A VERY ELEMENTARY PROOF THAT THE SOMOS 5 SEQUENCE IS INTEGER VALUED

Abstract. We give a short and elementary proof that every term in the Somos 5 sequence is integer valued, and is coprime to the proceeding two terms.

1. Introduction

Definition 1.1. The Somos 5 sequence is the sequence \( (a_n)_{n \in \mathbb{N}} \) defined by the rule
\[
a_n a_{n+5} = a_{n+1} a_{n+4} + a_{n+2} a_{n+3}
\]
with \( a_0 = a_1 = a_2 = a_3 = a_4 = 1 \).

The sequence starts 1, 1, 1, 1, 2, 3, 5, 11, \ldots. There are various proofs of the fact that it is integer valued. One is via elliptic curves \([2]\), another as a consequence of the Laurent phenomenon in cluster algebras \([1]\); presumably there are also many others that are unpublished or elsewhere in the literature. The purpose of this short note is to give a very elementary proof that furthermore establishes a stronger result, namely that each term in the Somos 5 sequence is integer valued, and coprime to the proceeding two terms.

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2. Proof

We require the following two well-known and easy lemmas. The first follows by inspection of the relevant prime decompositions.

Lemma 2.1. For \( a, b, x, y \in \mathbb{N} \), \( (a, x) = (a, y) = (b, x) = (b, y) = 1 \iff (ab, xy) = 1 \).

Lemma 2.2. For \( x, y \in \mathbb{N} \), \( (x, y) = 1 \iff (x + y, y) = 1 \).

Proof. \((\Rightarrow)\) By the assumption there exists \( p, q \in \mathbb{Z} \) such that \( px + qy = 1 \). Then \( p(x + y) - py + qy = 1 \) so that \( p(x + y) + (q - p)y = 1 \). Therefore \( (x + y, y) = 1 \).

\((\Leftarrow)\) By the assumption there exists \( m, n \in \mathbb{Z} \) such that \( m(x + y) + ny = 1 \). Then \( mx + (m + n)y = 1 \), so \( (x, y) = 1 \). \(\square\)

Recall that the Somos 5 sequence is denoted \( (a_n)_{n \in \mathbb{N}} \).

Notation 2.3. For \( n \geq 2 \), we define \( s_n := a_n^2 + a_{n-2} a_{n+2} \).

The following two results are elementary.

Lemma 2.4. For \( n \geq 4 \), \( a_{n-3} s_n = a_{n+1} a_{n-2} \).

Proof. We compute
\[
a_{n-3} s_n &= a_{n-3}(a_n^2 + a_{n-2} a_{n+2}) \\
&= a_{n-3} a_n^2 + a_{n-2} a_{n-3} a_{n+2} \\
&= a_{n-3} a_n^2 + a_{n-2} (a_{n+1} a_{n-2} + a_n a_{n-1}) \\
&= a_{n-2} a_{n+1} + a_n (a_{n-3} + a_{n-1} a_{n-2}) \\
&= a_{n-2}^2 a_{n+1} + a_n a_{n+1} a_{n-4} \\
&= a_{n+1} (a_{n-2}^2 + a_n a_{n-4}) \\
&= a_{n+1} s_{n-2}.
\] \(\square\)
Corollary 2.5. For all \( n \geq 2 \) we have
\[
 s_n = \begin{cases} 
 2a_{n+1}a_{n-1} & \text{for } n \text{ even} \\
 3a_{n+1}a_{n-1} & \text{for } n \text{ odd}.
\end{cases}
\]

Proof. We will show that this is true for \( n \) even. Certainly \( s_2 = 1^2 + 1 = 2 \) so the statement is true for \( n = 2 \). Now let \( n \geq 4 \) be even and suppose that the statement is true for all smaller even numbers. Then \( a_{n-3}s_n = a_{n+1}s_{n-2} \) by Lemma 2.4, and so by inductive hypothesis \( a_{n-3}s_n = a_{n+1}(2a_{n-1}a_{n-3}) \). Cancelling terms, \( s_n = 2a_{n+1}a_{n-1} \), proving the inductive step. The proof for \( n \) odd is identical. \( \square \)

This leads to the main result.

Theorem 2.6. In the Somos 5 sequence, each \( a_n \in \mathbb{Z} \), and further we have \( (a_n, a_{n-1}) = (a_n, a_{n-2}) = 1 \).

Proof. We prove the statement by induction, the statement being obvious by inspection for \( n \leq 6 \). Hence we consider \( n \geq 7 \), and suppose that the statement is true for smaller \( n \).

We first show that \( a_n \in \mathbb{Z} \). By Corollary 2.5 \( s_{n-2} = k a_{n-1} a_{n-3} \), where either \( k = 2 \) or \( k = 3 \), depending on whether \( n \) is even or odd. Regardless, \( k \in \mathbb{Z} \). Equating this expression with the definition of \( s_{n-2} \), rearranging we obtain
\[
a_{n-4}a_n = ka_{n-1}a_{n-3} - a_{n-2}^2.
\]
By the inductive hypothesis \( a_{n-1}, a_{n-3}, a_{n-2} \) and \( k \) are all integers, so it follows that \( a_{n-4}a_n \in \mathbb{Z} \). Further, by the definition of the Somos-5 sequence,
\[
a_n a_{n-5} = a_{n-1} a_{n-4} + a_{n-2} a_{n-3}
\]
and so since by inductive hypothesis \( a_{n-1}, a_{n-4}, a_{n-2} \) and \( a_{n-3} \) are all integers, we also see that \( a_{n-5}a_n \in \mathbb{Z} \). But \( (a_{n-5}, a_{n-4}) = 1 \) by inductive hypothesis, so there exists \( p, q \in \mathbb{Z} \) such that \( pa_{n-5} + qa_{n-4} = 1 \). Simply multiplying this equation by \( a_n \) gives
\[
a_n = pa_{n-5}a_n + qa_{n-4}a_n,
\]
which shows that \( a_n \in \mathbb{Z} \).

We next verify that \( (a_n, a_{n-1}) = (a_n, a_{n-2}) = 1 \). Certainly by inductive hypothesis we have
\[
(a_{n-1}, a_{n-2}) = (a_{n-1}, a_{n-3}) = (a_{n-4}, a_{n-2}) = (a_{n-4}, a_{n-3}) = 1.
\]
Hence, using Lemma 2.1,
\[
(a_{n-1}a_{n-4}, a_{n-2}a_{n-3}) = 1.
\]
Now using Lemma 2.2 with \( x := a_{n-1}a_{n-4} \) and \( y := a_{n-2}a_{n-3} \), we see that
\[
(a_{n-1}a_{n-4} + a_{n-2}a_{n-3}, a_{n-2}a_{n-3}) = 1,
\]
thus \( (a_n a_{n-5} + a_n a_{n-2} a_{n-3}) = 1 \) and so \( (a_n, a_{n-2}) = 1 \) by Lemma 2.1. By a very similar argument, using Lemma 2.2 with \( y := a_{n-1}a_{n-4} \) and \( x := a_{n-2}a_{n-3} \) we obtain \( (a_n a_{n-5} + a_{n-1}a_{n-4}) = 1 \) and so again by Lemma 2.1 \( (a_n, a_{n-1}) = 1 \), as required. \( \square \)

References
