\mathcal{S}, \mathcal{S})$.

To compute the higher Massey products, it will be convenient to consider the following:

$$h_{11} = \begin{array}{ccccccc} \mathcal{P}_1(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) & \longrightarrow & \mathcal{P}_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & \begin{pmatrix} 1 & 0 \\ 0 & (fe)^{n-1} \end{pmatrix} & & 0 \\ \mathcal{P}_1(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) & \longrightarrow & \mathcal{P}_1 \end{array}$$

$$h_{22} = \begin{array}{ccccccc} \mathcal{P}_2(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_1(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_1(-1) & \longrightarrow & \mathcal{P}_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & \begin{pmatrix} 1 & 0 \\ 0 & -(ef)^{n-1} \end{pmatrix} & & 0 \\ \mathcal{P}_2(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_1(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_1(-1) & \longrightarrow & \mathcal{P}_2 \end{array}$$

and further, provided $n \geq 2$, the maps g_{11}^j, g_{22}^j (for $0 \leq j \leq n - 2$) defined

$$g_{11}^j = \begin{array}{ccccccc} \mathcal{P}_1(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) & \longrightarrow & \mathcal{P}_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & \begin{pmatrix} 0 & 0 \\ 0 & (fe)^{n-2-j} \end{pmatrix} & & 0 \\ \mathcal{P}_1(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) & \longrightarrow & \mathcal{P}_1 \end{array}$$

$$g_{22}^j = \begin{array}{ccccccc} \mathcal{P}_2(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_1(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_1(-1) & \longrightarrow & \mathcal{P}_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & \begin{pmatrix} 0 & 0 \\ 0 & -(ef)^{n-2-j} \end{pmatrix} & & 0 \\ \mathcal{P}_2(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_1(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_1(-1) & \longrightarrow & \mathcal{P}_2 \end{array}$$

and also the maps k_{12}^j, k_{21}^j (for $0 \leq j \leq n-2$) defined

$$\begin{array}{ccccccc}
& & \mathcal{P}_1(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) \longrightarrow \mathcal{P}_1 \\
& & \downarrow & & \downarrow & & \downarrow \\
k_{12}^j = & & 0 & & \begin{pmatrix} 0 & 0 \\ 0 & -(ef)^{n-2-j}e \end{pmatrix} & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_2(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_1(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_1(-1) & \longrightarrow & \mathcal{P}_2 \\
& & \downarrow & & \downarrow & & \downarrow \\
k_{21}^j = & & 0 & & \begin{pmatrix} 0 & 0 \\ 0 & (fe)^{n-2-j}f \end{pmatrix} & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_1(-2n-2) & \longrightarrow & \mathcal{P}_0(-n-2) \oplus \mathcal{P}_2(-2n-1) & \longrightarrow & \mathcal{P}_0(-n) \oplus \mathcal{P}_2(-1) & \longrightarrow & \mathcal{P}_1
\end{array}$$

Write δ for the differential on $\text{End}_{\Lambda_n}(\mathcal{P})$.

Lemma 3.9. *If $n \geq 1$, then with notation as above, the following hold.*

- (1) $\delta h_{11} = \xi_{21} \circ \xi_{12}$ and $\delta h_{22} = \xi_{12} \circ \xi_{21}$.
- (2) *There are equalities*

$$\xi_{12}h_{11} + h_{22}\xi_{12} = \begin{cases} -\zeta_{12} & \text{if } n = 1 \\ \delta k_{12}^0 & \text{if } n > 1 \end{cases} \quad \xi_{21}h_{22} + h_{11}\xi_{21} = \begin{cases} \zeta_{21} & \text{if } n = 1 \\ \delta k_{21}^0 & \text{if } n > 1. \end{cases}$$

- (3) *If $n \geq 2$, then for all $0 \leq t \leq n-2$ there are equalities*

$$\xi_{12}g_{11}^t + g_{22}^t\xi_{12} = \begin{cases} -\zeta_{12} & \text{if } t = n-2 \\ \delta k_{12}^{t+1} & \text{if } t < n-2 \end{cases} \quad \xi_{21}g_{22}^t + g_{11}^t\xi_{21} = \begin{cases} \zeta_{21} & \text{if } t = n-2 \\ \delta k_{21}^{t+1} & \text{if } t < n-2. \end{cases}$$

Proof. All are direct verifications. For example, we see explicitly that $\xi_{21} \circ \xi_{12}$ equals

$$\begin{array}{ccccccc}
& & \mathcal{P}_1 & \xrightarrow{\begin{pmatrix} c_1 \\ f \end{pmatrix}} & \mathcal{P}_0 \oplus \mathcal{P}_2 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_2 \longrightarrow \mathcal{P}_1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \begin{pmatrix} c_1 \\ (fe)^{n-1}f \end{pmatrix} & & \begin{pmatrix} a_1 e \\ (a_1 e) \end{pmatrix} & & \begin{pmatrix} a_1 e \\ (a_1 e) \end{pmatrix} \\
& & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_2 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_2 & \longrightarrow & \mathcal{P}_1
\end{array}$$

This is homotopic to zero, via h_{11} , so $\delta h_{11} = \xi_{21}\xi_{12}$. Similarly, for the first claim in (2), we explicitly see that $\xi_{12}h_{11} + h_{22}\xi_{12}$ equals

$$\begin{array}{ccccccc}
& & \mathcal{P}_1 & \xrightarrow{\begin{pmatrix} c_1 \\ f \end{pmatrix}} & \mathcal{P}_0 \oplus \mathcal{P}_2 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_2 \longrightarrow \mathcal{P}_1 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \begin{pmatrix} 0 \\ -(ef)^{n-1} \end{pmatrix} & & \begin{pmatrix} 0 \\ -(fe)^{n-1} \end{pmatrix} & & \begin{pmatrix} 0 \\ -(fe)^{n-1} \end{pmatrix} \\
& & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_2 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_1 & \longrightarrow & \mathcal{P}_0 \oplus \mathcal{P}_1 & \xrightarrow{(c_2 f)} & \mathcal{P}_2
\end{array}$$

When $n = 1$ the vertical maps are $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $(0 \ -1)$, which is $-\zeta_{12}$. When $n > 1$ the above is homotopic to zero, via k_{12}^0 , so $\delta k_{12}^0 = \xi_{12}h_{11} + h_{22}\xi_{12}$. All other claims are similar. \square

To set notation, consider the n -fold Massey product $\langle a, b, \dots \rangle_n$.

Lemma 3.10. *If $n \geq 1$, then over any field \mathbb{K} , we have $\langle \xi_{12}, \xi_{21}, \dots, \xi_{12} \rangle_{2n+1} = -\zeta_{12}$ and $\langle \xi_{21}, \xi_{12}, \dots, \xi_{21} \rangle_{2n+1} = \zeta_{21}$.*

Proof. We prove the first, with the second being similar. As already remarked $[\xi_{12}][\xi_{21}] = 0$ and $[\xi_{21}][\xi_{12}] = 0$, and further $\delta h_{11} = \xi_{21}\xi_{12}$ and $\delta h_{22} = \xi_{12}\xi_{21}$ by Lemma 3.9(1). Hence by definition $[\xi_{12}h_{11} + h_{22}\xi_{12}] \in \langle \xi_{12}, \xi_{21}, \xi_{12} \rangle$. If $n = 1$, then the result follows by Lemma 3.9(2).

Hence we can assume that $n > 1$. Using Lemma 3.9(2)(3), we see directly that

$$[\xi_{12}g_{11}^0 + h_{22}k_{12}^0 + k_{12}^0h_{11} + g_{22}^0\xi_{12}] \in \langle \xi_{12}, \xi_{21}, \xi_{12}, \xi_{21}, \xi_{12} \rangle$$

But the middle $h_{22}k_{12}^0 + k_{12}^0h_{11} = 0$. Hence $[\xi_{12}g_{11}^0 + g_{22}^0\xi_{12}]$ belongs to the fivefold Massey product. If $n = 2$, we are done, again by Lemma 3.9(3).

The proof proceeds by induction. Using the equations in Lemma 3.9(2)(3), together with the fact that all middle terms involving compositions with g , h and k are zero, and we deduce that $[\xi_{12}g_{11}^{n-2} + g_{22}^{n-2}\xi_{12}] \in \langle \xi_{12}, \xi_{21}, \dots, \xi_{12} \rangle_{2n+1}$. The result follows by Lemma 3.9(3). \square

Proposition 3.11. *For $n > 0$, over any \mathbb{K} the algebra A_n is A_∞ -quasi-equivalent to the two-cycle quiver, with $\mu^1 = 0$, μ^2 being multiplication, and the unique higher multiplications being $\mu^{2n+1}(e, f, \dots) = e$ and $\mu^{2n+1}(f, e, \dots) = f$. The algebra A_0 is formal.*

Proof. Again, as in Remark 2.6 we cannot *a priori* make the A_∞ -structure both strictly cyclic and strictly unital over \mathbb{K} , so we work directly with the higher A_∞ products. We can however choose the higher multiplications so that $\mu^1 = 0$, μ^2 is multiplication, and all the μ^n preserve the Adams degree; see [39, 4], which works over any characteristic.

Recall that Ext^1 is generated by terms of Adams grade 1, Ext^2 by elements of Adams grade $2n + 1$, and Ext^3 by elements of Adams grade $2n + 2$. Hence, when $n = 0$, all higher multiplications μ^k with $k \geq 3$ are zero, just since they preserve the Adams grading and have ≥ 3 inputs.

When $n \geq 1$ we appeal to Lemma 3.10. Indeed, now for purely Adams grading reasons the only possible non-zero multiplications are

$$\mu^{2n+1}(\xi_{12}, \xi_{21}, \dots) \quad \text{and} \quad \mu^{2n+1}(\xi_{21}, \xi_{12}, \dots).$$

Higher multiplications give rise to Massey products, and the first non-trivial higher multiplication is a Massey product, up to sign. Hence

$$\mu^{2n+1}(\xi_{12}, \xi_{21}, \dots) = \pm\zeta_{12} \quad \text{and} \quad \mu^{2n+1}(\xi_{21}, \xi_{12}, \dots) = \pm\zeta_{21}.$$

Regardless of the signs, we can change variables $\xi \mapsto -\xi$ if necessary to ensure that

$$\mu^{2n+1}(\xi_{12}, \xi_{21}, \dots) = \zeta_{12} \quad \text{and} \quad \mu^{2n+1}(\xi_{21}, \xi_{12}, \dots) = \zeta_{21}. \quad \square$$

Remark 3.12. In a toric 3-fold, two intersecting rational curves cannot be flopped together, so none of the potentials $(ef)^{n+1}$ with $n > 0$ can arise from toric geometries. The zero potential, corresponding to $uv = xy^2$, is toric, and is known as the suspended pinch point.

3.4. **The categories \mathcal{C}_n and \mathcal{Q}_n .** As in the introduction, consider the full subcategory

$$\mathcal{C}_n = \langle \mathcal{O}_{C_1}(-1), \mathcal{O}_{C_2}(-1) \rangle \subset D^b(\text{coh } Y_n).$$

Since the distinct crepant resolutions are related by flops, in characteristic zero the quasi-equivalence class of \mathcal{C}_n is independent of the choice of resolution by [6]. In the positive characteristic case here, all NCCRs are derived equivalent, regardless of characteristic, and the mutation functors still provide derived equivalences between them.

Indeed, repeating Lemma 3.1 for the other resolutions described in Remark 3.2, we explicitly verify that each crepant resolution admits a tilting bundle with direct summand \mathcal{O} . The mutation functors on the resulting categories \mathcal{C} show that all are derived equivalent. In particular, even in characteristic p , in our particular setting the categories \mathcal{C} are independent of the chosen crepant resolution.

By definition, a strong generator for \mathcal{C}_n is given by the direct sum of the sheaves $\mathcal{O}_{C_i}(-1)$, where C_i are the irreducible components of $f^{-1}(0)$. Across the equivalence with Λ_n , this corresponds to the direct sum $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$. Thus we can encode the category \mathcal{C}_n by recording the A_∞ -structure on the algebra $A_n = \text{Ext}_{\Lambda_n}^*(\mathcal{S}, \mathcal{S})$.

The following establishes Theorem 1.1.

Corollary 3.13. *With notation as above, the following hold.*

- (1) *If $\mathbb{K} = \mathbb{C}$, then there are equivalences $\mathcal{Q}_0 \simeq \mathcal{C}_0$ and $\mathcal{Q}_k \simeq \mathcal{C}_1$ for all $k \geq 1$.*
- (2) *If $p > 2$ is prime and $\text{char } \mathbb{K} = p$, then there is an equivalence $\mathcal{Q}_p \simeq \mathcal{C}_p$.*

Proof. Part (1) is now an immediate consequence of Propositions 2.8 and 3.11, and Part (2) is an immediate consequence of Corollary 2.11 and Proposition 3.11. \square

Remark 3.14. The work of Abouzaid, Auroux and Katzarkov [2] provides SYZ-type mirrors to affine conic fibrations of the form $Y_H = \{uv = H(x, y)\}$. The torically non-generic, meaning some factors are not seen in $(\mathbb{C}^*)^2$, discriminant hypersurface $H = \{xy(x + y^n) = 0\} \subset \mathbb{C}^2$ has tropicalization with moment polytope

$$\{(p, q, s) \mid s \geq p + q + \max(p, nq)\}.$$

The corresponding toric variety is $\mathbb{C}^2 \times \mathbb{C}^*$, with co-ordinates (a, b, c) , and the mirror to Y_H following [2] is given by $\check{Y}_H = (\mathbb{C}^2 \setminus \{ab = 1\}) \times \mathbb{C}^*$ with superpotential $W = a + b^n \cdot c$. Whilst the corresponding Landau-Ginzburg category may contain spherical objects, the absence of any Lagrangian sphere in \check{Y}_H makes this arguably a less compelling mirror than the double bubble. One can perturb the defining equation to be torically generic, for instance considering $H' = \{(x-1)(y-1)(x-1+(y-1)^n) = 0\} \subset \mathbb{C}^2$. This yields a rather singular mirror $\check{Y}_{H'}$, given by deleting a hypersurface from the toric variety with moment polytope

$$\{(p, q, s) \mid s \geq \max(p, 0) + \max(q, 0) + \max(p, nq, 0)\}$$

and equipping the result with a superpotential associated to toric monomials defined by the points $(-1, 0, 0)$ and $(0, -1, 0)$.

4. AUTOEQUIVALENCES

4.1. Morse-Bott-Lefschetz presentations. Consider a Morse-Bott Lefschetz fibration $X \rightarrow \mathbb{C}$ with general fibre $(\mathbb{C}^*)^2$ and with critical fibres isomorphic to $(\mathbb{C}^*) \times (\mathbb{C} \vee \mathbb{C})$. The monodromy maps are fibred Dehn twists; these act trivially on the homology of the fibre, so the one-dimensional homology classes of $(\mathbb{C}^*)^2 \simeq T^2$ can be labelled consistently in the fibration. We will usually write a, b for a choice of meridian and longitude of the torus.

Lemma 4.1. *X admits a symplectic form for which parallel transport is globally defined for paths away from the critical values.*

Proof. We begin with a general statement about Hamiltonian group actions preserving the fibres of a symplectic fibration. Suppose $\pi: X \rightarrow B$ is a symplectic fibration and X admits a Hamiltonian action of a group G preserving the fibres of π . The symplectic structure on X gives a distinguished symplectic connection and hence locally defined symplectic parallel transport maps. We claim that these locally defined maps preserve the level sets of the moment map; in particular, if the moment map is proper, parallel transport is globally defined. To justify the claim, let $\gamma: [0, 1] \rightarrow B$ be a path and let V be the horizontal lift at a point of X of $\gamma'(t)$. Let H_ξ be a component of the moment map associated to a Lie algebra element $\xi \in \mathfrak{g}$, with associated Hamiltonian vector field v_ξ , so that $\iota_{v_\xi} \omega = dH_\xi$, and let ϕ_t be the Hamiltonian flow of v_ξ . Differentiating $\pi(\phi_t(x)) = \pi(x)$ with respect to t , we get $\pi_* v_\xi = 0$, so v_ξ is tangent to the fibres of π . Then

$$\mathcal{L}_V H_\xi = \iota_V dH_\xi = \omega(v_\xi, V) = 0,$$

since v_ξ is tangent to the fibres of π whilst V is symplectically orthogonal to them. This proves that each component of the moment map is preserved by the flow of V , i.e. by the symplectic parallel transport along γ .

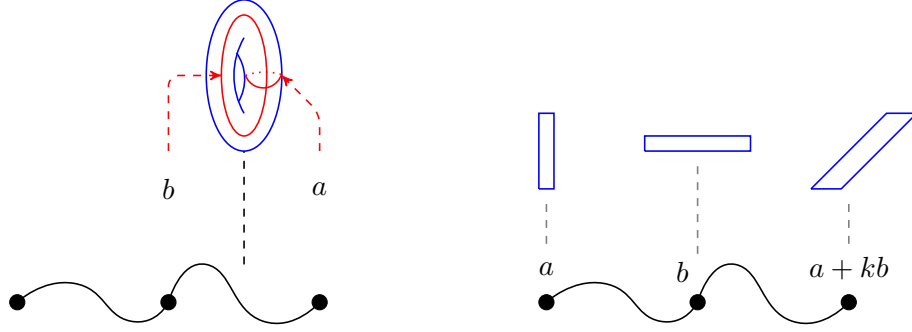
It will follow from Lemma 4.5 below that the local model of a Morse-Bott-Lefschetz fibration with one critical fibre admits a fibrewise Hamiltonian T^2 -action with proper moment map, hence has globally defined parallel transport. We may view X as a fibre sum of such local pieces by mapping the base $B = \mathbb{C}$ to a tree of discs each containing one critical point, and then patch the locally defined parallel transport maps. \square

Example 4.2. A path $\gamma \subset \mathbb{C}$ between two critical values (and meeting no critical values in its interior) defines a Lagrangian submanifold $L_\gamma \subset X$, formed of two Morse-Bott Lefschetz thimbles; by construction L_γ is obtained from gluing together two Lagrangian solid tori, hence is diffeomorphic to a Lens space or $S^1 \times S^2$. Denoting the standard meridian and longitude curves in the torus a and b , then the double bubble plumbings W_n are associated to fibrations with vanishing cycles $a, b, nb \pm a$ respectively. See Figure 1.

4.2. Affine realisations. Let $\{f(x_1, \dots, x_n) = 0\} \subset A \subset \mathbb{C}^n$ define an affine hypersurface $\{f = 0\}$ of an affine variety A . The *spinning* of f is the affine variety

$$\mathcal{S}(f) = \{(u, v, \mathbf{x}) \in \mathbb{C}^2 \times A \mid uv = f(x_1, \dots, x_n)\}$$

There is a natural projection $\mathcal{S}(f) \rightarrow A$ which is a conic bundle with the zero-locus of f as a Morse-Bott-Lefschetz discriminant. We equip both $\{f = 0\}$ and $\mathcal{S}(f)$ with the obvious Kähler structures induced from Euclidean space.


 FIGURE 1. Vanishing cycles for the Morse-Bott-Lefschetz presentation of W_k

Lemma 4.3. *A Lagrangian disc $D^n \subset \mathbb{C}^n$ with $\partial D^n \subset \{f = 0\}$ and whose interior is disjoint from that hypersurface naturally lifts to a Lagrangian $S^{n+1} \subset \mathcal{S}(f)$.*

Proof. This ‘suspension’ (or ‘spinning’) construction is standard, see [45, 2, 19]. \square

Lemma 4.4. *W_0 is an affine quartic surface.*

Proof. Begin with $T^*S^2 = \{xy + z^2 = 1\} = A \subset \mathbb{C}^3$, which has a standard Lefschetz fibration $T^*S^2 \xrightarrow{\pi_z} \mathbb{C}$ by projection to the z co-ordinate, with two critical fibres. Let $f = z|_A$. The spinning $\mathcal{S}(f)$ is then a \mathbb{C}^* -bundle over T^*S^2 with critical fibres along a smooth conic fibre $\mathbb{C}^* = \pi_z^{-1}(0) \subset T^*S^2$. The zero-section S^2 which is a matching sphere for the path $[-1, 1] \subset \mathbb{C}_z$ meets the critical fibre in a circle, and each of its two constituent discs spin to a Lagrangian 3-sphere. These two spheres meet cleanly along the unknotted circle $\mathbb{C}_z \cap S^2$. The Morse-Bott surgery of the two 3-spheres is the $S^1 \times S^2$ obtained from spinning up a perturbation of the zero-section in T^*S^2 which does not meet the critical fibre. It follows that the total space is the completion of a subdomain symplectomorphic to W_0 . The total space $\mathcal{S}(f)$ is defined by $\{xy + u^2v^2 = 1\} \subset \mathbb{C}^4$, which is an affine quartic. \square

The variety F of complete flags in \mathbb{C}^3 is a smooth hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$.

Lemma 4.5. *W_1 is an affine flag 3-fold.*

Proof. Fix $F = \{xx' + yy' + zz' = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2$ and consider the map $(\mathbb{P}^2 \times \mathbb{P}^2) \setminus \Theta \rightarrow \mathbb{P}^2$ given by

$$([x : y : z], ([x' : y' : z'])) \mapsto [xx' : yy' : zz']$$

which is defined away from the hexagon of lines $\Theta = \{xx' = yy' = zz' = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2$. We pull back a pencil of lines on \mathbb{P}^2 , and consider the pencil of $(1, 1)$ -divisors

$$D_{[\lambda:\mu]} = \{\lambda(xx' - zz') + \mu yy' = 0\} \cap F \subset F, \quad [\lambda : \mu] \in \mathbb{P}^1,$$

again with base locus Θ . Remove the smooth fibre $D_{[1:0]} \cap F$ from F , which is a del Pezzo surface of degree six representing $-K_F/2$. The projection $p = \lambda/\mu: F \setminus D_{[1:0]} \rightarrow \mathbb{C}$ is then a Morse-Bott-Lefschetz fibration with general fibre $(\mathbb{C}^*)^2$ as in Example 4.2, and with singular fibres over $\{0, \pm 1\}$. The map p is equivariant for a Hamiltonian T^2 -action

$$(\vartheta, \phi) \cdot ([x : y : z], [x' : y' : z']) = ([e^{i\vartheta}x : y : e^{i\phi}z], [e^{-i\vartheta}x' : y' : e^{-i\phi}z'])$$

on $\mathbb{P}^2 \times \mathbb{P}^2$, which has proper moment map restricted to W_1 . There are two Lagrangian 3-spheres which fibre, as Morse-Bott matching cycles, over the arcs $[-1, 0]$ and $[0, 1] \subset \mathbb{C}$. The circles which collapse at the critical fibres are those corresponding to $\vartheta = 0$ respectively $\phi = 0$ at ± 1 , and $\vartheta = \phi$ at 0. This shows that the two 3-spheres meet according to the W_1 -plumbing. \square

Remark 4.6. Jonny Evans (private communication) has shown that the union of the two matching spheres form a Lagrangian skeleton of W_1 , i.e. they are the critical locus of a global plurisubharmonic function. The weaker fact that W_1 embeds into the flag 3-fold follows from ‘semi-toric’ considerations, cf. [34, Figure 10(b)]. This exhibits a singular Lagrangian torus fibration on F in which two 3-spheres meeting cleanly along a circle are mapped to straight arcs in the base of the fibration (the ‘moment polytope’). By considering the slopes of those arcs in the affine structure in the base, one sees that the local model is that of W_1 .

Remark 4.7. We do not know if W_n is affine if $n > 1$, but one can use the description of Remark 3.14 to show that W_n embeds as a Stein subdomain of an affine variety. W_1 is a smoothing of the cone $\text{Cone}(-K_S)$ given by collapsing the zero-section in the anticanonical bundle over a del Pezzo surface S diffeomorphic to the 3-point blow-up of $\mathbb{C}P^2$. A smoothing of $\text{Cone}(-K_S^{\otimes n})$ gives an affine model for a plumbing of two Lens spaces.

Remark 4.8. \mathbb{P}^3 contains no Lagrangian sphere [43], whilst a quadric 3-fold cannot contain two Lagrangian 3-spheres meeting cleanly along a circle (an easy consequence of results in [47]). Degeneration techniques show a hypersurface $H_d \subset \mathbb{P}^4$ of degree $d \geq 4$, or a complete intersection $H_{d,d'} \subset \mathbb{P}^5$ of bidegrees $d, d' \geq 2$, contains two Lagrangian 3-spheres meeting cleanly in a circle.

4.3. A digression on knotting. The realisation of W_0 as an affine quartic yields an attractive local obstruction to knotting. Recall that a *dilation* on a Stein manifold Y is a class $b \in SH^1(Y)$ in the first symplectic cohomology with $\Delta(b) = 1$, where Δ is the BV -operator. If Y admits a dilation, it admits no exact Lagrangian $K(\pi, 1)$ by [46].

Lemma 4.9. W_0 admits a dilation.

Proof. We will construct an embedding of W_0 into the total space of a Lefschetz fibration over \mathbb{C} with fibre the A_3 -Milnor fibre. The result will then follow from [46, Proposition 7.3].

Inside the four-real-dimensional A_3 -Milnor fibre M we can find two Lagrangian matching 2-spheres which meet cleanly along a circle: if the critical points of the standard Lefschetz fibration $M \rightarrow \mathcal{C}$ lie at $\{1, 2, 3, 4\} \subset \mathcal{C}$ and the obvious real matching spheres are labelled S_1, S_2, S_3 then we can take the spheres S_1 and $S' = \tau_{S_2}^2(S_3)$. We now construct Z as the total space of a Lefschetz fibration with four singular fibres at $\{i, 2i, 3i, 4i\} \subset \mathcal{C}$, for which the associated vanishing cycles along paths parallel to $\mathbb{R}_{\geq 0}$ are respectively S_1, S_1, S', S'' , where S'' is chosen so that when the basis of vanishing paths is mutated so as to replace the fourth one with a path passing through $3i/2$ then the corresponding vanishing cycle agrees with S' . In this total space, the interval $[i, 2i] \subset i\mathbb{R}$ is a matching path, as is a path which meets that at its interior transversely once, and the corresponding spheres in the A_3 -fibre are identified with S_1 and S' . It is easy to check that the surgery is an $S^1 \times S^2$. \square

Proposition 4.10. *There is no Hamiltonian isotopy of the spheres $Q_i \subset W_0$ to a pair of spheres meeting cleanly in a circle knotted in either component.*

Proof. Suppose such an isotopy exists, yielding an embedding $W_n(\kappa_1, \kappa_2) \subset W_0$ with at least one κ_i non-trivial. Label the core spheres of the knotted clean intersection Q'_i . Since the Q'_i are isomorphic in the Fukaya category to the Q_i , necessarily $n = 0$ from (2.2). By compatibility of the unit and BV-operator with Viterbo restriction, we see that $W_0(\kappa_1, \kappa_2)$ also admits a dilation, hence contains no exact Lagrangian $K(\pi, 1)$. If κ_2 is the trivial knot, then the Bott surgery K of the isotoped spheres Q'_i is given by 0-surgery on κ_1 . If κ_1 is non-trivial, this is aspherical by Gabai's proof of Property R [17], a contradiction. So we can suppose both κ_i are non-trivial. Then the surgery admits an incompressible torus.

By a result of Ganatra-Pomerleano [19, Corollary 5.9], again building on [33], one sees that the surgery K of the Q'_i – whose cotangent bundle also admits a dilation, by the same restriction argument, and hence a quasi-dilation – is finitely covered by $S^1 \times \Sigma_g$ ($g \geq 1$) or by a connect sum of $S^1 \times S^2$'s. In the former case, K is aspherical, which is a contradiction. Thus K is finitely covered by $\#_k(S^1 \times S^2)$ for some $k \geq 0$. These are exactly the manifolds with no aspherical summand in their prime decomposition; since $H_1(K; \mathbb{Q}) \cong \mathbb{Q}$, we see that $K = (S^1 \times S^2) \# M$ for M a connect sum of spherical 3-manifolds. But this means $\pi_1(K)$ is a free product of \mathbb{Z} and a collection of finite groups, and this contains no $\mathbb{Z} \oplus \mathbb{Z}$ subgroup by Kurosh's theorem, contradicting existence of the incompressible torus. \square

We remark that there is no smooth obstruction to changing the knot type of the intersection.

After the first version of this note was circulated, Ganatra and Pomerleano constructed a quasi-dilation on W_1 in characteristic 3, which extends¹ Proposition 4.10 from W_0 to W_1 . When $p > 1$ is prime, one can rule out some knot types using the classification result Corollary 6.12, along the vein of the topological Corollaries given in the Introduction.

4.4. Monodromy. A Lagrangian sphere $L \subset X$ has an associated Dehn twist symplectomorphism τ_L . Keating [28] proved that the Dehn twists τ_{Q_1} and τ_{Q_2} generate a free group inside the mapping class group $\pi_0 \text{Symp}_{ct}(W_n(\kappa_1, \kappa_2))$. Since the intersection pairing on $H_3(W_n(\kappa_1, \kappa_2); \mathbb{Z})$ is trivial, the twists τ_{Q_i} act trivially on cohomology. The cone $K = \{Q_1 \xrightarrow{e} Q_2\}$ has surgery representative a Lagrangian Lens space, hence there is a geometric Dehn twist τ_K (cf. [41] for the Lens space case). The Lens space K represents the diagonal homology class in

$$H_3(W_n; \mathbb{Z}/2) \cong \mathbb{Z}/2\langle Q_1 \rangle \oplus \mathbb{Z}/2\langle Q_2 \rangle$$

so (even when $n = 1$ and K is a sphere) K is not quasi-isomorphic in $\mathcal{F}(W_n)$ to the image of Q_i under any element of $\langle \tau_{Q_1}, \tau_{Q_2} \rangle$, and τ_K does not belong to this group. The question arises as to which larger group naturally acts, i.e. what is the group $\langle \tau_{Q_1}, \tau_{Q_2}, \tau_K \rangle$.

Let $\text{Br}_3 = \langle a, b \mid aba = bab \rangle$ denote the braid group on 3 strings. The kernel of the natural surjection $\text{Br}_3 \rightarrow \text{Sym}_3$, which quotients out the subgroup normally generated by a^2 and b^2 , is called the *pure braid group* PBr_3 . One can also consider the preimage of a subgroup $S_2 \leq S_3$, to define the *mixed braid group* MBr_3 of braids which fix one end-point. The pure braid group PBr_3 is generated by either of the triples $\{a^2, b^2, ab^2a^{-1}\}$ or $\{a^2, b^2, (aba)^2\}$, the mixed braid group MBr_3 by $\{a^2, b^2, (aba)\}$.

Lemma 4.11. *There is a natural representation $\rho_0: \text{MBr}_3 \rightarrow \pi_0 \text{Symp}(W_0)$.*

¹The first version of this note used *ad hoc* methods to prove a weaker result, namely that the only possible non-trivial knot type that could occur in W_1 was the trefoil.

Proof. The construction of the affine quartic above has input (i) an element of $\text{Conf}_2(\mathbb{C})$, defining a degree two polynomial $p(z)$ with distinct roots and hence an affine surface $S_p = \{xy + p(z) = 0\}$ symplectically equivalent to T^*S^2 ; and a point $\lambda \in \mathbb{C} \setminus \{p^{-1}(0)\}$ defining a smooth conic fibre of the projection $S_p \rightarrow \mathbb{C}_z$ about which one can spin to produce the 3-fold. One therefore obtains a family of affine quartics over a bundle over $\text{Conf}_2(\mathbb{C}) \simeq S^1$ with fibre a twice-punctured plane. This parameter space has fundamental group MBr_3 . The representation is then obtained from symplectic parallel transport, well-defined since one can work relative to a family of compactification divisors with fixed topology.

The generator a^2 of MBr_3 is represented by the monodromy of a loop of affine varieties in which one of the Lefschetz critical points of $T^*S^2 \rightarrow \mathbb{C}$ rotates around the fixed fibre in which we spin to construct the threefold. One can therefore view this as the monodromy of a family of varieties over a disc, in which the central fibre is singular, obtained by spinning $\{uv = \pi\}$ along a *singular* conic fibre, so where $\pi^{-1}(0)$ is a critical fibre of the Lefschetz fibration $\pi: T^*S^2 \rightarrow \mathbb{C}$. The family of threefolds then acquires a nodal singularity $\{uv = x^2 + y^2\}$, so the monodromy is isotopic to the Dehn twist in the corresponding vanishing cycle. It follows that ρ_0 takes $\{a^2, b^2\}$ to the Dehn twists $\{\tau_{Q_1}, \tau_{Q_2}\}$.

The generator (aba) of MBr_3 is a lift, under the map $\text{MBr}_3 \rightarrow \text{Br}_2$ which forgets the fixed strand, of the generating half-twist exchanging the other two strands. Since the fixed strand corresponds to the location of the fibre over which we spin, this means that it is a lift to W_0 of the Dehn twist in the zero-section of T^*S^2 , under the natural map $W_0 \rightarrow T^*S^2$. \square

Remark 4.12. If $L = S^1 \times S^2 \subset X$ is a Lagrangian $S^1 \times S^2$, the non-compactly-supported symplectomorphism $\text{id} \times \tau_{S^2} \in \text{Symp}(T^*S^1 \times T^*S^2)$ need not extend to a global symplectomorphism of X ; our situation is special since W_0 admits a global map to T^*S^2 . This phenomenon is familiar on the B-side of the mirror, where there are objects \mathcal{E} such that

$$\text{Hom}^*(\mathcal{E}, \mathcal{E}) = \begin{cases} k[x] & * \in \{0, 2\} \\ 0 & \text{else.} \end{cases}$$

The existence of an associated autoequivalence is again delicate, because the object \mathcal{E} defining the autoequivalence admits an *infinite* filtration by copies of the underlying $(S^1 \times S^2)$ -type-object E , hence need not belong to the ‘compact’ category in question.

Consider a Morse-Bott-Lefschetz fibration $T^*S^3 \cong E_1$ as in Example 4.2, with singular fibres at $\{0, 1\}$ with vanishing cycles the usual meridional and longitude curves on the torus (call these a and b say). One can construct a family of Morse-Bott-Lefschetz fibrations E_λ over \mathbb{C}^* with singular fibres at 0 and λ , and there is a monodromy symplectomorphism $\phi_1: E_1 \rightarrow E_1$ associated to parallel transport around the unit circle.

Lemma 4.13. *The symplectomorphism $\phi_1: E_1 \rightarrow E_1$ is the Dehn twist.*

Proof. We have a $(\mathbb{C}^*)^2$ -Morse-Bott-Lefschetz fibration with two critical fibres, and one matching 3-sphere. Such can be obtained as the fibre product of two (\mathbb{C}^*) -Lefschetz fibrations $\pi_i: X_i \rightarrow \mathbb{C}$ whose critical values do not agree. The half-twist exchanging the critical values is realised by monodromy around a family in which the singular fibre is the self-fibre product

$$\{(u, v, x, y) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid \pi(u, v) = \pi(x, y)\}$$

of the standard Lefschetz singularity $\pi: \mathbb{C}^2 \rightarrow \mathbb{C}$, $\pi(u, v) = u^2 + v^2$. This is again a threefold ordinary double point. \square

This local model admits fibrewise \mathbb{Z}/k -actions which give a corresponding description for the cotangent bundle of a Lens space, and a similar monodromy viewpoint on the Lens space Dehn twist. Viewing $T^*S^3 = \{z_0z_1 + z_2z_3 = 1\} \subset \mathbb{C}^4$, the Morse-Bott-Lefschetz fibration is given by projection to z_0z_1 , and there is a fibrewise Hamiltonian action of T^2 generated by circle actions with weights $(1, -1, 0, 0)$ and $(0, 0, 1, -1)$. The \mathbb{Z}/k diagonal subgroup acts freely and has quotient $T^*(L(k, 1))$. Viewing the original T^2 as generated by the first and diagonal circle actions (so the vanishing cycles degenerating at the two Morse-Bott critical points are a and $b - a$), there is a moment map

$$(z_0, \dots, z_3) \mapsto (|z_0|^2 - |z_1|^2, |z_0|^2 - |z_1|^2 + |z_2|^2 - |z_3|^2)$$

The T^2 action descends to the quotient with moment map

$$(z_0, \dots, z_3) \mapsto (|z_0|^2 - |z_1|^2, (|z_0|^2 - |z_1|^2 + |z_2|^2 - |z_3|^2)/k)$$

and with vanishing cycles $a, kb - a$ (cf. Example 4.2). Let E_k denote this quotient fibration; there is again a symplectomorphism

$$\phi_k: E_k \rightarrow E_k$$

from parallel transport around the unit circle for the family of fibrations with critical fibres at 0 and $\lambda \in \mathbb{C}^*$.

Lemma 4.14. *The full twist monodromy $\phi_k: E_k \rightarrow E_k$ is Hamiltonian isotopic to the Lens space Dehn twist.*

Proof. The \mathbb{Z}/k -action discussed above exists in the total space of the family considered in Lemma 4.13. By construction, there is therefore a commuting diagram

$$\begin{array}{ccc} T^*S^3 & \xrightarrow{\phi_1} & T^*S^3 \\ \downarrow & & \downarrow \\ T^*L(k, 1) & \xrightarrow{\phi_k} & T^*L(k, 1) \end{array}$$

with vertical maps the \mathbb{Z}/k quotients introduced previously. Since ϕ_1 is the spherical twist, ϕ_k is the Lens space twist up to composition with a deck transformation; but both monodromies are compactly supported. \square

Lemma 4.15. *For $k \geq 1$, there is a natural representation $\rho_k: \text{PBr}_3 \rightarrow \pi_0 \text{Symp}(W_k)$, whose image is generated by (spherical and Lens space) Dehn twists in the core components and their surgery.*

Proof. As in Lemma 4.11, we obtain the representation ρ_k from parallel transport, now via the geometry of families of Morse-Bott-Lefschetz fibrations as in the set-up for Lemma 4.13 but for fibrations with three critical fibres. This immediately shows, when $k = 1$, for the representation ρ_1 the generating half-twists a^2, b^2, ab^2a^{-1} correspond to Dehn twists in Q_1, Q_2, K , viewed as monodromies of global degenerations. For higher k , an appeal to Lemma 4.14 shows that two of the pure braid group generators which perform a full twist in a pair of generators are taken to spherical twists in the cores, whilst the ‘third’ pure braid group generator acts as the Lens space twist. \square

Remark 4.16. For $k \geq 1$, each element of the image of ρ_k is isotopic to a compactly supported symplectomorphism (this is not true when $k = 0$). When $k = 1$, the monodromy can also be understood via transport in families of affine flag 3-folds as in Lemma 4.5. From that viewpoint, it is clear that the whole representation ρ_1 lifts to $\pi_0 \text{Symp}_{ct}(W_1)$.

Remark 4.17. Lemma 4.5 shows that the contact boundary of W_1 admits a free contact circle action, since it is the circle bundle over a smooth del Pezzo surface. If (Y, ξ) is a contact manifold carrying a contact circle action, there is an associated compactly supported symplectomorphism of the symplectization (by suspending the action), which here gives rise to a symplectomorphism $\tau_\partial \in \pi_0 \text{Symp}_{ct}(W_1)$ which is a “boundary twist”. The monodromy ρ_1 takes $(aba)^2$, which generates the centre of PBr_3 , to the boundary twist τ_∂ . This follows by considering a Lefschetz pencil of $(1, 1)$ -divisors in $\mathbb{P}^2 \times \mathbb{P}^2$ over \mathbb{P}^1 , with 3 critical fibres; the global monodromy rotates the normal bundle of the base locus.

4.5. A-side Twists. The Dehn twist in an exact Lagrangian sphere $L \cong S^3 \subset X$ acts by a twist functor on the Fukaya category:

$$(4.1) \quad \tau_L \simeq \text{Cone}(\text{hom}_{\mathcal{F}(X)}(L, \bullet) \otimes L \xrightarrow{ev} \bullet)$$

where (4.1) denotes an isomorphism in the category $nu\text{-fun}(\mathcal{F}(X), \mathcal{F}(X))$ of non-unital A_∞ -functors of $\mathcal{F}(X)$, see [44]. The action of a Lens space Dehn twist on the Fukaya category was analysed by Mak and Wu [41]. They prove that if $L \cong L(p, 1) \subset X$ is an exact Lagrangian Lens space and one works over a field \mathbb{K} of characteristic p , then for any other exact Lagrangian $K \subset X$ there is a quasi-isomorphism

$$(4.2) \quad \tau_L(K) \simeq \text{Cone}(\text{hom}_{\mathcal{F}(X)}(\mathcal{L}, K) \otimes_{\mathbb{K}[\pi_1(L)]} \mathcal{L} \xrightarrow{ev} K)$$

where \mathcal{L} denotes the universal local system over L and ev is now a suitable equivariant evaluation map, see [41] for details. We briefly recall that, fixing an isomorphism $\pi_1(L) = \mathbb{Z}/p$, there is an isomorphism of $\mathbb{K}[\pi_1(L)]$ -modules

$$\text{hom}_{\mathcal{F}}^*(\mathcal{L}, \mathcal{L}) = (C^*(\widehat{L}) \otimes \text{Hom}_{\mathbb{K}}(\mathbb{K}[\mathbb{Z}/p], \mathbb{K}[\mathbb{Z}/p]))^{\mathbb{Z}/p}$$

where $S^3 = \widehat{L} \rightarrow L$ is the universal cover, and we also note that [41, Lemma 2.13] gives an isomorphism of \mathbb{K} -algebras

$$(4.3) \quad HF^*(\mathcal{L}, \mathcal{L}) = H^*(S^3) \otimes_{\mathbb{K}} \mathbb{K}[\pi_1(L)].$$

Remark 4.18. It is presumably true, but not yet known, that (4.2) again arises from an exact triangle of functors.

The A_∞ -endomorphism algebra of an object E in a 3-CY category gives rise to a noncommutative deformation functor, and hence a universal object \mathcal{E} . We say that E is fat-spherical if $\text{Hom}^i(\mathcal{E}, E)$ is the cohomology of a sphere [48]. The dimension of this sphere can vary; in Remark 4.12 the fat-spherical object is the 2-sphere, whilst below for flopping contractions only the 3-sphere will turn out to be possible.

Below it will be a feature of our categories \mathcal{C}_k that the universal objects \mathcal{E} of our fat spherical objects will always turn out to satisfy

$$(4.4) \quad \text{Hom}^i(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{K}[x]/(x^n) & i = 0, 3 \\ 0 & \text{else,} \end{cases}$$

and indeed such an \mathcal{E} will be mirror to the universal local system \mathcal{L} of (4.3) (the universal object \mathcal{E} respectively \mathcal{L} arises via the same iterated mapping cone on E respectively L). The usual association of autoequivalences to spherical objects carries over to the fat-spherical case: any fat-spherical (or more precisely, its corresponding universal object) determines an autoequivalence, by e.g. [48].

If $\text{char } \mathbb{K} = 0$, then the Lens space $L(p, 1)$ supports p distinct rank one local systems associated to the unitary characters of $\pi_1(L)$; each of these determines a spherical object $S_i \in \mathcal{F}(X; \mathbb{K})$. These spherical objects are pairwise orthogonal, and so their associated twists commute. [41] implies that

- when $\text{char}(\mathbb{K}) = p$, the Lens space twist acts on objects of the Fukaya category by the fat spherical twist associated to the object \mathcal{L} ;
- when $\text{char}(\mathbb{K}) = 0$, the Lens space twist acts on objects of $\mathcal{F}(X)$ as the product of spherical twists $T_{S_1} \dots T_{S_p}$.

4.6. Generation. If $\mathcal{D} \subset \mathcal{F}(X)$ is a full subcategory, we say \mathcal{D} split-generates if every object of $D^\pi \mathcal{F}(X)$ is quasi-equivalent to an object of $D^\pi \mathcal{D}$. Let \mathcal{Q}_n denote the full subcategory of $D^\pi \mathcal{F}(W_n)$ split-generated by the two core spheres Q_1 and Q_2 .

Lemma 4.19. *Let $n = 0$. The core spheres $\{Q_i\}$ do not split-generate $\mathcal{F}(W_n)$.*

Proof. Consider the Morse-Bott surgery $K = \{Q_1 \xrightarrow{k} Q_2\}$, diffeomorphic to $S^1 \times S^2$. Let $\xi \rightarrow K$ be a rank one local system with non-trivial monodromy; this defines a Lagrangian brane $\xi \rightarrow K$ which is an object of $\mathcal{F}(W_0)$. The endomorphisms of this object are given by

$$HF^*((\xi \rightarrow K), (\xi \rightarrow K)) \cong H^*(K, \text{End}(\xi)) \cong H^*(K)$$

so $\xi \rightarrow K$ is a non-zero object. From the description of W_n in terms of a $(\mathbb{C}^*)^2$ -Morse-Bott-Lefschetz fibration in Example 4.2, one sees that one can construct the surgery K so that there are clean intersections $K \cap Q_i \cong S^1$ for each i , where moreover the S^1 is a homology generator for $H_1(K; \mathbb{Z})$. The Floer cohomology

$$HF^*(Q_i, (\xi \rightarrow K)) \cong H^*(S^1, \xi)$$

which vanishes for non-trivial ξ . We have therefore produced a non-trivial object of $\mathcal{F}(W_0)$ which is orthogonal to both the Q_i . \square

When $n > 1$, the same proof goes through over \mathbb{C} : the cone K is a Lens space which is split-generated by the Q_i when equipped with the trivial local system, but not when equipped with a local system associated to a non-trivial character of \mathbb{Z}/n .

Lemma 4.20. *Let $n = 1$. The core spheres $\{Q_i\}$ split-generate $\mathcal{F}(W_1)$.*

Proof. This follows from Remark 4.17 and [44, Corollary 5.8]; a “boundary twist”, which acts on $\mathcal{F}(W_1)$ by a non-trivial shift, can be expressed in terms of Dehn twists in spheres generated by \mathcal{Q}_1 (compare to Remark 4.17). \square

We now give a version of Lemma 4.20 which does not rely on having an affine model for W_k when $k \neq \{0, 1\}$.

Let $\mathcal{W}(W_k; \mathbb{K})$ denote the wrapped Fukaya category of the Stein manifold W_k , defined over the field \mathbb{K} . The wrapped category is invariant under not necessarily compactly supported Hamiltonian isotopies, so there is a representation

$$(4.5) \quad \pi_0 \text{Symp}(W_k) \longrightarrow \text{Auteq}(\mathcal{W}(W_k; \mathbb{K})) / \langle [2] \rangle$$

into the quotient of the autoequivalence group by twice the shift. This lifts to a representation of the central \mathbb{Z} -extension $\pi_0 \text{Symp}_{gr}(W_k)$ of isotopy classes of graded symplectomorphisms into $\text{Auteq}(\mathcal{W}(W_k; \mathbb{K}))$. The following result should be compared to Remark 4.17.

Lemma 4.21. *Suppose $k = 1$ and $\text{char } \mathbb{K} = 0$ or $k > 2$ is prime and \mathbb{K} has characteristic k . The centre of the pure braid group PBr_3 acts on $\mathcal{W}(W_k; \mathbb{K})$ by a non-trivial shift.*

Proof. The centre ρ of the braid group is given by the product of the twists in the two core components and a (fat) spherical twist in the surgery. Working with not necessarily compactly supported representatives coming from parallel transport, this is represented by the ‘boundary twist’ in $\text{PBr}_3 \simeq \pi_0 \text{Diff}(D^2; \{p, q, r\})$. Thus ρ is geometrically a lift to the total space of the Morse-Bott-Lefschetz fibration W_k of the Dehn twist near infinity in the base; undoing that Dehn twist by an isotopy which rotates infinity clockwise shows that ρ is trivial in $\pi_0 \text{Symp}(W_k)$. Then (4.5) shows that it acts on the wrapped category by a shift. The fact that the shift is non-trivial can then be checked on any object, and ρ shifts the core spheres by $[2]$. \square

Corollary 4.22. *Let $k > 2$ be prime and suppose $\text{char } \mathbb{K} = k$. Then the core spheres Q_i split-generate the compact category $\mathcal{F}(W_k; \mathbb{K})$.*

Proof. Whenever a composition of (fat) spherical twists acts by a shift, the corresponding objects split-generate, compare to [44]. This means the core spheres together with the fat-spherical object (whose universal object is) \mathcal{L} underlying the Lens space twist split-generate. In characteristic k , the Lens space $L = L(k, 1)$ has a non-trivial class $\eta \in HF^1(L, L) = \mathbb{K}$ and hence a graded self-extension $L \xrightarrow{\eta} L$. When $k = 2$, this represents the universal local system \mathcal{L} associated to L . More generally, the universal local system \mathcal{L} over L has endomorphisms as in (4.4); it determines the underlying fat-spherical L of the associated autoequivalence; and the general theory of fat-sphericals implies that \mathcal{L} is filtered by copies of L . Therefore the core spheres together with L split-generate. But the Lens space itself is manifestly generated by the cores. \square

Remark 4.23. When $k > 2$ and $\text{char } \mathbb{K} = 0$ the same argument and the description of the Lens space twist at the end of Section 4.5 shows that $\mathcal{F}(W_k)$ is split-generated by the cores Q_i and the collection of branes given by equipping the Lens space with a non-trivial rank one local system.

Remark 4.24. If $k > 1$, then the Lens space Dehn twist $\tau_L \in \pi_0 \text{Symp}_{ct}(W_k)$ acts on $\mathcal{F}(W_k; \mathbb{C})$ by a product of spherical twists. However, the individual twist equivalences $T_{S_i} \in \text{Auteq}(\mathcal{F}(W_k; \mathbb{C}))$ have no obvious geometric origin, i.e. they are not visibly in the image of $\pi_0 \text{Symp}_{ct}(W_k)$.

Remark 4.25. Proposition 2.8 gives an equivalence over \mathbb{C}

$$\mathcal{Q}_1 \simeq \mathcal{Q}_k \quad k \geq 2$$

However, this is *not* PBr_3 -equivariant for the parallel transport actions constructed above. Indeed, the generators a^2, b^2, ab^2a^{-1} of PBr_3 act by spherical twists on \mathcal{Q}_1 , but – as autoequivalences of the compact category $\mathrm{Tw} \mathcal{F}(W_k)$ – two act by spherical twists and one by a product of k spherical twists in orthogonal objects for $k > 1$. When $k > 1$, the representation

$$\mathrm{PBr}_3 \rightarrow \mathrm{Auteq}(\mathrm{Tw} \mathcal{F}(W_k))$$

is not especially natural from the viewpoint of the subcategory $\mathrm{Tw} \mathcal{Q}_k \subset \mathrm{Tw} \mathcal{F}(W_k)$, since that subcategory does not contain the objects given by the Lens space with non-trivial local systems. The equivalence $\mathcal{Q}_1 \simeq \mathcal{Q}_k$ induces a different PBr_3 action on \mathcal{Q}_k , in which all three ‘standard’ generators are spherical twists: the Lens space twist is replaced just by T_{S_1} where S_1 is the spherical object corresponding to the trivial local system.

4.7. B-side twists. We now revert back to the specific crepant resolution $Y_k \rightarrow \mathrm{Spec} R_k$ constructed in Lemma 3.1, for any \mathbb{K} , and any $k \geq 0$.

For any \mathbb{K} and any $k \geq 0$, the threefold Y_k has spherical objects $\mathcal{S}_i = \mathcal{O}_{C_i}(-1)$ for $i = 1, 2$ corresponding to the two $(-1, -1)$ -curves in Lemma 3.1. Both give rise to spherical twists. As on the A-side, the third generator depends on whether $k = 0$ or not.

- When $k = 0$, and $\mathrm{char} \mathbb{K} = 0$, we can consider the twist autoequivalence over the affine line $\mathbb{K}[x]$ considered in [13, Example 6.4], which we denote τ_{mix} .
- When $k > 0$, regardless of the characteristic, since the mutation functors give derived equivalences between the NCCRs it can be seen directly that the mapping cone $\mathcal{S}_1 \rightarrow \mathcal{S}_2$ on the non-trivial morphism is fat-spherical. We write τ_{fat} for its associated twist, which is spherical over $\mathbb{K}[x]/x^k$. Hence this third generator is a ‘geniune’ spherical object if and only if $k = 1$.

Paralleling the monodromy actions by symplectomorphisms gives the following.

Lemma 4.26. *With notation as above:*

- (1) *The category \mathcal{C}_0 admits an action of the mixed braid group MBr_3 .*
- (2) *For $k > 0$, the category \mathcal{C}_k admits an action of the pure braid group PBr_3 .*

In both cases, one may take the spherical twists in the objects $\mathcal{O}_{C_i}(-1)$ to correspond to the elements a^2 and b^2 . When $k = 0$ the third generator is τ_{mix} which corresponds to aba , and when $k > 0$ the third generator is τ_{fat} which corresponds to $(aba)^2$.

Proof. Part (1) is a special case of results in [11]; see also [13, Example 6.4]. The generators of the mixed braid group action are the spherical twists in the two $(-1, -1)$ -curves in the resolution, and the family twist τ_{mix} . Since the resolution curves move in a surface, the local geometry is modelled on the product of the resolution of a surface singularity with $\mathrm{Spec} \mathbb{C}[x]$, and the family twist is essentially the pullback of the twist from one factor. In terms of the usual braid group generators, these correspond to $\{a^2, b^2, aba\}$.

When $k > 0$ and $\mathrm{char} \mathbb{K} = 0$ this is contained in [14]; the pure braid group generators can be taken either to be given by the spherical twists in the two resolution curves and the spherical twist in their cone, or the spherical twists in the resolution curves and the autoequivalence associated to their union. These correspond to the braid group elements $\{a^2, b^2, ab^2a^{-1}\}$ and $\{a^2, b^2, (aba)^2\}$ respectively. When $\mathrm{char} \mathbb{K} = p$ these results still hold, since they only rely on the mutation functors and the tilting order, which still exist. \square

Proposition 4.27. *The following statements hold.*

- (1) *If $\mathbb{K} = \mathbb{C}$, then the equivalence of categories $\mathcal{Q}_1 \simeq \mathcal{C}_1$ of Theorem 1.1 entwines the pure braid group actions, whilst the equivalence $\mathcal{Q}_0 \simeq \mathcal{C}_0$ entwines the mixed braid group actions on the level of objects.*
- (2) *If $\text{char } \mathbb{K} = p > 2$ then the equivalence $\mathcal{Q}_p \simeq \mathcal{C}_p$ of Theorem 1.1 entwines the action of the pure braid group actions, on the level of objects.*

Proof. In Lemma 4.15 and 4.11 we showed that the representations $\text{MBr}_3 \rightarrow \pi_0 \text{Symp}(W_0)$ respectively $\text{PBr}_3 \rightarrow \pi_0 \text{Symp}(W_k)$ take the generators a^2, b^2 to the Dehn twists in the core spheres τ_{Q_1}, τ_{Q_2} . The equivalence of Theorem 1.1 take the sphericals Q_i to the simple modules corresponding to the two vertices of the cyclic quiver. In turn, these go to the sheaves $\mathcal{O}_{C_i}(-1)$, hence take the twists τ_{Q_i} to the spherical twists in these sheaves.

For W_1 , when $\mathbb{K} = \mathbb{C}$ the third generator is again a spherical twist on both sides (now in the cone), and the result follows. When $\text{char } \mathbb{K} = p$, the third generator on both sides is now fat spherical. However, the results of [41] are sufficient only to infer that the Lens space twist τ_L and the appropriate mapping cone agree object-wise. Hence, we can only conclude that the actions entwine on the level of objects.

For W_0 , the generator aba of MBr_3 is the lift of the Dehn twist in T^*S^2 for W_0 and is the ‘family spherical twist’ for \mathcal{C}_0 . In both cases, this is a two-dimensional twist which exchanges the two simples and shifts each of them by $[-1]$, which is sufficient to determine the action on objects. \square

Remark 4.28. One could probably upgrade the statement for $\mathcal{C}_0, \mathcal{Q}_0$ to a full entwining of actions by proving mirror symmetry for slightly larger categories which incorporates the non-compact object defining the appropriate twist, cf. Remark 4.12.

4.8. Faithfulness. Faithfulness of the pure braid group action for flopping contractions was first established in [21], for the case $\mathbb{K} = \mathbb{C}$. When $\text{char } \mathbb{K} = p$ the faithfulness still holds, since all techniques used only rely on properties of noncommutative resolutions, mutation functors, torsion pairs and the tilting order, which are characteristic independent.

Recall that the group $\text{Symp}(W_p)$ has a central \mathbb{Z} -extension of ‘graded’ symplectomorphisms, and there is a corresponding graded symplectic mapping class group $\pi_0 \text{Symp}_{gr}(W_p)$ which acts by autoequivalences on the compact Fukaya category (the usual mapping class group acts by autoequivalences up to shift).

Corollary 4.29. *If $p = 1$ or $p > 2$ is prime, then the natural representation $\rho_p: \text{PBr}_3 \rightarrow \pi_0 \text{Symp}_{gr}(W_p)$ is faithful.*

Proof. Appealing to Corollary 3.13, $\mathcal{Q}_1 \simeq \mathcal{C}_1$ in characteristic zero, or $\mathcal{Q}_p \simeq \mathcal{C}_p$ in characteristic $p > 2$. Either way, by [21] and the above paragraph, the natural homomorphism

$$\text{PBr}_3 \rightarrow \text{Auteq } \mathcal{C}_k$$

from Lemma 4.26 is injective. Even although in Proposition 4.27 we don’t know functorially that the pure braid actions entwine under the equivalence $\mathcal{Q}_p \simeq \mathcal{C}_p$, it is still true that the natural homomorphism

$$(4.6) \quad \text{PBr}_3 \rightarrow \text{Auteq } \mathcal{Q}_p$$

is injective. This is since injectivity in [21] is determined by Deligne normal form and dimensions of Ext groups, and this information is determined entirely by the action of the functors on objects. Now by Lemma 4.15 the injective homomorphism (4.6) factors as

$$\mathrm{PBr}_3 \rightarrow \pi_0 \mathrm{Symp}_{gr}(W_k) \rightarrow \mathrm{Auteq} \mathcal{Q}_p,$$

and so it follows that the first map must also be injective. \square

Using Remark 4.16, one can improve this when $k = 1$ to a faithful representation $\mathrm{PBr}_3 \rightarrow \pi_0 \mathrm{Symp}_{ct}(W_1)$.

5. THE WRAPPED FUKAYA CATEGORY AND KOSZUL DUALITY

In this section we lift the equivalence $\mathcal{Q}_k \simeq \mathcal{C}_k$, over the appropriate coefficient field, to yield a new description and comparison of the wrapped category of W_k and the relative singularity category of Y_k . This involves A_∞ -Koszul duality, which we now briefly recall, following [39], [27] and [15].

For an A_∞ -category \mathcal{A} , recall that we have categories of proper and perfect modules

$$\mathcal{A}^{per} \subset \mathcal{A}^{mod} \supset \mathcal{A}^{prop}.$$

We say \mathcal{A} is homologically smooth if the diagonal bimodule is perfect; in this case, $\mathcal{A}^{prop} \subset \mathcal{A}^{per}$. We also have the subcategory \mathcal{A}^c of compact objects, i.e. ones for which taking morphisms from the object commutes with arbitrary direct sums. If \mathcal{A} is compactly generated, in the sense that it admits a weak generator which is a sum of compact objects, then a compact object $V \in \mathcal{A}$ split-generates \mathcal{A}^c if and only if it is a weak generator of \mathcal{A} (i.e. no non-zero object of \mathcal{A} is orthogonal to $V[j]$ for all j).

5.1. A-side Koszul duality. Consider a Stein manifold X containing distinguished compact Lagrangians L_i and dual non-compact Lagrangian discs T_i ; we will always consider a plumbing with the L_i being cores and the T_i co-cores (cotangent fibres). Note that, in this setting, the $\{T_i\}$ are known to generate the wrapped Fukaya category $\mathcal{W}(X)$ by [9]. We have compact and wrapped endomorphism algebras $\mathcal{Q} = \bigoplus_{i,j} CF^*(L_i, L_j)$ and $\mathcal{W} = \bigoplus_{i,j} CW^*(T_i, T_j)$, which are A_∞ -algebras naturally augmented over the semisimple ring $\mathbf{k} = \bigoplus_i \mathbb{K}$ generated by the identity idempotents of the objects. It is known that the wrapped Fukaya category of any Stein manifold is homologically smooth, by deep results of Chantraine, Dimitroglou Rizell, Ghiggini and Golovko [9].

We impose the following *hypotheses*: $H^*(\mathcal{Q})$ is concentrated in non-negative degrees, $H^0(\mathcal{Q}) = \mathbf{k}$, and $H^*(\mathcal{W})$ is concentrated in non-positive degrees. The first two hold for the double bubble plumbings by Theorem 1.1, whilst the last holds for the wrapped category of a double bubble by [3].

Given that, Ekholm and Lekili prove that there is an equivalence

$$(5.1) \quad \mathcal{Q} \simeq \mathrm{RHom}_{\mathcal{W}}(\mathbf{k}, \mathbf{k})$$

and there is a natural map to a certain completion

$$(5.2) \quad \mathcal{W} \rightarrow \widehat{\mathcal{W}} \simeq \mathrm{RHom}_{\mathcal{Q}}(\mathbf{k}, \mathbf{k}).$$

When no completion is necessary, we say the compact and wrapped categories are Koszul dual; in general, the compact category is dual to $\widehat{\mathcal{W}}$.

Returning to double bubble plumbings, by Corollary 4.22 the category \mathcal{Q}_k is generated by the cores, which furthermore split-generates if we work in the right characteristic. The dual collection of cotangent fibres gives a generating subcategory $\mathcal{W}_k \subset \mathcal{W}(W_k)$. The following argument follows [37, Proposition 6.5].

Proposition 5.1. *Let $k = 1$ and $\mathbb{K} = \mathbb{C}$ or $k > 2$ be prime and $\text{char } \mathbb{K} = k$. Then \mathcal{Q}_k is Koszul dual to \mathcal{W}_k .*

Proof. The key geometric ingredient is Lemma 4.21, which implies following an argument of Lekili and Ueda [36, Proposition 7.7] that the diagonal bimodule over the wrapped category $\Delta_{\mathcal{W}_k}$ can be expressed as a colimit of bimodules which are images of Yoneda modules $Z \otimes Z'$ of objects in $\mathcal{Q}_k^{op} \otimes \mathcal{Q}_k$. The usual Beilinson-type ‘resolution of the diagonal’ argument then yields a fully faithful embedding

$$(5.3) \quad \mathcal{W} \hookrightarrow \mathcal{F}^{mod}.$$

We obtain functors

$$(5.4) \quad \mathcal{W}^{per} \hookrightarrow \mathcal{F}^{prop} \simeq (\widehat{\mathcal{W}})^{per}$$

where the first arrow comes from (5.3), noting that wrapped Floer cohomology is finite-dimensional when either Lagrangian is compact, and the second equivalence is induced by the Koszul duality functor $L \mapsto \text{RHom}_{\mathcal{Q}}(L, \mathbf{k})$.

The subcategory of compact objects $\mathcal{W}^c \subset \mathcal{W}^{per}$ is exactly \mathcal{W}^{prop} . The Yoneda modules over the core components $\mathcal{Y}_{Q_1} \oplus \mathcal{Y}_{Q_2}$ are objects of \mathcal{W}^{prop} which give a set of weak generators of \mathcal{W}^{per} , via resolution of the diagonal. Under the first inclusion of (5.4), these become compact weak generators for \mathcal{F}^{prop} ; indeed, this is a compactly generated category (compactly generated by \mathcal{F}^{per}), and $\mathcal{Y}_{Q_1} \oplus \mathcal{Y}_{Q_2}$ is a split-generator for \mathcal{F}^{per} , so weak generation of \mathcal{F}^{prop} follows from the remarks at the beginning of this section. By [40, Theorem 2.2], it follows that there is an embedding

$$\mathcal{F}^{prop} \rightarrow \mathcal{W}^{per}$$

and so the first arrow of (5.4) is also an equivalence. Hence, both arrows in (5.4) are equivalences, so $\mathcal{W}^{per} \simeq \widehat{\mathcal{W}}^{per}$, which establishes the required duality. \square

Remark 5.2. A sufficient criterion for \mathcal{Q} and \mathcal{W} to be Koszul dual (no completion required) is that $H^*(\mathcal{W})$ is concentrated in non-positive degrees and finite-dimensional in each degree; in fact, an argument due to Kalck-Yang [27] show that it is even sufficient to have $H^0(\mathcal{W})$ be finite-dimensional. The double-bubble plumbing W_k can be presented as the result of Morse-Bott surgery on T^*S^3 , attaching the cotangent bundle of a solid torus to the Legendrian conormal torus of an unknot $S^1 \subset S^3$. This gives a description of the wrapped Floer complex of the cocores in terms of binormal chords between an unknot and a distinct point (corresponding to the handle that is attached to yield T^*S^3 from $T^*\mathbb{R}^3$). Assuming an extension of the ‘surgery’ relation between Lagrangian and Legendrian invariants from [5, 15] from the case of handle attachment to Morse-Bott handle attachment, one can presumably use such a presentation to establish finite-dimensionality of $H^0(\mathcal{W})$ directly. This would yield the Koszul duality statement without requiring the detour via Lemma 4.21. In contrast, both CW^0 and HW^0 are infinite-dimensional for the plumbing W_0 .

5.2. B-side Koszul duality. For any $k \geq 1$ consider again the flopping contraction $Y_k \rightarrow \text{Spec}(R_k)$ over a field \mathbb{K} . In this subsection we will show that \mathcal{C}_k is Koszul dual to the relative singularity $D^b(\text{coh } Y_k)/\langle \mathcal{O}_{Y_k} \rangle$, following [27]. An advantage of the B-side is that this Koszul duality works over all $k \geq 1$ and all \mathbb{K} .

Since we are working algebraically (instead of complete locally as in [27]), we must first resolve the slightly subtle issue of whether our categories are idempotent complete. This is taken care of in the following two results.

Lemma 5.3. *Suppose that R is a 3-sCY excellent normal domain with a unique singular point, which admits an NCCR $\text{End}_R(M)$ where M decomposes into rank one R -modules. Then $\underline{\text{CM}} R \rightarrow \underline{\text{CM}} \widehat{R}$ is an equivalence, and in particular, $\underline{\text{CM}} R$ is idempotent complete.*

Proof. Since the assumptions imply that R is an isolated G-ring, it is known generally [29, A.1] that the completion functor $F: \underline{\text{CM}} R \rightarrow \underline{\text{CM}} \widehat{R}$ is an equivalence up to summands, so we just need to prove that F is essentially surjective. Consider an arbitrary $X \in \underline{\text{CM}} \widehat{R}$. Since NCCRs pass to the completion FM is a cluster tilting object [23, 5.5], so by [23, 4.12] there exists a triangle

$$Y_0 \xrightarrow{h} Y_1 \rightarrow X \rightarrow$$

with $Y_i \in \text{add } FM$. By assumption M decomposes into rank one modules $M \cong \bigoplus L_i$ say, so $FM \cong \bigoplus FL_i$. Since each FL_i has rank one, each is still indecomposable. Hence by Krull-Schmidt both Y_i are direct sums of the form $\bigoplus (FL_i)^{\oplus a_i}$, and thus are in the image of F . Say $Y_i = FM_i$. We already know that F is fully faithful, so $h = Fg$ for some $g: M_0 \rightarrow M_1$. Set $Y = \text{Cone } g$, then it follows $FY \cong X$, since both are the cone on $h = Fg$. \square

Corollary 5.4. *For all $k \geq 1$, and any \mathbb{K} , the singularity category $\underline{\text{CM}} R_k$ and the relative singularity category $D^b(\text{coh } Y_k)/\langle \mathcal{O}_{Y_k} \rangle$ are both idempotent complete.*

Proof. Since R_k admits an NCCR by Proposition 3.3 which is clearly a direct sum of rank one R_k -modules, using Lemma 5.3 we see that $\underline{\text{CM}} R_k$ is idempotent complete. The fact that the relative singularity is idempotent complete follows immediately from this using Schlichting's negative K-Theory (see [8, Theorem 3.2]). \square

It then follows that the category \mathcal{C}_k and the relative singularity category are Koszul dual.

Proposition 5.5. *For any $k \geq 1$ and any \mathbb{K} , $\mathcal{C}_k \simeq \text{per}(A)$ and $D^b(\text{coh } Y_k)/\langle \mathcal{O}_{Y_k} \rangle \simeq \text{per}(A^!)$, with $A^{!!} = A$ up to quasi-isomorphism.*

Proof. By [27, Corollary 2.12(a)], there is a DG-algebra B such that $\text{per}(B)$ is equivalent to the idempotent completion of the relative singularity category. Hence, using Corollary 5.4, $\text{per}(B) \simeq D^b(\text{coh } Y_k)/\langle \mathcal{O}_{Y_k} \rangle$ itself.

Consider the A_∞ -algebra $A = \mathbf{R}\text{End}_\Lambda(\mathcal{S})$. We already know that $\mathcal{C}_k \simeq \text{per}(A)$, and further that $A^{!!} = A$ (see e.g. [27, 2.7]), so it suffices to show that $A^! = B$. It is easy to check that $B^! = A$ up to quasi-isomorphism (see e.g. [27, 2.6]), thus $B^{!!} = A^!$. Hence it remains to justify why $B = B^{!!}$, which is much harder.

As in the A-side (Remark 5.2), the difficulty is in proving that the non-positive DGA B is finite dimensional in each degree; given this, the fact $B^{!!} = B$ is standard [27, Theorem 2.8(b)]. The key point here is that we need the auxiliary data in the form of the existence

of the NCCR Λ , which allows us to say that all $\Lambda/\Lambda e\Lambda$ -modules have finite projective dimension as a Λ -module. We can then appeal directly to [27, Corollary 2.13(i)] to deduce that B is finite dimensional in each degree, with the two main points being that $H^0(B)$ is the contraction algebra, and thus is finite dimensional since $k \geq 1$, and $D_{fd}(B) \subseteq \text{per}(B)$ due to the existence of the NCCR. \square

5.3. Matching the A- and B-sides. The previous subsections §5.1 and §5.2 combine to give the following, which lifts Theorem 1.1 to include non-compact objects.

Corollary 5.6. *For $k = 1$ and $\mathbb{K} = \mathbb{C}$ or $k > 2$ prime and \mathbb{K} of characteristic k , there is an equivalence of categories $\mathcal{W}(W_k; \mathbb{K}) \simeq D^b(\text{coh } Y_k)/\langle \mathcal{O}_{Y_k} \rangle$.*

Proof. Corollary 3.13 is proved by establishing that $\mathcal{Q}_k \simeq \text{per}(A) \simeq \mathcal{C}_k$. Hence combining Propositions 5.1 and 5.5 it follows that $\mathcal{W}(W_k; \mathbb{K}) \simeq \text{per}(A^!) \simeq D^b(\text{coh } Y_k)/\langle \mathcal{O}_{Y_k} \rangle$. \square

Corollary 5.7. *Suppose $k = 1$ and $\mathbb{K} = \mathbb{C}$ or $k > 2$ is prime and \mathbb{K} has characteristic k . The wrapped Floer cohomology algebra $HW^*(T_{q_0}^* \oplus T_{q_1}^*, T_{q_0}^* \oplus T_{q_1}^*; \mathbb{K})$ of the cocores to the core components is the Ginzburg dg-algebra of the potential $(ef)^{k+1}$ on the two-cycle quiver.*

We remind the reader of our standing abuse of notation in the finite characteristic case, where the Ginzburg algebra is not strictly well-defined given issues with cyclic permutation and differentiation. But the coherent interpretation of the statement is correct, in particular, HW^0 is the corresponding Jacobian algebra, i.e. the quotient of the path algebra by the ideal killing paths of length $2k + 1$.

6. CLASSIFICATION OF SPHERICAL OBJECTS

In this section, we consider the 3-fold flopping contraction $f: Y_k \rightarrow \text{Spec } R_k$, with $k \geq 1$, over any \mathbb{K} , and we classify all fat-spherical objects in the category \mathcal{C}_k . Our main result is that there are only three, up to shift and up to the action of the pure braid group.

To ease notation, throughout this section we fix some $k \geq 1$ and henceforth drop all mention of it from the notation. The flopping contraction $f: Y \rightarrow \text{Spec } R$ has two $(-1, -1)$ -curves above the origin, say C_1 and C_2 , meeting at a point and as in §3.2, there is an equivalence

$$(6.1) \quad \Psi: D^b(\text{coh } Y) \xrightarrow{\sim} D^b(\text{mod } \Lambda)$$

where Λ is the NCCR from Lemma 3.5. There is an idempotent $e \in \Lambda$, corresponding to the vertex labelled 0, such that $\Lambda_{\text{con}} = \Lambda/\langle e \rangle$ is the algebra

$$(6.2) \quad \begin{array}{ccc} & a & \\ & \curvearrowright & \\ 1 & & 2 \\ & \curvearrowleft & \\ & b & \end{array}$$

such that $(ab)^k a = 0$ and $b(ab)^k = 0$. Since f is a flopping contraction, we have

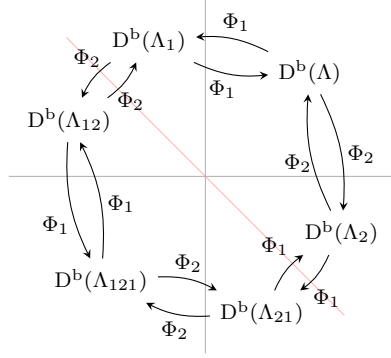
$$\mathcal{C} = \{a \in D^b(\text{coh } Y) \mid \mathbf{R}f_* a = 0\},$$

and across (6.1), \mathcal{C} corresponds to those complexes whose cohomology groups are all annihilated by e , or in other words, those complexes whose cohomology groups are Λ_{con} -modules.

Remark 6.1. There are various t-structures on \mathcal{C} . However, throughout this section, for $a \in \mathcal{C}$ we will always write $H^*(-)$ for cohomology with respect to the standard t-structure in $D^b(\text{mod } \Lambda)$. Equivalently, we could always work in $D^b(\text{coh } Y)$, and take cohomology with respect to the perverse t-structure (with perversity $p = -1$). Further, for $a \in \text{mod } \Lambda_{\text{con}}$, we will often write $\text{Hom}_{\mathcal{C}}(a, a)$ for $\text{Hom}_{\mathcal{C}}(\Psi^{-1}a, \Psi^{-1}a)$, which is the same as $\text{Hom}_{\Lambda}(a, a)$.

A key feature of below is that whilst the normal bundle of a curves being flopped does not change, the normal bundle of other curves it intersects with can change. Algebraically, this corresponds to the fact that the quiver mutation of (6.2) changes the algebra (and introduces a loop) if and only if $k > 1$. Thus, below the complexity of our argument increases slightly in the $k > 1$ case, although the main ideas remain the same.

6.1. Mutation Functors. We recall next the *mutation functors* Φ_1 and Φ_2 corresponding to mutation between the NCCRs of R at vertex 1, and vertex 2 respectively. These form a representation of the Deligne groupoid, as follows:



where the algebras $\Lambda, \Lambda_1, \Lambda_2, \Lambda_{12}, \Lambda_{21}, \Lambda_{121}$ are the six NCCRs corresponding to the six crepant resolutions of $\text{Spec } R$, and Λ is the NCCR in Lemma 3.5. It is possible to present the other NCCRs as quivers with relations, but we refrain from doing this here.

As explained in [21, Remark 4.7], using only the tilting order, there is a functorial isomorphism $\Phi_1\Phi_2\Phi_1 \cong \Phi_2\Phi_1\Phi_2$. By slight abuse of notation, we write $\mathcal{S}_1, \mathcal{S}_2$ for the simple modules for the NCCRs, regardless of which NCCR is under consideration. The labelling is consistent, so that mutation Φ_i always shifts \mathcal{S}_i .

Algebraically, the mutation functors can always be described as $\mathbf{R}\text{Hom}(T_i, -)$ where T_i is a certain tilting module of projective dimension one. From our perspective, the known facts that we require are contained in the following two lemmas.

Lemma 6.2. *Let x be a finite length module viewed as a complex in homological degree zero. Then for $i \in \{1, 2\}$, the following statements hold.*

- (1) $H^j(\Phi_i(x)) = 0$ unless $j = 0, 1$.
- (2) $H^j(\Phi_i^{-1}(x)) = 0$ unless $j = -1, 0$.
- (3) If $\text{Hom}(x, \mathcal{S}_i) = 0$ then $H^j(\Phi_i(x)) = 0$ for all $j \neq 0$.
- (4) If $\text{Hom}(\mathcal{S}_i, x) = 0$ then $H^j(\Phi_i^{-1}(x)) = 0$ for all $j \neq 0$.

Proof. (1) & (2) are a direct consequence of the fact that T_i is a tilting module, and so has projective dimension one.

(3) & (4) are [51, Corollary 5.10], which needs x to be finite length. □

As notation, set $M_1 = \text{Cone}(\mathcal{S}_1[-1] \rightarrow \mathcal{S}_2)$, and $M_2 = \text{Cone}(\mathcal{S}_2[-1] \rightarrow \mathcal{S}_1)$.

Lemma 6.3. *The following statements hold.*

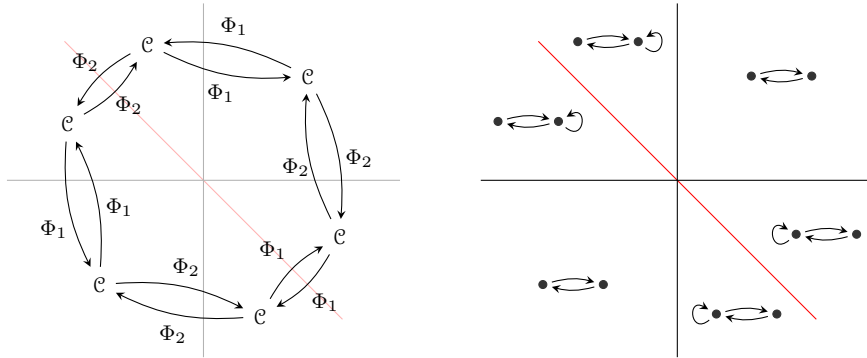
- (1) $\Phi_1(\mathcal{S}_1) = \mathcal{S}_1[-1]$, $\Phi_1(\mathcal{S}_2) = M_1$ and $\Phi_1(M_2) = \mathcal{S}_2$.
- (2) $\Phi_2(\mathcal{S}_2) = \mathcal{S}_2[-1]$, $\Phi_2(\mathcal{S}_1) = M_2$ and $\Phi_2(M_1) = \mathcal{S}_1$.
- (3) *The composition $\Phi_1\Phi_2\Phi_1$ sends $\mathcal{S}_1 \mapsto \mathcal{S}_2[-1]$, $\mathcal{S}_2 \mapsto \mathcal{S}_1[-1]$, and further if $a \in \text{mod } \Lambda_{\text{con}}$, then $a \mapsto x[-1]$ for some $x \in \text{mod } \Lambda_{\text{con}}$.*

Proof. (1) The fact that $\Phi_i(\mathcal{S}_i) = \mathcal{S}_i[-1]$ is [51, 4.15]. The fact that $\Phi_1(\mathcal{S}_2) = M_1$ follows using the same method: by torsion pairs $\Phi_1(\mathcal{S}_2)$ is a module in degree zero, and the dimension vector of this module can be determined using the exchange sequences, which are [25, 5.26]. The final statement $\Phi_1(M_2) = \mathcal{S}_2$ follows from the first two facts, and applying Φ_1 to the exact sequence $0 \rightarrow \mathcal{S}_1 \rightarrow M_2 \rightarrow \mathcal{S}_2 \rightarrow 0$. All of Part (2) follows by symmetry. For (3), the statement that $\mathcal{S}_1 \mapsto \mathcal{S}_2[-1]$ follows from [14, (3.I)], which shows that $\Phi_1\Phi_2\Phi_1$ is isomorphic to a direct mutation functor, together with [51, 4.15] which shows simples shift under mutation. The fact that \mathcal{S}_1 gets sent to $\mathcal{S}_2[-1]$, as opposed to $\mathcal{S}_1[-1]$, is due to our global numbering conventions here, and is explained by the fact $(\vartheta_1, \vartheta_2) \mapsto (-\vartheta_2, -\vartheta_1)$ underneath [14, (3.I)]. The other statement $\mathcal{S}_2 \mapsto \mathcal{S}_1[-1]$ follows by symmetry. The final statement follows, since every module is filtered by simples. \square

6.2. Representation Theory of Contraction Algebras. Our proof of the classification of fat-spherical objects will track through the functors Φ_i in the groupoid above. As such, at a technical level the categories change mildly, although of course all are equivalent.

The six crepant resolutions of $\text{Spec } R$ give rise to six versions of the category \mathcal{C} , all of which are equivalent. To avoid a proliferation of notation, we will mildly abuse notation, and refer to each simply as \mathcal{C} . Further, each crepant resolution has a tilting bundle giving an equivalence to the corresponding NCCR, and again we will abuse notation and write Ψ for this equivalence, regardless of which crepant resolution we are considering.

Each of the categories \mathcal{C} , across the relevant Ψ , is equivalent to the subcategory of the derived category of the NCCR whose cohomology is a module for the corresponding contraction algebra. Since the mutation functors restrict to an equivalence on these subcategories, we summarise this information in the following diagram. The right hand side shows the quiver of the corresponding contraction algebra, where the loops exist if and only if $k > 1$.



Also considering the relations (which we have suppressed), by symmetry, up to isomorphism there are only two algebras in the right hand side above, namely

$$\begin{array}{ccc}
 \begin{array}{c} \bullet \xrightarrow{a} \bullet \\ \bullet \xleftarrow{b} \bullet \end{array} & & \begin{array}{c} \bullet \xrightarrow{a} \bullet \xrightarrow{y} \bullet \\ \bullet \xleftarrow{b} \bullet \end{array} \\
 (ab)^k a = 0 = b(ab)^k & & y^k = ba, \quad ay = 0 = yb.
 \end{array}$$

We write Λ_{con} for the first, and Γ_{con} for the second. Note that both depend on k , and they are isomorphic if and only if $k = 1$. Further, $\dim_{\mathbb{K}} \Lambda_{\text{con}} = 2 + 4k$ and $\dim_{\mathbb{K}} \Gamma_{\text{con}} = k + 5$.

Remark 6.4. The difference between $k = 1$ and $k > 1$ is also manifest symplectically. The Stein manifold W_k has a Lagrangian skeleton $Q_0 \cup Q_1$, but also a Lagrangian skeleton given by the union of one Q_i and the Morse-Bott surgery K ; cf. Example 4.2, where K fibres over a path in the base of the Morse-Bott-Lefschetz fibration of Figure 1 between the outermost two critical points. When $k \neq 1$ and this is a Lens space, the endomorphism algebra of the second core (and its wrapped counterpart) differs from that of the first, even though they are derived equivalent.

For our purposes, we only require the following two facts, which follow from an analysis of the representation theory. These will turn out to be the key to understanding how to iteratively apply mutation functors later to ‘improve’ fat spherical objects.

Proposition 6.5. *Suppose that $a, b \in \text{mod } \Lambda_{\text{con}}$ satisfies $\text{Hom}_e(a, b) = 0$. Then either a or b is filtered by a unique simple. The same result holds for Γ_{con} .*

Proof. We give the proof for $k = 1$ here. Indeed, when $k = 1$ there are only six indecomposable Λ_{con} -modules, namely the two projective Λ_{con} -modules, say \mathcal{Q}_1 and \mathcal{Q}_2 , and the modules $\mathcal{S}_1, \mathcal{S}_2, M_1$ and M_2 . As can easily be checked directly using computer algebra (e.g. QPA in GAP), the dimension of the $\text{Hom}(X, Y)$ with X a member of the rows, and Y a member of the columns, is summarised in the following table

$$\begin{array}{c}
 (6.3) \quad \begin{array}{ccccccc}
 & \mathcal{Q}_1 & \mathcal{Q}_2 & M_1 & M_2 & \mathcal{S}_1 & \mathcal{S}_2 \\
 \mathcal{Q}_1 & 2 & 1 & 1 & 1 & 1 & 0 \\
 \mathcal{Q}_2 & 1 & 2 & 1 & 1 & 0 & 1 \\
 M_1 & 1 & 1 & 1 & 1 & 1 & 0 \\
 M_2 & 1 & 1 & 1 & 1 & 0 & 1 \\
 \mathcal{S}_1 & 1 & 0 & 0 & 1 & 1 & 0 \\
 \mathcal{S}_2 & 0 & 1 & 1 & 0 & 0 & 1
 \end{array}
 \end{array}$$

Now, if \mathcal{Q}_1 is a summand of a , since by assumption $\text{Hom}(a, b) = 0$, the only possibility for b is $\mathcal{S}_2^{\oplus t}$, regardless of what other summands a has. Ditto M_1 . Similarly, if \mathcal{Q}_2 is a summand of a , then b is $\mathcal{S}_1^{\oplus t}$. Ditto M_2 . The only remaining options for a are the remaining rows and combinations of them, so either $\mathcal{S}_1^{\oplus s}, \mathcal{S}_1^{\oplus t}$, or $\mathcal{S}_1^{\oplus s} \oplus \mathcal{S}_2^{\oplus t}$. In the latter case, if a module is orthogonal to both simples, it is zero. Hence the last two cases are a is either $\mathcal{S}_1^{\oplus s}$ or $\mathcal{S}_2^{\oplus t}$.

The general $k > 1$ proof is similar, but the number of indecomposable modules increases. Since these method of proof uses knitting on the AR quiver (which gives an independent check on the computer algebra claim above), and is quite different to the rest of this section, we delay the full proof until Section 6.5. \square

The following is rather similar.

Lemma 6.6. *There are only four modules $X \in \text{mod } \Lambda_{\text{con}}$ for which $\text{End}_{\mathcal{C}}(X) = \mathbb{K}$. They are: $\mathcal{S}_1, \mathcal{S}_2, M_1$ and M_2 . The same statement holds for $X \in \text{mod } \Gamma_{\text{con}}$.*

Proof. For the case $k = 1$, this follows directly from observing the diagonal entries in the table (6.3). The $k > 1$ case again is similar, and uses knitting, and consequently we again delay the proof until Section 6.5. \square

6.3. Induction on Length. With the mild abuse of notation as outlined in Section 6.2, for $a \in \mathcal{C}$, write t for the maximum j such that $H^j(\Psi a) \neq 0$, and b for the minimum j such that $H^j(\Psi a) \neq 0$. The *homological length* of $a \in \mathcal{C}$ is defined to be $\ell(a) = t - b$. In particular, $\ell(a) = 0$ if and only if Ψa is in a single homological degree, which holds if and only if Ψa is a shift of a Λ_{con} -module (or Γ_{con} -module, depending which \mathcal{C} we are considering).

Remark 6.7. The homological length used here is very different to the length invariant used to induct in the surfaces case [22]. Indeed, comparing to Remark 6.1, in the two-dimensional case there are again two natural 2-CY categories: the derived category of modules over the preprojective algebra, and the derived category of the resolution (the analogues in our setting are respectively $D^b(\text{mod } \Lambda)$ and $D^b(\text{coh } Y)$). *Op. cit.* works with the latter rather than the former, which leads to a more complicated induction.

Lemma 6.8. *Suppose that $a \in \mathcal{C}$ satisfies $\text{Hom}_{\mathcal{C}}(a, a[i]) = 0$ for all $i < 0$. If $\ell(a) > 0$, then $\text{Hom}_{\mathcal{C}}(H^t(\Psi a), H^b(\Psi a)) = 0$.*

Proof. This follows from the spectral sequence

$$E_2^{p,q} = \bigoplus_i \text{Hom}^p(H^i(\Psi a), H^{i+q}(\Psi a)) \Rightarrow \text{Hom}^{p+q}(\Psi a, \Psi a).$$

The bottom left of the spectral sequence is $E_2^{0,-\ell(a)} = \text{Hom}(H^t(\Psi a), H^b(\Psi a))$, and so this survives to E_∞ , to give part of $\text{Hom}_{D^b}(\Psi a, \Psi a[-\ell(a)])$. Since by assumption Ψa has no negative Exts, and $\ell(a) > 0$, it follows that $E_2^{0,-\ell(a)} = 0$, proving the assertion. \square

The following very simple spectral sequence argument is one of our key technical lemmas. It demonstrates why we don't study the twist functors directly in dimension three: in contrast, the spectral sequence for the twists does not degenerate immediately, since it has $E_2^{0,q}, E_2^{1,q}$, and $E_2^{2,q}$. Thus, our strategy is to use the mutation functors to simplify first (as for surfaces), then afterwards put everything in terms of the twists.

Lemma 6.9. *If $a \in \mathcal{C}$ with $\ell(a) > 0$, write $H^t = H^t(\Psi a)$ for the top cohomology, and $H^b = H^b(\Psi a)$ for the bottom. Then for $i \in \{0, 1\}$, the following statements hold.*

- (1) *If $H^1(\Phi_i(H^t)) = 0$ and $H^0(\Phi_i(H^b)) = 0$, then $\ell(\Phi_i(a)) < \ell(a)$.*
- (2) *If $H^0(\Phi_i^{-1}(H^t)) = 0$ and $H^{-1}(\Phi_i^{-1}(H^b)) = 0$, then $\ell(\Phi_i^{-1}(a)) < \ell(a)$.*

Proof. (1) Consider the spectral sequence

$$E_2^{p,q} = H^p(\Phi_i(H^q(\Psi a))) \Rightarrow H^{p+q}(\Phi_i(\Psi a))$$

By Lemma 6.2(1), only $E_2^{0,q}$ and $E_2^{1,q}$ can possibly be non-zero, so the spectral sequence degenerates immediately. The bottom left entry is $H^0(\Phi_i(H^b))$ and the top right entry is $H^1(\Phi_i(H^t))$. If these are both zero, it follows that in the resulting complex $\Phi_i(\Psi a)$, the

bottom cohomology is in homological degree $b+1$, and the top cohomology is in homological degree t . Hence the complex has become smaller, and $\ell(\Phi_i(\Psi a)) < \ell(a)$.

(2) This is very similar, using

$$E_2^{p,q} = H^p(\Phi_i^{-1}(H^q(\Psi a))) \Rightarrow H^{p+q}(\Phi_i^{-1}(\Psi a)).$$

Now by Lemma 6.2(2) the only non-zero terms are $E_2^{-1,q}$ and $E_2^{0,q}$; the spectral sequence still degenerates immediately. \square

The following is the main technical result of this section, and gives a constructive way of simplifying the complex a by applying mutation functors.

Theorem 6.10. *Suppose that $a \in \mathcal{C}$ satisfies $\text{Hom}_{\mathcal{C}}(a, a[i]) = 0$ for all $i < 0$, and $\ell(a) > 0$.*

- (1) *Either $H^t(\Psi a)$ or $H^b(\Psi a)$ is filtered by a unique simple module \mathcal{S}_i , with $i \in \{0, 1\}$.*
- (2) *When $H^b(\Psi a)$ is filtered only by \mathcal{S}_i , applying Φ_i decreases the length ℓ .*
- (3) *When $H^t(\Psi a)$ is filtered only by \mathcal{S}_i , applying Φ_i^{-1} decreases the length ℓ .*

Proof. (1) By Lemma 6.8, $\text{Hom}_{\mathcal{C}}(H^t, H^b) = 0$, so the result follows from the key representation theory fact in Proposition 6.5.

(2) Since $\Phi_i(\mathcal{S}_i) = \mathcal{S}_i[-1]$ by Lemma 6.3, by induction on the length of the filtration of H^b , it follows that $H^0(\Phi_i(H^b)) = 0$. Further, $\text{Hom}(H^t, H^b) = 0$ implies $\text{Hom}(H^t, \mathcal{S}_i) = 0$, so by Lemma 6.2(3) $H^1(\Phi_i(H^t)) = 0$. Combining, using Lemma 6.9(1) we see that $\ell(\Phi_i(a)) < \ell(a)$.

(3) This is very similar, using $\Phi_i^{-1}(\mathcal{S}_i) = \mathcal{S}_i[1]$, and Lemmas 6.2(4) and 6.9(2). \square

Corollary 6.11. *Suppose that $a \in \mathcal{C}$ satisfies $\text{Hom}_{\mathcal{C}}(a, a[i]) = 0$ for all $i < 0$. Then there exists a composition of mutation functors $F: D^b(\Lambda) \rightarrow D^b(\Lambda_\beta)$ such that $Fa \cong a'$, where a' is a module for the contraction algebra of Λ_β .*

Proof. By inducting along Theorem 6.10, it is easy to see that there exists a composition on mutation functors G such that $\ell(Ga) = 0$. Hence $G(\Psi a) \cong a'[j]$ where a' is a module in degree zero, and $j \in \mathbb{Z}$. Since $\Psi a \in \mathcal{C}$, and the mutation functors preserve \mathcal{C} , necessarily a' is a module for the relevant contraction algebra.

Now by Lemma 6.3(3) $\Phi_1\Phi_2\Phi_1 \cong \Phi_2\Phi_1\Phi_2$ sends every contraction algebra module to a shift of some contraction algebra module. Hence, applying $\Phi_1\Phi_2\Phi_1$ or its inverse repeatedly if necessary, we can move $G(\Psi a)$ to homological degree zero. \square

The following is the main result of this subsection.

Corollary 6.12. *Suppose that $a \in \mathcal{C}$ satisfies $\text{Hom}_{\mathcal{C}}(a, a[i]) = 0$ for all $i < 0$, and further $\dim_k \text{Hom}_{\mathcal{C}}(a, a) = 1$. Then the following statements hold.*

- (1) *There exists a composition of mutation functors $F: D^b(\Lambda) \rightarrow D^b(\Lambda_\beta)$ such that Fa is either \mathcal{S}_1 or \mathcal{S}_2 .*
- (2) *The complex a is fat-spherical, and up to the action of the pure braid group, is either $\mathcal{S}_1, \mathcal{S}_2, M_1$, or their shifts by $[1]$.*

Proof. (1) By Corollary 6.11, we can find a composition of mutation functors $F: D^b(\Lambda) \rightarrow D^b(\Lambda_\beta)$ such that $Fa \cong a'$, where a' is a $(\Lambda_\beta)_{\text{con}}$ -module in homological degree zero. But

$$\text{Hom}_{\mathcal{C}}(a', a') \cong \text{Hom}_{\mathcal{C}}(a, a) = \mathbb{K},$$

so Lemma 6.6 forces a' to be either M_1 , M_2 , \mathcal{S}_1 or \mathcal{S}_2 . Since $\Phi_2(M_1) \cong \mathcal{S}_1$ and $\Phi_1(M_2) \cong \mathcal{S}_2$ by Lemma 6.3, in all cases overall we can find a composition of mutation functors taking a to either \mathcal{S}_1 or \mathcal{S}_2 .

(2) By (1) there exists F such that Fa is either \mathcal{S}_1 and \mathcal{S}_2 . Depending on the NCCR Λ_β , the \mathcal{S}_i can be either spherical or fat-spherical (when $k = 1$ they are always spherical). Since autoequivalences preserve both, a itself must also be spherical, or fat-spherical. For the last statement, consider the shortest positive chain of mutation functors $\Phi_\alpha: D^b(\Lambda) \rightarrow D^b(\Lambda_\beta)$. Since Fa is a simple, by torsion pairs [21, (5.B)] $\Phi_\alpha^{-1}Fa$ is either a module, or a shift [1] of a module. Since $\text{End}_{\mathcal{C}}(\Phi_\alpha^{-1}Fa)$ is one-dimensional, by Lemma 6.6 it is either M_1 , M_2 , \mathcal{S}_1 or \mathcal{S}_2 , or their shift [1]. Given $\Phi_1\Phi_1(M_2) = M_1$ by Lemma 6.3(1), the result follows. \square

6.4. Symplectic applications. We collect a number of symplectic applications of Corollary 6.12, which in particular establish several claims from the Introduction. Recall that our basic equivalence $\mathcal{Q}_p \simeq \mathcal{C}_p$ concerns \mathbb{Z} -graded categories, and an exact Lagrangian $L \subset W_p$ (more generally, an exact Lagrangian equipped with a local system) defines an object of the \mathbb{Z} -graded Fukaya category precisely when it has vanishing Maslov class. The basic translation of the Corollary is as follows, where by a trivial local system we mean no local system.

Corollary 6.13. *Let $p = 1$ or $p > 2$ be prime. Let $L \subset W_p$ be a closed connected exact Lagrangian submanifold with vanishing Maslov class, equipped with a rank one \mathbb{K} -local system. Then L is quasi-isomorphic to a pure braid image of one of the two core components or the Morse-Bott surgery of the core components.*

Proof. By exactness, for any rank one local system $\xi \rightarrow L$, the endomorphisms of the corresponding Lagrangian brane reproduce those of the trivial local system, which are in turn given by ordinary cohomology, $HF^*(\xi, \xi) \cong HF^*(L, L) \cong H^*(L; \mathbb{K})$. The hypotheses imply that $HF^{<0}(L, L) = 0$ and $HF^0(L, L) \cong \mathbb{K}$. Under the mirror equivalence $\mathcal{F}(W_p; \mathbb{K}) = D^\pi \mathcal{Q}_p \simeq \mathcal{C}_p$, valid over a field of the appropriate characteristic, L (or more generally $\xi \rightarrow L$) defines an object $\mathcal{E}_L \in \mathcal{C}_p$ satisfying the hypotheses of Corollary 6.12. The second statement of the Corollary shows that \mathcal{E}_L is a pure braid image of one of the simples $\mathcal{S}_1, \mathcal{S}_2$ or the module M_1 obtained as the mapping cone. The equivalence $\mathcal{Q}_p \simeq \mathcal{C}_p$ identifies the cores Q_i with the simples \mathcal{S}_i , so the result follows. \square

It follows that every two graded exact Lagrangians in W_p , with $p = 1$ or $p > 2$ prime, have non-empty intersection, which is already not obvious.

Remark 6.14. We established in Proposition 2.8 that, in characteristic zero, there is an equivalence $\mathcal{Q}_p \simeq \mathcal{Q}_1$ for any $p > 0$. Nonetheless, one cannot obtain restrictions on Lagrangian submanifolds of W_p from this, because for $p > 1$ the subcategory $\mathcal{Q}_p \subset \mathcal{F}(W_p; \mathbb{K})$ is not split-generating if $\text{char}(\mathbb{K}) = 0$, and not invariant under the action of the pure braid group, cf. the discussion after Lemma 4.19. The symplectic applications for Lagrangians in W_p with $p \neq 1$ rely essentially on the equivalence of categories in finite characteristic.

Note that in the case $p > 2$, the endomorphisms of the core components and of their surgery themselves differ: the surgery is fat spherical but not spherical.

Corollary 6.15. *If $p > 2$ is prime, then every Lagrangian sphere in W_p admits an orientation such that, after applying a pure braid, it becomes quasi-isomorphic to one of the*

two oriented core components. If $p = 1$, the same is true, except that it may also be quasi-isomorphic to a surgery of the two cores.

Identify $H_3(W_p; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators the core components Q_i , with either choice of orientation. The following is the homological analogue for W_p of the ‘nearby Lagrangian conjecture’, which implies that an exact Lagrangian in the cotangent bundle is homologous to the zero-section. Note, however, that we require a hypothesis on the Maslov class.

Corollary 6.16. *Let $p = 1$ or $p > 2$ be prime. If $L \subset W_p$ is an exact Lagrangian submanifold with vanishing Maslov class, then $\pm[L] \in \{(1, 0), (0, 1), (1, \pm 1)\} \subset \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. This follows immediately from Corollary 6.12 and the fact that the pure braid group acts trivially on $H_3(W_p; \mathbb{Z})$. \square

Corollary 6.17. *If $p > 2$ is prime, then any exact graded Lagrangian $L \subset W_p$ satisfies $H^*(L; \mathbb{K}) = H^*(S^3; \mathbb{K})$ if $\pm[L] \in \{(1, 0), (0, 1)\} \subset H_3(W_p; \mathbb{Z})$, and $H^*(L; \mathbb{K}) = H^*(L(p, 1); \mathbb{K})$ if $\pm[L] \in \{(1, \pm 1)\} \subset H_3(W_p; \mathbb{Z})$.*

Corollary 6.18. *If $p = 1$ or $p > 2$ is prime, there is no exact Lagrangian $S^1 \times S^2 \subset W_p$ with vanishing Maslov class.*

Proof. Although the graded rings $H^*(L(p, 1); \mathbb{Z}/p)$ and $H^*(S^1 \times S^2; \mathbb{Z}/p)$ agree, the former has a non-trivial μ^3 -operation (Massey product), whilst the latter is formal. Since the A_∞ -structure in the endomorphism algebra of an exact Lagrangian reproduces the classical structure on cohomology, the result then follows as before. \square

Note that W_0 does contain an exact graded Lagrangian $S^1 \times S^2$, namely the surgery of the two core spheres.

6.5. Proof of 6.5 and 6.6. In this subsection, which is logically independent of the remainder of the paper, we give the full proofs of the two representation theory facts presented in Section 6.2. Namely, we establish that if two modules for the contraction algebras Λ_{con} or Γ_{con} are orthogonal, then one is filtered by a single simple, and furthermore we prove that both Λ_{con} and Γ_{con} admit precisely four bricks.

Our method to approach both problems uses Auslander–Reiten (AR) theory.

Lemma 6.19. *The AR quivers of the contraction algebras Λ_{con} and Γ_{con} are illustrated in Figure 2, where the numbers denote the dimension vectors of the corresponding representation. In each case, the left and right dotted arrows are identified, and so AR quiver is really drawn on a cylinder.*

Proof. Both Λ_{con} and Γ_{con} are Brauer tree algebras of finite representation type. Further, Λ_{con} is a symmetric Nakayama algebra. The above AR quivers are well-known, and can be extracted from e.g. [18]. \square

Proof of 6.5. The proof for Λ_{con} and for Γ_{con} is slightly different. For Λ_{con} , we claim that if X and Y are indecomposable Λ_{con} -modules which are not simple, then $\text{Hom}(X, Y) \neq 0$. This is just clear from the shape of the AR quiver, and the fact that the simples are in the

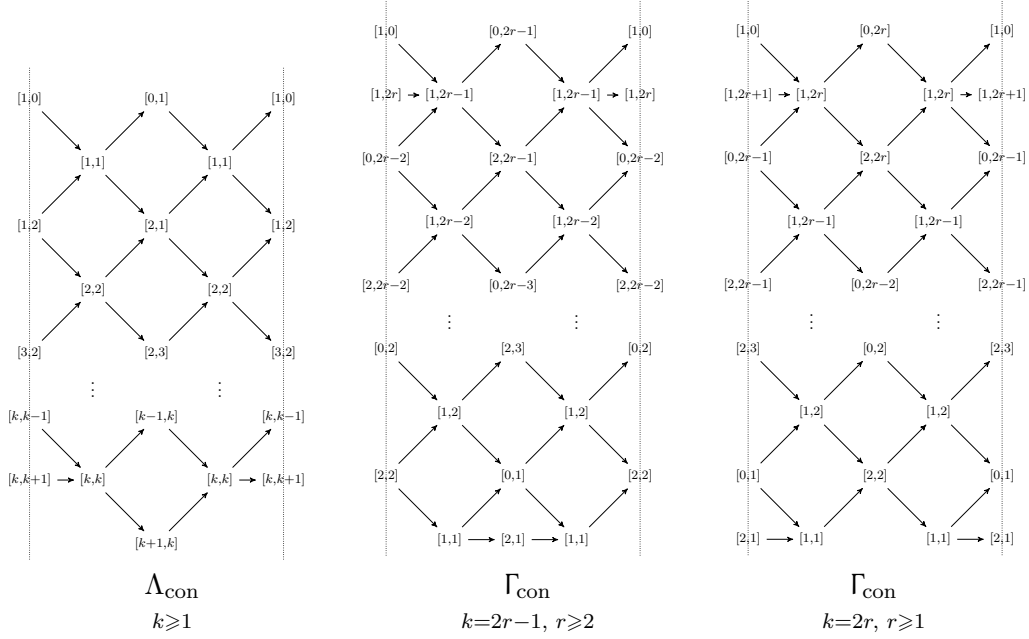
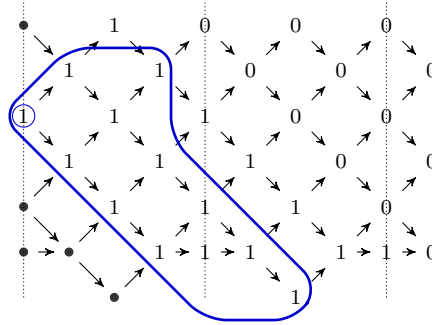


FIGURE 2. AR quivers of Λ_{con} and Γ_{con} .

top row. For example, when $k = 3$, the blue region in the following knitting calculation:

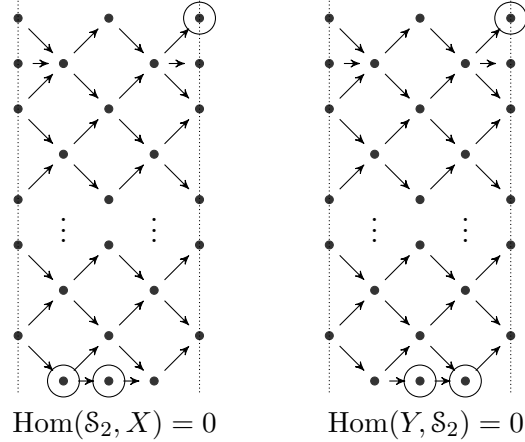


shows that there is a non-zero homomorphism from the circled representation to every other indecomposable non-simple module. By the highly regular shape of the AR quiver, the calculation for every other non-simple representation is very similar.

Now suppose that $a, b \in \text{mod } \Lambda_{\text{con}}$ satisfies $\text{Hom}(a, b) = 0$. By above, if a contains a non-simple module, then b cannot. But then $b = \mathcal{S}_1^{\oplus s} \oplus \mathcal{S}_2^{\oplus t}$. Since no non-zero module can be orthogonal to both simples, either $b = \mathcal{S}_1^{\oplus s}$ or $b = \mathcal{S}_2^{\oplus t}$. The only other case is when a only contains simple modules; for a similar reason either $a = \mathcal{S}_1^{\oplus s}$ or $a = \mathcal{S}_2^{\oplus t}$.

For Γ_{con} , the proof is slightly different. The trick here is that, due to the asymmetry in the quiver, \mathcal{S}_2 injects into almost all modules, and almost all modules surject onto \mathcal{S}_2 . Indeed, the left hand side of the following digram circles all indecomposable modules X for which $\text{Hom}(\mathcal{S}_2, X) = 0$, and the right hand side circles all indecomposable modules Y for which

$\text{Hom}(Y, \mathcal{S}_2) = 0$. We illustrate the case when k is odd; the k -even case is similar. Both of these calculations can be achieved via knitting (see below for a very similar calculation).

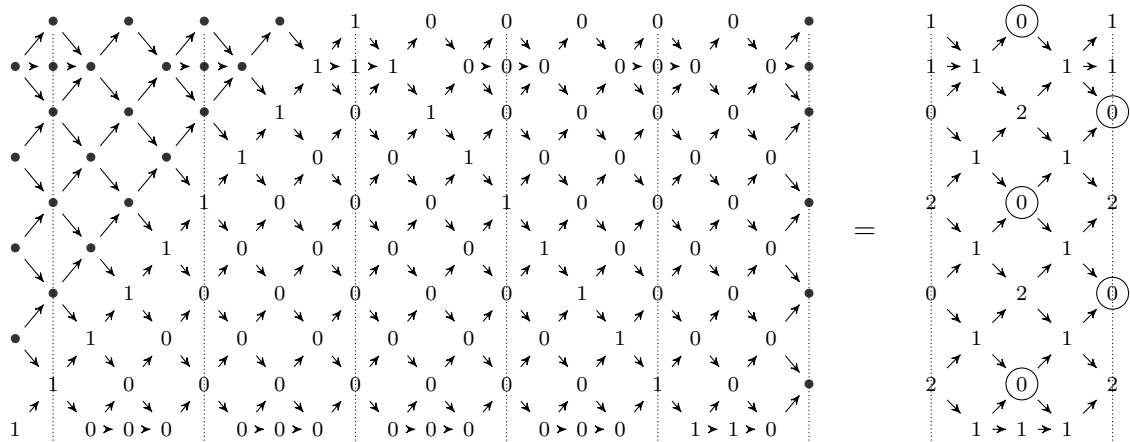


Write N for the indecomposable module in the middle of the bottom row, which is circled in both sides above. Now, we are given a, b such that $\text{Hom}(a, b) = 0$. If a has a summand from the complement of the right hand circles, and b has a summand from the complement of the left hand circles, then $a \rightarrow \mathcal{S}_2 \hookrightarrow b$, which is a non-zero homomorphism. Hence we can assume that either $a = \mathcal{S}_1^{\oplus s} \oplus M_1^{\oplus t} \oplus N^{\oplus u}$ or $b = \mathcal{S}_1^{\oplus s} \oplus M_2^{\oplus t} \oplus N^{\oplus u}$.

Now, observe that, for any $x \in \text{mod } \Gamma_{\text{con}}$,

$$\begin{aligned}
 (\text{by positions in AR quiver}) \quad & \text{Hom}(M_1, x) = 0 \iff \text{Hom}(N, x) = 0 \\
 (\Gamma_{\text{con}} \text{ is 0-CY, and } N \text{ is projective}) \quad & \iff \text{Hom}(x, N) = 0 \\
 (\text{by positions in the AR quiver}) \quad & \iff \text{Hom}(x, M_2) = 0,
 \end{aligned}$$

and furthermore the following knitting calculation (illustrated for $k = 5$) shows that the first condition, and hence all conditions, is equivalent to x being filtered by \mathcal{S}_2 .

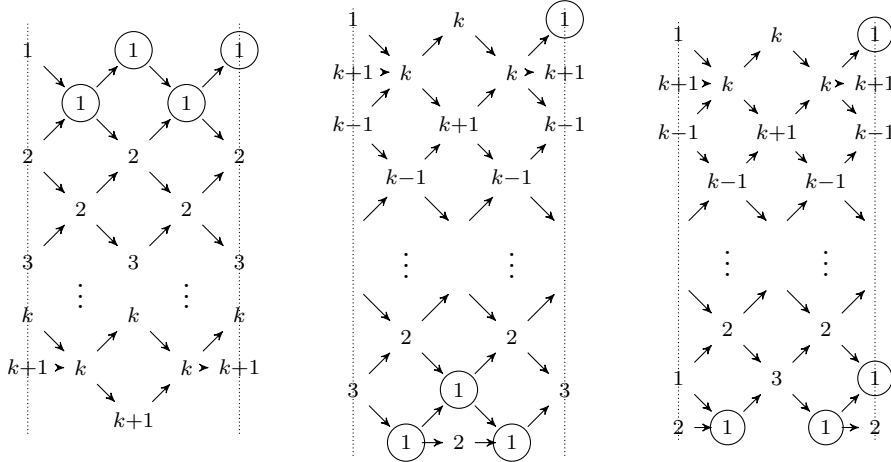


In the right hand side of the above, we have circled the modules x for which $\text{Hom}(M_1, x) = 0$. By inspection of the dimension vectors in the AR quiver in Figure 2, the circled modules are precisely those modules filtered by \mathcal{S}_2 .

In the case that $a = \mathcal{S}_1^{\oplus s} \oplus M_1^{\oplus t} \oplus N^{\oplus u}$, it follows from the preceding paragraph that if either $u \geq 1$ or $t \geq 1$, then b is filtered by \mathcal{S}_2 . Hence we can assume that $u = t = 0$, in which case $a = \mathcal{S}_1^{\oplus s}$, and so a is filtered by \mathcal{S}_1 .

In the case that $b = \mathcal{S}_1^{\oplus s} \oplus M_2^{\oplus t} \oplus N^{\oplus u}$, in a very similar way, it follows from the preceding paragraph that if either $u \geq 1$ or $t \geq 1$, then a is filtered by \mathcal{S}_2 . Hence we can assume that $u = t = 0$, in which case $b = \mathcal{S}_1^{\oplus s}$, and so b is filtered by \mathcal{S}_1 . \square

Proof of 6.6. Running the knitting algorithm as above repeatedly, it is easy (but long) to calculate that the dimension of the self-Hom of each module is as follows:



In all cases, there are precisely four modules with one-dimensional endomorphism ring, which we have circled. By inspection of the AR quivers in Lemma 6.19, the circled modules are either simple, or have dimension vector $(1, 1)$. By uniqueness of the non-split extensions between \mathcal{S}_1 and \mathcal{S}_2 , the result follows. \square

7. A REALISATION QUESTION

Theorem 1.1 sits within a much wider conjectural framework; we briefly sketch this here, together with some obstructions and subtleties to lifting our main results into this more general framework.

As demonstrated by Theorem 1.1, it may simply not be feasible to expect mirror symmetry statements for *all* flops without changing the characteristic of the field. In addition to this issue, 3-fold flops have moduli, and on the A-side it is not so clear how continuous parameters can enter into the plumbing of spheres, even if we iterate. As such, following a philosophy that the mirror to the exact form should be some point in complex structure moduli space which is picked out arithmetically, we propose that the existence of the flop equation (3.1) defined over \mathbb{Z} , not just \mathbb{C} , is one of the key features of this paper.

Much more generally, we pose the following ‘realisation question’, which is open on both the algebraic and symplectic sides. In both parts we abuse notation, as above Corollary 2.11, when describing A_∞ -structures over fields possibly of prime characteristic.

Question 7.1. *For the quivers Q in Figure 3, and for those potentials f defined over \mathbb{Z} for which the Jacobi algebra J_f is finite dimensional (over some field \mathbb{K}), does there exist:*

- (1) A 3-fold flop $g: Y \rightarrow \text{Spec } R$ over \mathbb{K} , where Y is smooth and defined over \mathbb{Z} , such that its contraction algebra is isomorphic to J_f , and the A_∞ -structure on the Ext-algebra of the curves is also described by f ?
- (2) An exact symplectic manifold W_f with $n = |Q_0|$ distinguished Lagrangians L_1, \dots, L_n , over which each L_i is fat-spherical and the A_∞ -algebra $\bigoplus_{i,j} HF^*(L_i, L_j)$ is described by f ?

An obvious challenge is to find some symplectic-geometric mechanism which distinguishes this particular collection of quivers with potential. The paper [10] gives some evidence towards Question 7.1 on the B-side, and there is substantial computer algebra evidence towards Question 7.1(1) for the two-loop quiver \bar{D}_1 , which is already highly non-trivial. The fact that the symplectic topology predicted Lemma 6.6 gives some hope for a general interplay between both sides of the mirror.

For any given potential f , whenever both parts of 7.1 exist, over the appropriate field \mathbb{K} the top horizontal row in the following is an equivalence of categories, and it should extend via Koszul duality to an equivalence in the dotted bottom horizontal row, modulo Remark 5.2.

$$\begin{array}{ccc}
 \mathcal{C} := \{a \mid \mathbf{R}g_*a = 0\} & \xleftarrow{\sim} & \mathcal{L} := \langle L_1, \dots, L_n \rangle \\
 \downarrow & & \downarrow \\
 \text{Db}(\text{coh } Y) / \langle \mathcal{O}_Y \rangle & \xleftarrow{\sim} & \mathcal{W}(W_f; \mathbb{K})
 \end{array}$$

Furthermore, on the B-side, there is a known real simplicial hyperplane arrangement \mathcal{H} which induces a faithful group action

$$\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}) \hookrightarrow \text{Auteq } \mathcal{L}.$$

When \mathcal{H} is an ADE root system, $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})$ is isomorphic to the pure braid group, but in general \mathcal{H} need not be Coxeter, and further higher length braid relations exist.

Question 7.1 is very subtle, even for A_2 and A_3 . Most notably, on the B-side the dual graph changes under flop, and so the families in Figure 3 can be related via mutation. More plainly, even if we just add loops to the oriented two-cycle, we get *much* more complicated geometry than we consider in this paper.

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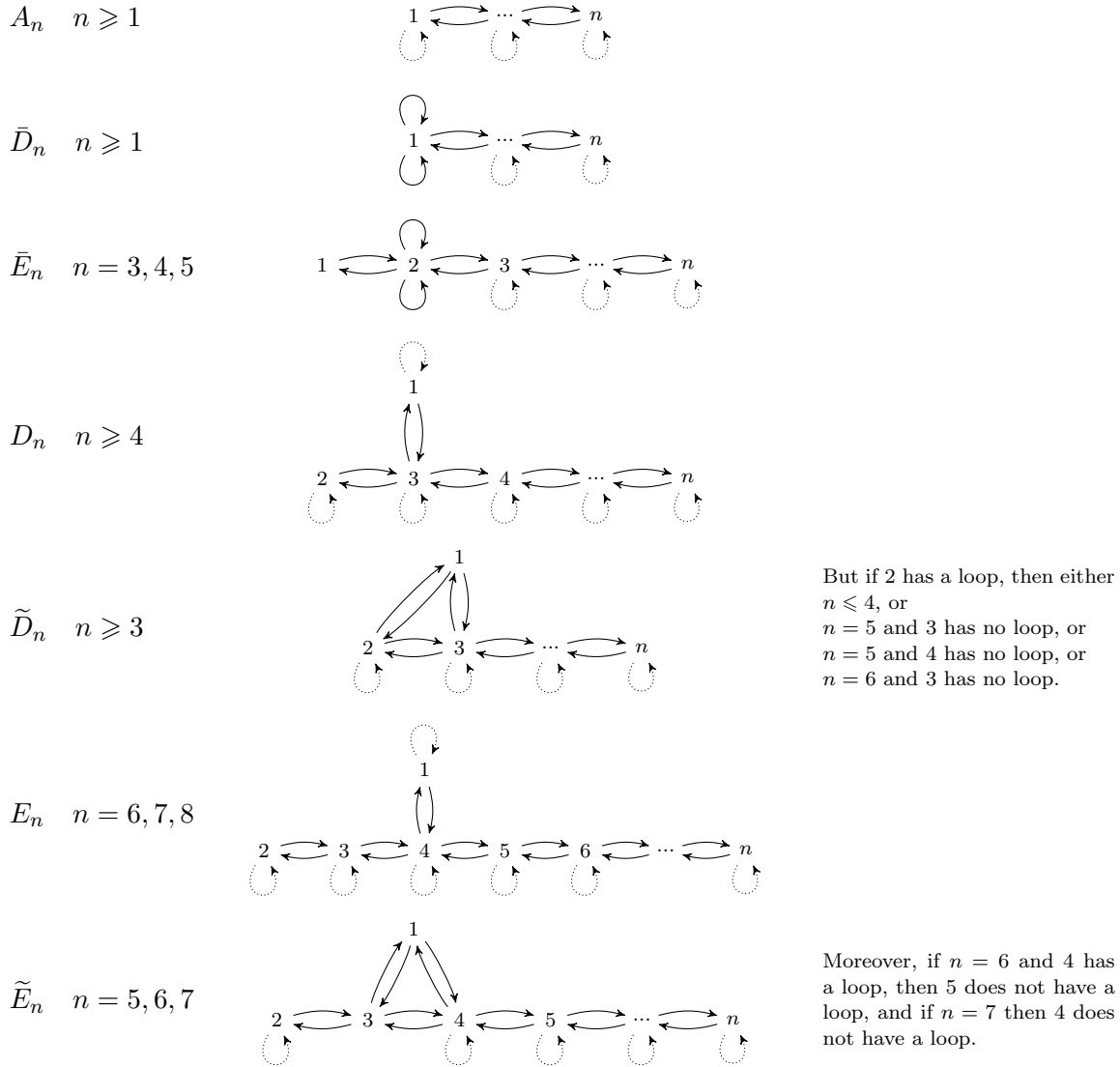


FIGURE 3. Possible quivers of contraction algebras of smooth 3-fold flops, as a consequence of [42, 5.5]. There is choice, as a single loop may or may not occur wherever a dotted arrow is present.

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