DERIVED DEFORMATION THEORY OF CREPANT CURVES

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ABSTRACT. This paper determines the full derived deformation theory of certain smooth rational curves C in Calabi-Yau 3-folds, by determining all higher A_∞-products in its controlling DG-algebra. This geometric setup includes very general cases where C does not contract, cases where the curve neighbourhood is not rational, all known simple smooth 3-fold flops, and all known divisorial contractions to curves. As a corollary, it is shown that the noncommutative deformation theory of C can be described as a superpotential algebra derived from what we call free necklace polynomials, which are elements in the free algebra obtained via a closed formula from combinatorial gluing data. The description of these polynomials, together with the above results, establishes a suitably interpreted string theory prediction due to Ferrari [F], Aspinwall–Katz [AK] and Curto–Morrison [CM]. Perhaps most significantly, the main results give both the language and evidence to finally formulate new contractibility conjectures for rational curves in CY 3-folds, which lift Artin's celebrated results [A] from surfaces.

CONTENTS

1. INTRODUCTION

Rational curves in Calabi–Yau 3-folds $C \subset \mathfrak{X}$ are fundamental building blocks of geometry, and yet some of even their most basic properties remain wide open. The main reason, in contrast to when $\mathfrak{X}$ is a surface, is that various key properties of C in $\mathfrak{X}$ including its formal neighbourhood, its contractibility and its deformation theory, are structural and not combinatorial. The controlling structure of all three properties now lies in homological and noncommutative algebra.

The purpose of this paper is describe the full, derived, deformation theory of C in $\mathfrak{X}$, for the general curves described in §1.1 below. This information is then used to (1) give the first general closed-formula description of the associated noncommutative deformation algebra of C, (2) build, in the context of mirror symmetry, a B-side geometric model from the purely algebraic data of a given superpotential, and (3) use the evidence from these results, and others, to finally be in a position to formulate and conjecture contractibility criteria for formal neighbourhoods of curves in 3-folds. Noncommutative deformations are the language and framework needed in order to lift Artin's 60-year old [A] work from surfaces to 3-folds.

1.1. **Geometric Setup.** For many reasons, some explained in §1.5, the crucial open and key benchmarking case for all the problems listed above is when $C \cong \mathbb{P}^1$ is a single curve, with normal bundle $\mathcal{O}(-3) \oplus \mathcal{O}(1)$. This is a vast set of curves, an uncharted zoo, known to contain an array of different and surprising geometric behaviour. It is thus important to understand such curves, and to establish their deformation properties.

Deformation theory is local, and so this problem at once reduces to a local model, patching together two copies of affine space (after completion, when necessary). In full generality, there is no known description of this model. However inside this general situation sits a very large subset, which is expected to be representative of general behaviour. This is provided by the string theory literature [F, K2, CM], where a smooth rational curve C inside a 3-fold $\mathcal{X}$ is built as follows.

Set $\lambda_{00} = \lambda_{10} = \lambda_{01} = 0$, and choose finitely many scalars λ_{jk} where $j, k \geq 0$ and $j + k \geq 2$, Given this choice of data, consider the scheme $\mathcal{X}$ defined the two open patches $\mathcal{U}_1 = \mathbb{A}^3_{a,v_2,v_1}$ and $\mathcal{U}_2 = \mathbb{A}^3_{b,w_2,w_1}$ glued along

$$(a, v_2, v_1) \overset{a \neq 0}{\longleftrightarrow} (a^{-1},\, a^{-1}v_2,\, a^3 v_1 + \sum_{j,k \geq 0} \lambda_{jk} a^{2-k} v_2^{j+k-1}). \tag{1.A}$$

The locus $v_2 = v_1 = 0$ glues with the locus $w_2 = w_1 = 0$ to create a curve $C \cong \mathbb{P}^1 \subset \mathcal{X}$.

Both the scheme $\mathcal{X}$, and the properties of the curve $C \subset \mathcal{X}$, depend heavily on the choice of λ_{jk}, but this is suppressed from the notation. We show in 2.3 that $C \subset \mathcal{X}$ has normal bundle $\mathcal{O}(-3) \oplus \mathcal{O}(1)$ if and only if there are no small degree terms in (1.A). Given our motivation, for the remainder of this introduction we assume that this condition holds.

1.2. **Free Necklace Polynomials.** To state our main results requires some new noncommutative objects, which we briefly summarise here. Given two integers $j, k \geq 0$ such that $n := j + k \geq 1$, consider the set $\mathrm{Neck}_{j,k}$ consisting of coloured n-gons, or necklaces, where there are precisely j nodes shaded $\bullet$ and precisely k unshaded nodes $\circ$.

The cyclic group $\mathbb{Z}_n$ acts on the set $\mathrm{Neck}_{j,k}$ via clockwise rotation by $2\pi/n$, partitioning $\mathrm{Neck}_{j,k}$ into a set orbits, written $\mathrm{Orb}_{j,k}$. To each $m \in \mathrm{Orb}_{j,k}$ we then prescribe a monomial p_m in the free algebra $\mathbb{C}\langle x, y\rangle$. This is achieved by choosing a representative of the orbit m, then starting in the bottom left corner and working clockwise, writing x for $\bullet$ and y for $\circ$. As an example, $\mathbb{Z}_6$ partitions the 15 elements of $\mathrm{Neck}_{4,2}$ into three orbits: the following illustrates representatives of the three orbits, together with the corresponding monomial $p_m \in \mathbb{C}\langle x, y\rangle$, and the number of elements in that orbit.

$$\begin{array}{ccc} xxxxyy & xxxyxy & xxyxxy \\ 6 & 6 & 3 \end{array}$$
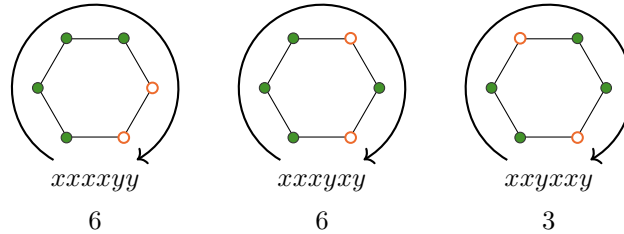
Now, given j, k, the *free necklace polynomial* $\mathcal{N}_{j,k}(x, y)$ is defined to be

$$\mathcal{N}_{j,k}(x, y) := \frac{1}{j+k} \sum_{m \in \mathrm{Orb}_{j,k}} |m| \cdot p_m,$$

which is a well defined element of $\mathbb{C}\langle x, y\rangle$, up to cyclic permutation.

For its commutative version, consider $\mathcal{N}^{\mathrm{ab}}_{j,k}$, which is defined to be the image of $\mathcal{N}_{j,k}$ under the abelianisation map $\mathbb{C}\langle x, y\rangle \to \mathbb{C}[x, y]$. The difference between $\mathcal{N}_{j,k}$ and $\mathcal{N}^{\mathrm{ab}}_{j,k}$ becomes stark as $j + k$ increases. To calibrate, $\mathcal{N}_{4,2} = x^4 y^2 + x^3 y x y + \frac{1}{2} x^2 y x^2 y$ whilst $\mathcal{N}^{\mathrm{ab}}_{4,2} = \frac{5}{2} x^4 y^2$.

1.3. **Main Results.** Given a curve C in $\mathcal{X}$ locally modelled on (1.A), as explained in §3.4–3.5 there is a DG-algebra $\mathcal{C}$ that controls the deformation theory of $\mathcal{O}_C$, in the sense that its A_∞-products describe the prorepresenting hull of the deformation functor. There are many such abstract models for $\mathcal{C}$, with the challenge being to construct one where the A_∞-products can be calculated.

Consider the graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A_i$ where

$$A_i = \begin{cases} \mathbb{C} & \text{if } i = 0, 3 \\ \mathbb{C}^2 & \text{if } i = 1, 2 \\ 0 & \text{else.} \end{cases}$$

Write x, y for the basis of A_1, and X, Y for the basis of A_2, and $\mathfrak{s}$ for the basis of A_3. The following is our main result.

Theorem 1.1 (5.13)**.** *Given a curve* C *in* $\mathfrak{X}$ *locally modelled on* (1.A)*, the following defines an* A_∞*-structure* $\{\mathsf{m}_n\}_{n \geq 2}$ *on the vector space* A *above. Furthermore, the resulting* A_∞*-algebra is quasi-isomorphic to the DG-algebra* $\mathcal{C} = \mathbf{R}\mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_{\mathrm{C}}, \mathcal{O}_{\mathrm{C}})$.

(1) *For any* $n \geq 2$ *and any decomposition* $n = j + k$ *with* $j, k \geq 0$,

$$\mathsf{m}_n(\underbrace{\mathsf{x}, \ldots, \mathsf{x}}_{j}, \underbrace{\mathsf{y}, \ldots, \mathsf{y}}_{k}) = \lambda_{j+1, k}\, \mathsf{X} + \lambda_{j, k+1}\, \mathsf{Y},$$

where the λ*'s are the coefficients from the glue in* (1.A)*.*

(2) *More generally,* m_n *with* $n \geq 2$ *applied only to degree one inputs (so, combinations of* x *and* y*) does not depend on the order of those degree one inputs, and thus is determined by* (1) *above.*

(3) *The only other non-zero products are*

$$\begin{aligned} \mathsf{m}_2(\mathsf{x}, \mathsf{X}) &= \mathfrak{s} & \mathsf{m}_2(\mathsf{y}, \mathsf{Y}) &= \mathfrak{s} \\ \mathsf{m}_2(\mathsf{X}, \mathsf{x}) &= -\mathfrak{s} & \mathsf{m}_2(\mathsf{Y}, \mathsf{y}) &= -\mathfrak{s}. \end{aligned}$$

Retrospectively, the fact that the order of the inputs does not matter in 1.1(2) should be viewed as the remnants of all the data being determined from the geometric setup (1.A), which is described from commutative gluing data.

There are two main remarkable aspects of 1.1. Perhaps the first is that describing the full A_∞-structure is possible at all, never mind with such precision on the coefficients λ_{jk}. The second is that the proof goes counter to expectations: we do not assume the abstract existence of an A_∞-structure then argue it has nice properties. Instead, the A_∞-structure is constructed from the ground up, using the Kadeishvili algorithm [K1]. One of the main novelties is that it is possible to construct, in §3.1, a uniform locally free resolution $\mathcal{E}$ of $\mathcal{O}_{\mathrm{C}}$. In turn this allows us to choose a uniform basis, which makes the computation of the full A_∞-structure possible.

Although the A_∞-structure in 1.1 turns out to be a cyclic A_∞-structure in the sense of Kontsevich–Soibelman [KS], this is a consequence of the proof, not an input. It is also unsurprising that there is some reasonable description of at least part of the A_∞-structure, given that the physics literature [AK, CM, F] predicts both a commutative and also some form of 'matrix' (=noncommutative) superpotential. As sketched in §6.5, the matrix prediction of [F] turns out to be consistent with, and mathematically explained by, the main result 1.1. Much of this paper, and our broader work, is inspired by this physics prediction.

1.4. **Corollaries.** The first main consequences of 1.1 are to deformation theory. Given a curve C in $\mathfrak{X}$ locally described by (1.A), the sheaf $\mathcal{O}_{\mathrm{C}} \in \mathrm{coh}\, \mathfrak{X}$ has an associated noncommutative (NC) deformation functor, recalled in §6.1. This functor always admits a prorepresenting hull, Λ_{def}, called the NC deformation algebra (see §6.1).

Abstractly Λ_{def} is always a superpotential algebra, however describing the superpotential has been a key open question. The following gives the first closed-formula description.

Corollary 1.2 (6.4)**.** *The* NC *deformation algebra* Λ_{def} *of* $\mathcal{O}_{\mathrm{C}} \in \mathrm{coh}\, \mathfrak{X}$ *is described by*

$$\Lambda_{\mathrm{def}} \cong \mathcal{J}\mathrm{ac}(\mathsf{W}) = \frac{\mathbb{C}\langle\!\langle x, y \rangle\!\rangle}{(\!(\delta_x \mathsf{W}, \delta_y \mathsf{W})\!)}$$

where $\mathsf{W} = \sum \lambda_{jk}\, \mathcal{N}_{j,k} \in \mathbb{C}\langle\!\langle x, y \rangle\!\rangle$ *is the sum of free necklace polynomials, and the* λ_{jk} *are the data in the glue* (1.A)*.*

The above then immediately implies a classical 1972 result of Namba [N, K2], proved using complex analysis methods and the existence of Kuranishi spaces. As is standard, taking the abelianisation of Λ_{def}, which just means formally commuting the variables, recovers commutative deformations of $\mathcal{O}_{\mathrm{C}}$. After formally commuting variables, each p_m in $\mathcal{N}_{j,k}(x, y)$ becomes $x^j y^k$, and so $\mathcal{N}_{j,k}(x, y)^{\mathrm{ab}} = \frac{1}{j+k}\binom{j+k}{k} x^j y^k$.

Corollary 1.3 (6.5, Namba, Katz)**.** *Classical commutative deformations of $\mathcal{O}_C \in \operatorname{coh} \mathcal{X}$ are prorepresented by*

$$\Lambda_{\mathrm{def}}^{\mathrm{ab}} \cong \mathcal{J}\mathrm{ac}(\mathsf{W})^{\mathrm{ab}} = \frac{\mathbb{C}[\![x,y]\!]}{(\delta_x \mathsf{V}, \delta_y \mathsf{V})}$$

where $\mathsf{V} = \sum \lambda_{jk} \, \mathcal{N}_{j,k}^{\mathrm{ab}} = \sum \frac{\lambda_{jk}}{j+k} \binom{j+k}{k} x^j y^k$.

The other consequence of 1.1 is categorical. Given a quiver with superpotential (Q, W), Ginzburg associates a 3-CY category $\mathcal{D}_\mathsf{W}$. It is a basic question in mirror symmetry to find geometric models for such categories, on both the A- and B-sides.

Corollary 1.4 (6.6)**.** *Let* $\mathsf{W} \in \mathbb{C}\langle x, y\rangle$ *and consider the associated* 3-CY *category* $\mathcal{D}_\mathsf{W}$*. If there exist scalars* λ_{jk} *for which* $\mathsf{W} = \sum \lambda_{jk} \, \mathcal{N}_{j,k}$*, then there exists a 3-fold* $\mathcal{X}$ *and smooth rational curve* $C \subset \mathcal{X}$ *such that*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathcal{X}) \supset \langle \mathcal{O}_C \rangle \cong \mathcal{D}_\mathsf{W}.$$

The above gives the first hint that perhaps not all superpotentials can be realised, whilst simultaneously not giving any hint on what general reasonable restrictions might be, given that necklace polynomials are not preserved under automorphisms. After imposing strong extra conditions such as $\dim_\mathbb{C} \Lambda_{\mathrm{def}} < \infty$, the realisation problem is expected to be true [BW, 1.11].

1.5. **Contractibility.** In 1962 Artin [A] established that the contractibility of curves in smooth surfaces is (suitably locally) a combinatorial problem. This is no longer true for 3-folds. It has been an open question as to what should replace this in higher dimension, and only special cases such as smooth curves currently have a criterion [J].

Corollary 1.2 gives a presentation of Λ_{def}. Based on empirical evidence from many computer algebra calculations [BCP] on these presentations, and from theoretical evidence in [BW], we conjecture that in the 3-fold setting Artin's combinatorics are replaced by numerical properties of Λ_{def}. One of the advantages of the NC deformation theory technology is that it finally permits clean statements for multi-curves.

Conjecture 1.5 ('3-fold Artin contractibility')**.** *Let* C *be connected union of* n *smooth rational curves in a* CY 3-*fold, and write* Λ_{def} *for its multi-pointed* NC *deformation algebra. Then, in a formal neighbourhood of* C*, the following are equivalent.*

(1) C *contracts to a point, with an isomorphism elsewhere.*
(2) $\dim_\mathbb{C} \Lambda_{\mathrm{def}} < \infty$*.*

This conjecture is a reinterpretation of Artin's results for surfaces, where crepant curves contract if and only if they are in ADE formation. The NC deformation algebra in this case is the preprojective algebra, which is finite dimensional if and only if the curve configuration is ADE. Note also for 3-folds that the conjecture is numerical, but not combinatorial: to illustrate this, whilst the superpotential determined in 1.1 is a combinatorial object, determining whether the numerical condition $\dim_\mathbb{C} \Lambda_{\mathrm{def}} < \infty$ holds is much more subtle.

The direction $(\Rightarrow)$ of 1.5 is known, by [DW]. In the case of single curves, [BW, 1.10] shows that the only case of $(\Leftarrow)$ that remains open is when the normal bundle is $\mathcal{O}(-3) \oplus \mathcal{O}(1)$. The results of this paper give strong evidence towards, but do not quite yet prove, that final case.

Conventions. We work over an algebraically closed field $\mathbb{K}$. Commas will suppressed on the subscripts on scalars wherever possible, so e.g. $\lambda_{j,k} = \lambda_{jk}$. A further restriction to $\mathbb{C}$, or any algebraically closed field of characteristic zero, is made in §6. This is needed for the deformation theory to work nicely, and also to be able to integrate relations to obtain a superpotential, where the free necklace polynomials first appear.

2. Setup and Preliminaries

This section introduces $C \subset \mathcal{X}$, describes its basic properties, and sets notation that will be used throughout.

2.1. **Gluing of a $(-3,1)$ neighbourhood.** Choose finitely many non-zero scalars $\lambda_{jk} \in \mathbb{K}$ for all $j, k \geq 0$, where $\lambda_{0,0} = \lambda_{1,0} = \lambda_{0,1} = 0$, and consider the scheme $\mathcal{X}$ defined by gluing the two open patches $\mathcal{U}_1 = \mathbb{A}^3_{a,v_2,v_1}$ and $\mathcal{U}_2 = \mathbb{A}^3_{b,w_2,w_1}$ along

$$(a, v_2, v_1) \xmapsto{a \neq 0} (a^{-1}, \, a^{-1}v_2, \, a^3 v_1 + \sum_{j,k \geq 0} \lambda_{jk} a^{2-k} v_2^{j+k-1}). \tag{2.A}$$

It is easily checked that the inverse map is given by

$$(b, w_2, w_1) \xmapsto{b \neq 0} (b^{-1}, \, b^{-1}w_2, \, b^3 w_1 - \sum_{j,k \geq 0} \lambda_{jk} b^{2-j} w_2^{j+k-1}). \tag{2.B}$$

Write $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ for the above open cover, with common open $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$.

The ideal (v_1, v_2) glues with the ideal (w_1, w_2) to give an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$, the quotient of which defines $C \subset \mathcal{X}$.

Remark 2.1. If *all* $\lambda_{j,k}$ are zero, then $\mathcal{X}$ is the total space of the vector bundle $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Furthermore, in this case the zero section $C = \mathbb{P}^1$ is very well known to have non-commutative deformation algebra $\mathbb{K}\langle\!\langle x, y \rangle\!\rangle$, and commutative deformation algebra $\mathbb{K}[x, y]$. In §6 we will interpret this as being given by the zero potential.

Lemma 2.2. $\mathcal{X}$ *is quasi-compact and separated.*

Proof. Since $\mathcal{X}$ has the finite open cover $\mathcal{U}_1 \cup \mathcal{U}_2$, where each $\mathcal{U}_i$ is Spec of a polynomial ring, it follows that $\mathcal{X}$ is a noetherian scheme [H, p83], and thus in particular is quasi-compact [H, 3.1.1, Ex I.1.7(b)].

Now $\mathcal{X}$ is obtained from the separated $\mathbb{K}$-schemes $\mathcal{U}_1 = \mathbb{A}^3$ and $\mathcal{U}_2 = \mathbb{A}^3$, glued together via the open subsets $\phi \colon \{a \neq 0\} \to \{b \neq 0\}$ in (2.A) above. Now $\mathcal{X} \times_{\mathbb{K}} \mathcal{X}$ is covered by $\{\mathcal{U}_1 \times \mathcal{U}_1, \mathcal{U}_1 \times \mathcal{U}_2, \mathcal{U}_2 \times \mathcal{U}_1, \mathcal{U}_2 \times \mathcal{U}_2\}$ [S, Tag 01JS], and so by the standard [S, Tag 01KJ] the scheme $\mathcal{X}$ is separated if and only if the 'diagonal' map

$$\mathcal{U}_{12} \to \mathcal{U}_1 \times \mathcal{U}_2$$

is a closed immersion. Since $\mathcal{U}_1$ and $\mathcal{U}_2$ are affine, this condition is equivalent to the surjectivity of the map

$$\mathbb{K}[a, v_2, v_1] \otimes_{\mathbb{K}} \mathbb{K}[b, w_2, w_1] \to \mathbb{K}[b^{\pm 1}, w_2, w_1]$$

sending $a \otimes b \mapsto \phi(a)b$. Since clearly $a^t \otimes w_1^{i_1} w_2^{i_2} \mapsto b^{-t} w_1^{i_1} w_2^{i_2}$ and $1 \otimes b^t w_1^{i_1} w_2^{i_2} \mapsto b^t w_1^{i_1} w_2^{i_2}$, the map clearly hits the monomial basis of $\mathbb{K}[b^{\pm 1}, w_2, w_1]$, and thus is surjective. $\qquad\square$

Lemma 2.3 (Ferrari [F])**.** *With the notation as above,*

$$N_{C|\mathcal{X}} \cong \mathcal{O}(-3) \oplus \mathcal{O}(1) \iff \lambda_{20} = \lambda_{11} = \lambda_{02} = 0.$$

In the remaining case where some $\lambda_{20}, \lambda_{11}, \lambda_{02}$ is nonzero, set $\Delta = \lambda_{11}^2 - \lambda_{20}\lambda_{02}$. Then

$$N_{C|\mathcal{X}} \cong \begin{cases} \mathcal{O}(-2) \oplus \mathcal{O} & \iff \Delta = 0 \\ \mathcal{O}(-1) \oplus \mathcal{O}(-1) & \iff \Delta \neq 0. \end{cases}$$

Proof. Working mod $\mathcal{I}^2 = (v_1, v_2)^2$, the glue (1.A) becomes

$$(a, v_2, v_1) \xleftrightarrow{a \neq 0} (a^{-1}, \, a^{-1}v_2, \, a^3 v_1 + \sum_{j+k \leq 2} \lambda_{jk} a^{2-k} v_2^{j+k-1}). \tag{2.C}$$

Since $\lambda_{00} = \lambda_{10} = \lambda_{01} = 0$ by convention, by inspection of the known gluing of the total space $\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, certainly the above curve has normal bundle $\mathcal{O}(-3) \oplus \mathcal{O}(1)$ if the displayed sum in the right hand side of (2.C) is zero, equivalently $\lambda_{20} = \lambda_{11} = \lambda_{02} = 0$.

If one of $\lambda_{20}, \lambda_{11}, \lambda_{02}$ is nonzero, then mod $\mathcal{I}^2$ the gluing is

$$(a, v_2, v_1) \xleftrightarrow{a \neq 0} (a^{-1}, \, a^{-1}v_2, \, a^3 v_1 + (\lambda_{20}a^2 + \lambda_{11}a + \lambda_{02})v_2). \tag{2.D}$$

It is then a result of Ferrari [F, Appendix B] that the normal bundle is $(r-1, -r-1)$, where r is the corank of the quadratic form $\left(\begin{smallmatrix} \lambda_{20} & \lambda_{11} \\ \lambda_{11} & \lambda_{02} \end{smallmatrix}\right)$, and so the result follows. $\qquad\square$

2.2. **The A and B Polynomials.** From here on, we will consider the following setup.

Setup 2.4. Consider the glue (2.A), which by definition is given by specifying finitely many nonzero λ_{jk}. We will assume that:

(1) $\lambda_{00} = \lambda_{10} = \lambda_{01} = 0$, to ensure the existence of a closed curve.
(2) $\lambda_{20} = \lambda_{11} = \lambda_{02} = 0$, by 2.3 to ensure that the normal bundle is $\mathcal{O}(-3) \oplus \mathcal{O}(1)$.
(3) That not all λ_{jk} are zero, to exclude the easy degenerate case described in 2.1.

Under 2.4, the following three constants $\mathsf{t}, \mathsf{r}, \mathsf{s} \in \mathbb{N}$ will naturally appear throughout the analysis, as will the following polynomials A and B.

Notation 2.5. Set $\mathsf{t} = \min\{j + k \mid \lambda_{j,k} \neq 0\}$. By 2.4(3) t is defined, and by 2.4(1)(2), $\mathsf{t} \geq 3$. Furthermore, we may write

$$\sum \lambda_{jk} a^{2-k} v_2^{j+k-1} = a^{2-\mathsf{r}} v_2^{\mathsf{t}-1} A$$
$$\sum \lambda_{jk} b^{2-j} w_2^{j+k-1} = b^{2-\mathsf{s}} w_2^{\mathsf{t}-1} B \tag{2.E}$$

where $A = A(a, v_2) \in \mathcal{O}_{\mathcal{U}_1}$ and $B = B(b, w_2) \in \mathcal{O}_{\mathcal{U}_2}$ are *polynomials*, and neither a nor v_2 divides A, and neither b nor w_2 divides B. In particular, $2 - \mathsf{r} \in \mathbb{Z}$ is the most negative power of a appearing in (2.A) and $2 - \mathsf{s}$ is the most negative power of b in (2.B). Again using 2.4(1)(2), both $\mathsf{r} \geq 0$ and $\mathsf{s} \geq 0$.

Lemma 2.6. *Restricting $A \in \mathcal{O}_{\mathcal{U}_1}$ and $B \in \mathcal{O}_{\mathcal{U}_2}$ to the common open subset $\mathcal{O}_{\mathcal{U}_{12}}$, then $B = b^{\mathsf{r}+\mathsf{s}-\mathsf{t}} A$, equivalently $A = a^{\mathsf{r}+\mathsf{s}-\mathsf{t}} B$.*

Proof. In the notation of 2.5, the final components of the glue equations (2.A) and (2.B) can be written

$$b^{2-\mathsf{r}} w_1 = a^{\mathsf{r}+1} v_1 + v_2^{\mathsf{t}-1} A$$
$$a^{2-\mathsf{s}} v_1 = b^{\mathsf{s}+1} w_1 - w_2^{\mathsf{t}-1} B. \tag{2.F}$$

Rearranging for A and B gives the result. $\square$

The following more refined notation will be needed for inductive arguments later.

Notation 2.7. Given (2.E), for all $i \geq 3$ define $A_i \in \mathbb{K}[a]$ and $B_i \in \mathbb{K}[b]$ by

$$A_i = \lambda_{i,0} a^{\mathsf{r}} + \lambda_{i-1,1} a^{\mathsf{r}-1} + \ldots + \lambda_{0,i} a^{\mathsf{r}-i}$$
$$B_i = \lambda_{0,i} b^{\mathsf{s}} + \lambda_{1,i-1} b^{\mathsf{s}-1} + \ldots + \lambda_{i,0} b^{\mathsf{s}-i}.$$

By definition of t (in 2.5), $A_3 = \ldots = A_{\mathsf{t}-1} = 0$ and $B_3 = \ldots = B_{\mathsf{t}-1} = 0$. The following is the graded piece analogue of 2.6.

Lemma 2.8. *For all $i \geq 3$, we have $A_i = a^{\mathsf{r}+\mathsf{s}-i} B_i$ and so $v_2^{i-\mathsf{t}} A_i = a^{\mathsf{r}+\mathsf{s}-\mathsf{t}} w_2^{i-\mathsf{t}} B_i$.*

Proof. The statement $A_i = a^{\mathsf{r}+\mathsf{s}-i} B_i$ is immediate by inspection, since $a = b^{-1}$ on $\mathcal{U}_{12}$. The second statement follows, since by the glue (2.A) $w_2 = a^{-1} v_2$. $\square$

Inspecting the graded pieces of (2.E), note that

$$\sum_{j+k=i} \lambda_{jk} a^{2-k} v_2^{j+k-1} = a^{2-\mathsf{r}} v_2^{i-1} A_i \quad \text{and} \quad \sum_{j+k=i} \lambda_{jk} b^{2-j} w_2^{j+k-1} = b^{2-\mathsf{s}} w_2^{i-1} B_i.$$

Consequently, the A and B defined in 2.5 can be written

$$A = A_{\mathsf{t}} + v_2 A_{\mathsf{t}+1} + v_2^2 A_{\mathsf{t}+2} + \ldots$$
$$= \underbrace{v_2^{3-\mathsf{t}} A_3 + v_2^{4-\mathsf{t}} A_4 + \ldots + v_2^{-1} A_{\mathsf{t}-1}}_{=0} + \underbrace{A_{\mathsf{t}}}_{\neq 0} + v_2 A_{\mathsf{t}+1} + v_2^2 A_{\mathsf{t}+2} + \ldots \tag{2.G}$$

and similarly $B = w_2^{3-\mathsf{t}} B_3 + \ldots + w_2^{-1} B_{\mathsf{t}-1} + B_{\mathsf{t}} + w_2 B_{\mathsf{t}+1} + w_2^2 B_{\mathsf{t}+2} + \ldots$ with $B_t \neq 0$ where all terms $w_2^{-i} B_{t-i}$ with negative w_2 exponents are zero.

2.3. **Sheaves on the neighbourhood.** For $n \in \mathbb{Z}$, consider the locally free sheaves $\mathcal{O}(n)$ defined on $\mathcal{X}$ by taking the rank one free module on $\mathcal{U}_2$ and the rank one free module on $\mathcal{U}_1$, and gluing them on the common intersection $\mathcal{U}_{12}$ via the isomorphism

$$\begin{aligned}
\mathbb{K}[b^{\pm 1}, w_2, w_1] &\to \mathbb{K}[a^{\pm 1}, v_2, v_1] \\
f(b, w_2, w_1) &\mapsto a^n \cdot f(a^{-1}, a^{-1}v_2, a^3 v_1 + \textstyle\sum_{j,k \geq 0} \lambda_{jk} a^{2-k} v_2^{j+k-1}).
\end{aligned}$$

A map between sheaves $\varphi \colon \mathcal{O}(m) \to \mathcal{O}(n)$ is determined by two polynomial maps in a diagram

$$
\begin{array}{ccc}
\text{On } \mathcal{U}_1: & \mathcal{O}_{\mathcal{U}_1}(m) \xrightarrow{\;\varphi_1(a,v_2,v_1)\;} \mathcal{O}_{\mathcal{U}_1}(n) & \\[4pt]
 & {\scriptstyle a^m}\Big\uparrow \qquad\qquad\qquad {\scriptstyle a^n}\Big\uparrow & \quad\text{(2.H)} \\[4pt]
\text{On } \mathcal{U}_2: & \mathcal{O}_{\mathcal{U}_2}(m) \xrightarrow{\;\varphi_2(b,w_2,w_1)\;} \mathcal{O}_{\mathcal{U}_2}(n) &
\end{array}
$$

that commutes after substituting for glueing expressions (2.A). Although it is natural to define φ_1 and φ_2 in the coordinates on their patch, we may use expressions in any of the variables that remain regular on the given patch and commute up to the glue, e.g. in 3.1 below. The above picture will be abbreviated

$$\varphi \colon \mathcal{O}(m) \xrightarrow[\varphi_2(b,w_2,w_1)]{\varphi_1(a,v_2,v_1)} \mathcal{O}(n)$$

leaving the transition maps implicit, and similarly maps between direct sums of such bundles will be represented

$$\bigoplus \mathcal{O}(m_i) \xrightarrow[\Phi_2(b,w_2,w_1)]{\Phi_1(a,v_2,v_1)} \bigoplus \mathcal{O}(n_j)$$

where Φ_i are represented by matrices representing the maps on the two charts, and the transition maps are given by the matrices $\operatorname{diag}(a^{m_i})$ and $\operatorname{diag}(a^{n_j})$.

3. Constructing the DG-model

For any choice of scalars λ_{jk} satisfying 2.4, consider the scheme $\mathcal{X}$ defined by the gluing rule in (2.A), which contains $C \cong \mathbb{P}^1$. This section builds a particular DG-algebra that controls the deformation theory of $\mathcal{O}_C$.

In §3.1 below, which is the key new construction, we exhibit a uniform locally free resolution $\mathcal{E}$ of $\mathcal{O}_C$. Using this, together with standard results involving homological DG-algebras (in §3.2) and Čech enhancements (in §3.4), we then exhibit a complex $(\mathcal{C}, \mathbb{D})$ which computes the modules $\operatorname{Ext}^i_{\mathcal{X}}(\mathcal{O}_C, \mathcal{O}_C)$. After some some sign adjustments to multiplication (in §3.5), we obtain an explicit DG-algebra $(\mathcal{C}, \star, \mathbb{D})$ that models $\mathbf{R}\operatorname{Hom}_{\mathcal{X}}(\mathcal{O}_C, \mathcal{O}_C)$.

3.1. **Locally Free Resolution of $\mathcal{O}_C$.** This subsection constructs a locally free resolution of the structure sheaf $\mathcal{O}_C$. Recall the polynomials $A \in \mathbb{C}[a, v_2]$ and $B \in \mathbb{C}[b, w_2]$ from 2.5 and the abbreviated notation for maps of sheaves on $\mathcal{X} = \mathcal{U}_1 \cup \mathcal{U}_2$ from §2.3.

Theorem 3.1. *Under Setup* 2.4, *the following is a locally free resolution of* $\mathcal{O}_C$.

$$0 \longrightarrow \mathcal{O}(-\mathsf{r}-\mathsf{s}) \xrightarrow[\left(\begin{smallmatrix} w_2 \\ -1 \\ -b^{\mathsf{s}+1}\end{smallmatrix}\right)]{\left(\begin{smallmatrix} v_2 \\ -a^{\mathsf{r}+1} \\ -1\end{smallmatrix}\right)} \begin{matrix}\mathcal{O}(1-\mathsf{r}-\mathsf{s}) \\ \oplus \\ \mathcal{O}(1-\mathsf{s}) \\ \oplus \\ \mathcal{O}(1-\mathsf{r})\end{matrix} \xrightarrow[\left(\begin{smallmatrix} 1 & w_2 & 0 \\ w_2^{\mathsf{t}-2}B & -a^{2-\mathsf{s}}v_1 & w_1 \\ -b^{\mathsf{s}+1} & 0 & -w_2\end{smallmatrix}\right)]{\left(\begin{smallmatrix} a^{\mathsf{r}+1} & v_2 & 0 \\ v_2^{\mathsf{t}-2}A & -v_1 & b^{2-\mathsf{r}}w_1 \\ -1 & 0 & -v_2\end{smallmatrix}\right)} \begin{matrix}\mathcal{O}(2-\mathsf{s}) \\ \oplus \\ \mathcal{O}(-1) \\ \oplus \\ \mathcal{O}(2-\mathsf{r})\end{matrix} \xrightarrow[\left(a^{2-\mathsf{s}}v_1 \;\; w_2 \;\; w_1\right)]{\left(v_1 \;\; v_2 \;\; b^{2-\mathsf{r}}w_1\right)} \mathcal{O}$$

where in the top $b^{2-\mathsf{r}}w_1 = a^{\mathsf{r}+1}v_1 + v_2^{\mathsf{t}-1}A$, *and in the bottom* $a^{2-\mathsf{s}}v_1 = b^{\mathsf{s}+1}w_1 - w_2^{\mathsf{t}-1}B$.

Proof. We first check that the given maps on charts glue to maps of sheaves as indicated. On the left, since $w_2 = a^{-1}v_2$ and $b = a^{-1}$, we have

$$\begin{pmatrix} a^{1-\mathsf{r}-\mathsf{s}} & 0 & 0 \\ 0 & a^{1-\mathsf{s}} & 0 \\ 0 & 0 & a^{1-\mathsf{r}}\end{pmatrix}\begin{pmatrix} a^{-1}v_2 \\ -1 \\ -a^{-\mathsf{s}-1}\end{pmatrix} = \begin{pmatrix} a^{-\mathsf{r}-\mathsf{s}}v_2 \\ -a^{1-\mathsf{s}} \\ -a^{-\mathsf{r}-\mathsf{s}}\end{pmatrix} = \begin{pmatrix} v_2 \\ -a^{\mathsf{r}+1} \\ -1\end{pmatrix}\cdot a^{-\mathsf{r}-\mathsf{s}}$$

so the local charts glue to a map $\mathcal{O}(-\mathsf{r}-\mathsf{s}) \to \mathcal{O}(1-\mathsf{r}-\mathsf{s}) \oplus \mathcal{O}(1-\mathsf{s}) \oplus \mathcal{O}(1-\mathsf{r})$.

For the middle map, substituting also for $B = a^{\mathsf{t}-\mathsf{r}-\mathsf{s}}A$ by 2.6, we have

$$\begin{pmatrix} a^{\mathsf{r}+1} & v_2 & 0 \\ v_2^{\mathsf{t}-2}A & -v_1 & b^{2-\mathsf{r}}w_1 \\ -1 & 0 & -v_2 \end{pmatrix} \begin{pmatrix} a^{1-\mathsf{r}-\mathsf{s}} & 0 & 0 \\ 0 & a^{1-\mathsf{s}} & 0 \\ 0 & 0 & a^{1-\mathsf{r}} \end{pmatrix} = \begin{pmatrix} a^{2-\mathsf{s}} & a^{1-\mathsf{s}}v_2 & 0 \\ a^{1-\mathsf{r}-\mathsf{s}}v_2^{\mathsf{t}-2}A & -a^{1-\mathsf{s}}v_1 & a^{-1}w_1 \\ -a^{1-\mathsf{r}-\mathsf{s}} & 0 & -a^{1-\mathsf{r}}v_2 \end{pmatrix}$$

$$= \begin{pmatrix} a^{2-\mathsf{s}} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a^{2-\mathsf{r}} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}v_2 & 0 \\ a^{2-\mathsf{r}-\mathsf{s}}v_2^{\mathsf{t}-2}A & -a^{2-\mathsf{s}}v_1 & w_1 \\ -a^{-\mathsf{s}-1} & 0 & -a^{-1}v_2 \end{pmatrix}$$

so the local charts glue to a map as indicated. Finally, on the right,

$$\begin{pmatrix} v_1 & v_2 & a^{\mathsf{r}-2}w_1 \end{pmatrix} \begin{pmatrix} a^{2-\mathsf{s}} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a^{2-\mathsf{r}} \end{pmatrix} = \begin{pmatrix} a^{2-\mathsf{s}}v_1 & a^{-1}v_2 & w_1 \end{pmatrix}$$

so the local charts glue to a map to the trivial bundle. The sequence of maps of sheaves is clearly a complex, so it remains to check that it is exact and has cokernel $\mathcal{O}_C$.

Exactness of the complex is local, and it is enough to check exactness on the two patches separately. We do this for the first patch in coordinates $\mathcal{O}_{\mathcal{U}_1} = \mathbb{C}[a, v_2, v_1]$, with the second patch being similar. The left-most nonzero map is clearly injective. If $(r_1, r_2, r_3)^T$ belongs to the kernel of $\mathcal{O}^3 \to \mathcal{O}^3$, then $r_1 = -v_2 r_3$ and

$$0 = a^{\mathsf{r}+1}r_1 + v_2 r_2 = (-a^{\mathsf{r}+1}r_3 + r_2)v_2$$

so that $r_2 = a^{\mathsf{r}+1}r_3$, as $\mathcal{O}_{\mathcal{U}_1}$ is a domain. Thus $(r_1, r_2, r_3)^T$ lies in the image and the complex is exact at this point.

If $(r_1, r_2, r_3)^T$ belongs to the kernel of the right-hand map $\mathcal{O}^3 \to \mathcal{O}$ then by (2.F)

$$v_1 r_1 + v_2 r_2 + (a^{\mathsf{r}+1}v_1 + v_2^{\mathsf{t}-1}A)r_3 = 0.$$

Rewriting this as $(r_1 + a^{\mathsf{r}+1}r_3)v_1 = (-r_2 - v_2^{\mathsf{t}-2}Ar_3)v_2$ shows that $r_1 + a^{\mathsf{r}+1}r_3 = Cv_2$ and $-r_2 - v_2^{\mathsf{t}-2}Ar_3 = Cv_1$ for some $C \in \mathcal{O}_{\mathcal{U}_1}$, since $\mathcal{O}_{\mathcal{U}_1}$ is a UFD. Thus $(r_1, r_2, r_3)^T$ is the image of $(-r_3, C, 0)^T$, as required.

Finally, it is clear that $\mathcal{O}_C$ is the cokernel, since the equation $b^{2-\mathsf{r}}w_1 = a^{\mathsf{r}+1}v_1 + v_2^{\mathsf{t}-1}A$, together with $\mathsf{t} \geq 2$, shows that $b^{2-\mathsf{r}}w_1 \in (v_1, v_2) = I_C$. □

3.2. **Homological DG-algebra.** To immediately ease notation, write

$$\mathcal{E} := \quad 0 \to \mathcal{E}_3 \xrightarrow{\mathsf{d}_3} \mathcal{E}_2 \xrightarrow{\mathsf{d}_2} \mathcal{E}_1 \xrightarrow{\mathsf{d}_1} \mathcal{E}_0 \to 0$$

for the complex of locally free sheaves on $\mathcal{X}$ constructed in 3.1, and consider the homological DG-algebra $(\mathcal{E}\mathrm{nd}_{\mathcal{X}}(\mathcal{E}), \circ, \delta)$ defined by $\mathcal{E}\mathrm{nd}_{\mathcal{X}}(\mathcal{E}) = \bigoplus_{i \in \mathbb{Z}} \mathcal{H}\mathrm{om}^i(\mathcal{E}, \mathcal{E})$, where

$$\mathcal{H}\mathrm{om}^i(\mathcal{E}, \mathcal{E}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}\mathrm{om}(\mathcal{E}_n, \mathcal{E}_{n-i}). \tag{3.A}$$

By (3.A), a homogeneous element $\mathsf{b} \in \mathcal{H}\mathrm{om}^i(\mathcal{E}, \mathcal{E})$ of $\mathcal{E}\mathrm{nd}_{\mathcal{X}}(\mathcal{E})$ decomposes as $\mathsf{b} = (\mathsf{b}_n)_{n \in \mathbb{Z}}$ with $\mathsf{b}_n \in \mathcal{H}\mathrm{om}(\mathcal{E}_n, \mathcal{E}_{n-i})$, and we will refer to b_n as the *nth component* of the homogenous element b.

Composition $\circ$ gives rise to a product on $\mathcal{E}\mathrm{nd}_{\mathcal{X}}(\mathcal{E})$ which preserves the grading. Furthermore, there is a differential $\delta \colon \mathcal{H}\mathrm{om}^i(\mathcal{E}, \mathcal{E}) \to \mathcal{H}\mathrm{om}^{i+1}(\mathcal{E}, \mathcal{E})$ is defined by the nth component being

$$(\delta \mathsf{a})_n := \mathsf{d}_{n-i} \circ \mathsf{a}_n - (-1)^i \mathsf{a}_{n-1} \circ \mathsf{d}_n. \tag{3.B}$$

In the following picture, visually this is down-across $-(-1)^i$ across-down.

$$
\begin{array}{ccc}
\mathcal{E}_n & \xrightarrow{\mathsf{d}_n} & \mathcal{E}_{n-1} \\
\downarrow{\scriptstyle \mathsf{a}_n} & & \downarrow{\scriptstyle \mathsf{a}_{n-1}} \\
\mathcal{E}_{n-i} & \xrightarrow[\mathsf{d}_{n-i}]{} & \mathcal{E}_{n-i-1}
\end{array}
$$

Lemma 3.2. $\delta^2(\mathsf{a}) = 0$ *and* $\delta(\mathsf{a}_{|\mathcal{U}}) = \delta(\mathsf{a})_{|\mathcal{U}}$ *for any* $\mathsf{a} \in \mathcal{H}\mathrm{om}^i(\mathcal{E}, \mathcal{E})$ *and any open* $\mathcal{U} \subset \mathcal{X}$.

Proof. The first claim is standard, and the second claim is the statement that d_n and a are maps of sheaves, so are defined by composition on the open $\mathcal{U}$. □

3.3. **Čech complex.** This subsection sets notation for the Čech complex $(\check{\mathrm{C}}(\mathcal{U}, \mathcal{F}), \mathfrak{d})$ of a sheaf $\mathcal{F}$ on $\mathcal{X}$ relative to the open cover $\mathcal{X} = \mathcal{U}_1 \cup \mathcal{U}_2$. Recall that $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$.

The only nonzero terms in the Čech complex are $\check{\mathrm{C}}^0 = \mathcal{F}|_{\mathcal{U}_1} \oplus \mathcal{F}|_{\mathcal{U}_2}$ and $\check{\mathrm{C}}^1 = \mathcal{F}|_{\mathcal{U}_{12}}$. For $(\mathsf{a}_1, \mathsf{a}_2) \in \check{\mathrm{C}}^0$, the coboundary $\mathfrak{d} \colon \check{\mathrm{C}}^0 \to \check{\mathrm{C}}^1$ is the map

$$\mathfrak{d}(\mathsf{a}_1, \mathsf{a}_2) = \mathsf{a}_{1|\mathcal{U}_{12}} - \mathsf{a}_{2|\mathcal{U}_{12}}. \tag{3.C}$$

3.4. **The total complex.** Following [AK], consider $\mathcal{C} = \bigoplus_{n \in \mathbb{Z}} \mathcal{C}_n$ where

$$\mathcal{C}_n := \bigoplus_{p+q=n} \check{\mathrm{C}}^p(\mathcal{U}, \mathcal{Hom}^q(\mathcal{E}, \mathcal{E})),$$

with differential $\mathbb{D} = \mathfrak{d} + (-1)^p \delta$ described below.

Unpacking this, since only $\check{\mathrm{C}}^0$ and $\check{\mathrm{C}}^1$ are nonzero, a homogenous element $\mathsf{a} \in \mathcal{C}_i$ is a triple of vectors of homomorphisms

$$\mathsf{a} = (\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_{12})$$

where

$$\mathsf{a}_1 \in \bigoplus_{n \in \mathbb{Z}} \mathcal{Hom}_{\mathcal{U}_1}(\mathcal{E}_n, \mathcal{E}_{n-i}), \;\; \mathsf{a}_2 \in \bigoplus_{n \in \mathbb{Z}} \mathcal{Hom}_{\mathcal{U}_2}(\mathcal{E}_n, \mathcal{E}_{n-i}), \;\; \mathsf{a}_{12} \in \bigoplus_{n \in \mathbb{Z}} \mathcal{Hom}_{\mathcal{U}_{12}}(\mathcal{E}_n, \mathcal{E}_{n-i+1})$$

The differential $\mathbb{D} \colon \mathcal{C}_i \to \mathcal{C}_{i+1}$ is defined by

$$(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_{12}) \mapsto (\delta \mathsf{a}_1, \delta \mathsf{a}_2, (\mathsf{a}_{1\mathcal{U}_{12}} - \mathsf{a}_{2|\mathcal{U}_{12}}) - \delta \mathsf{a}_{12}) \tag{3.D}$$

where δ is from (3.B) and we have applied $\mathfrak{d}$ from (3.C).

As is well-known, the complex $(\mathcal{C}, \mathbb{D})$ computes the cohomology $\mathrm{Ext}_{\mathcal{X}}^i(\mathcal{O}_{\mathrm{C}}, \mathcal{O}_{\mathrm{C}})$.

Proposition 3.3. $\mathrm{H}^i(\mathcal{C}) \cong \mathrm{Ext}_{\mathcal{X}}^i(\mathcal{O}_{\mathrm{C}}, \mathcal{O}_{\mathrm{C}})$ *for all* $i \in \mathbb{Z}$.

Proof. $(\mathcal{C}, \mathbb{D})$ as defined above is equal to the complex $\mathrm{Hom}_{\mathrm{D}_\infty^{\mathrm{b}}(\mathcal{X})}^\bullet(\mathcal{E}, \mathcal{E})$ as defined in [A+, §8.2.1]. Thus

$$\mathrm{H}^i(\mathcal{C}) \cong \mathrm{H}^i(\mathrm{Hom}_{\mathrm{D}_\infty^{\mathrm{b}}(\mathcal{X})}^\bullet(\mathcal{E}, \mathcal{E})) \cong \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{X})}(\mathcal{E}, \mathcal{E}[i]) \cong \mathrm{Ext}_{\mathcal{X}}^i(\mathcal{O}_{\mathrm{C}}, \mathcal{O}_{\mathrm{C}}),$$

where the middle isomorphism is e.g. [A+, p587]. $\square$

3.5. **The Čech enhancement.** Using the well-known Čech enhancement described in e.g. [A+, p587] or [CS, §1.2], the complex $(\mathcal{C}, \mathbb{D})$ can be upgraded to a DG-algebra. The only slightly subtle point is the sign on the composition, which we explicitly recall here.

Definition 3.4. Given homogeneous $\mathsf{a} = (\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_{12}) \in \mathcal{C}_i$ and $\mathsf{b} = (\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_{12}) \in \mathcal{C}_j$, define

$$\mathsf{a} \star \mathsf{b} := (\mathsf{a}_1 \circ \mathsf{b}_1, \;\; \mathsf{a}_2 \circ \mathsf{b}_2, \;\; \mathsf{a}_{12} \circ \mathsf{b}_{2|\mathcal{U}_{12}} + (-1)^i \mathsf{a}_{1|\mathcal{U}_{12}} \circ \mathsf{b}_{12}) \in \mathcal{C}_{i+j} \tag{3.E}$$

and extend $\star$ to all of $\mathcal{C}$ by linearity.

To set notation for the next result, choose an injective resolution

$$0 \to \mathcal{O}_{\mathrm{C}} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \dots$$

of $\mathcal{O}_{\mathrm{C}}$. Then consider the DG-algebra $\mathrm{End}_{\mathcal{X}}^{\mathrm{DG}}(\mathcal{I}) = \bigoplus_{t \in \mathbb{Z}} \mathrm{End}_{\mathcal{X}}^{\mathrm{DG}}(\mathcal{I})_t$, where

$$\mathrm{End}_{\mathcal{X}}^{\mathrm{DG}}(\mathcal{I})_t := \{(f_s)_{s \in \mathbb{Z}} \mid f_s \colon \mathcal{I}_s \to \mathcal{I}_{s+t}\}.$$

Multiplication is given by composition, and the differential is defined as in §3.2.

Proposition 3.5. $(\mathcal{C}, \star, \mathbb{D})$ *is a* DG-*algebra, and this is quasi-isomorphic to the* DG-*algebra* $\mathrm{End}_{\mathcal{X}}^{\mathrm{DG}}(\mathcal{I})$.

Proof. The first statement can be checked manually. Alternatively, as in the proof of 3.3, $(\mathcal{C}, \mathbb{D})$ equals the complex $\mathrm{Hom}_{\mathrm{D}_\infty^{\mathrm{b}}(\mathcal{X})}^\bullet(\mathcal{E}, \mathcal{E})$. Now $\mathrm{D}_\infty^{\mathrm{b}}(\mathcal{X})$ is in fact a DG category, under the composition described in [A+, p587]. In our restricted setting, with only two open affine sets, this translates precisely into the operation $\star$ defined above (see also [AK]). The second statement follows since $\mathcal{X}$ is quasi-compact and separated by 2.2, so the Čech DG enhancement of perfect complexes is quasi-equivalent to the injective DG enhancement of perfect complexes; see [CS, §1.2] or [LS, 3.19]. $\square$

4. Generators and Homotopies

This section constructs certain elements of the DG-algebra $(\mathcal{C}, \star, \mathbb{D})$ from §3.5. In §4.1–4.4 some key elements of $\mathcal{C}$ are introduced. Various easy relations between these elements involving both $\mathbb{D}$ and $\star$ are then computed in §4.5 and §4.6. Inductive notation is introduced in §4.7. The totality of elements considered is summarised in §4.8. The whole section is elementary, essentially being nothing more than multiplication of matrices, with care taken about degrees.

To set and ease notation throughout, under Setup 2.4 write

$$0 \to \mathcal{E}_3 \xrightarrow{d_3} \mathcal{E}_2 \xrightarrow{d_2} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \to 0$$

for the complex of locally free sheaves on $\mathcal{X}$ constructed in 3.1 that resolves $\mathcal{O}_C$.

4.1. **Degree One Generators.** Consider $\mathsf{x} := (\mathsf{x}_1, \mathsf{x}_2, 0) \in \mathcal{C}_1$, where

$$
\mathsf{x}_1 =
\begin{array}{ccccc}
 & \mathcal{E}_3 & \xrightarrow{\begin{pmatrix} v_2 \\ -a^{r+1} \\ -1 \end{pmatrix}} & \mathcal{E}_2 & \xrightarrow{\begin{pmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{pmatrix}} & \mathcal{E}_1 \to \mathcal{E}_0 \\
\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \Big\downarrow & & \begin{pmatrix} 0 & -1 & 0 \\ v_2^{t-3}A & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Big\downarrow & & \Big\downarrow (0\ -1\ 0) \\
\mathcal{E}_3 \to \mathcal{E}_2 & \xrightarrow{\begin{pmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{pmatrix}} & \mathcal{E}_1 & \xrightarrow{(v_1\ v_2\ b^{2-r}w_1)} & \mathcal{E}_0
\end{array}
$$

$$
\mathsf{x}_2 =
\begin{array}{ccccc}
 & \mathcal{E}_3 & \xrightarrow{\begin{pmatrix} w_2 \\ -1 \\ -b^{s+1} \end{pmatrix}} & \mathcal{E}_2 & \xrightarrow{\begin{pmatrix} 1 & w_2 & 0 \\ w_2^{t-2}B & -a^{2-s}v_1 & w_1 \\ -b^{s+1} & 0 & -w_2 \end{pmatrix}} & \mathcal{E}_1 \to \mathcal{E}_0 \\
b\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \Big\downarrow & & b\begin{pmatrix} 0 & -1 & 0 \\ w_2^{t-3}B & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Big\downarrow & & \Big\downarrow b(0\ -1\ 0) \\
\mathcal{E}_3 \to \mathcal{E}_2 & \xrightarrow{\begin{pmatrix} 1 & w_2 & 0 \\ w_2^{t-2}B & -a^{2-s}v_1 & w_1 \\ -b^{s+1} & 0 & -w_2 \end{pmatrix}} & \mathcal{E}_1 & \xrightarrow{(a^{2-s}v_1\ w_2\ w_1)} & \mathcal{E}_0
\end{array}
$$

Under the sign convention defining δ in (3.B), which for odd degree is down-across+across-down, $\delta(\mathsf{x}_1) = 0 = \delta(\mathsf{x}_2)$. Further, it follows from (2.H) and 2.6 that each pair of vertical maps glue to give a global map, so $\mathfrak{d}(\mathsf{x}_1, \mathsf{x}_2) = 0$. Hence $\mathbb{D}\mathsf{x} = 0$.

Similarly, write $\mathsf{y} = (\mathsf{y}_1, \mathsf{y}_2, 0) \in \mathcal{C}_1$ where y_1 is defined by multiplying all the vertical maps of x_1 by a, and y_2 by dividing all the vertical maps of x_2 by b. It follows, in a similar way, that $\mathbb{D}(\mathsf{y}) = 0$.

Lemma 4.1. *With notation as above, x and y give linearly independent elements of* $\mathrm{H}^1(\mathcal{C}) \cong \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \cong \mathbb{C}^2$.

Proof. The first isomorphism holds by 3.3 and the second holds since $\mathcal{N}_{C|\mathcal{X}} \cong \mathcal{O}(-3) \oplus \mathcal{O}(1)$.

Both x and y are closed, as observed above, so suppose there exists $\alpha = (\alpha_1, \alpha_2, \alpha_{12}) \in \mathcal{C}_0$ such that $\mathbb{D}\alpha = \lambda\mathsf{x} + \mu\mathsf{y}$ for some $\lambda, \mu \in \mathbb{K}$. Restricting to $\mathcal{U}_1$, we have $\delta(\alpha_1) = \lambda\mathsf{x}_1 + \mu\mathsf{y}_1$, and by (3.B) and 3.1 the component $\mathcal{E}_1 \to \mathcal{E}_0$ of this element is

$$\delta(\alpha_1) = (v_1, v_2, a^{r+1}v_1 + v_2^{t-1}A)M + c(v_1, v_2, a^{r+1}v_1 + v_2^{t-1}A)$$

for some 3×3 matrix M with entries in $\mathcal{O}_{\mathcal{U}_1}$, and some $c \in \mathcal{O}_{\mathcal{U}_1}$. In particular, all entries of $\delta(\alpha_1)$ lie in the ideal $\langle v_1, v_2 \rangle \subset \mathcal{O}_{\mathcal{U}_1}$ and so vanish on C. In contrast, the second entry $-(\lambda + \mu a)$ of $\lambda\mathsf{x}_1 + \mu\mathsf{y}_1 = \lambda(0, -1, 0) + \mu(0, -a, 0)$ does not vanish on C unless $\lambda = \mu = 0$. $\square$

4.2. **Shapes and Degree One Homotopies.** Alongside closed elements of $\mathcal{C}$ such as x and y of §4.1 that appear in Theorem 1.1, it is convenient to define simpler elements of $\mathcal{C}$ that we call *shapes*, which are akin to elementary matrices. As we will see repeatedly, these shapes will help distinguish between identities that hold for trivial reasons — matrix

products vanishing because all terms have a factor of zero — and those that vanish because some nontrivial cancellation occurs.

With this in mind, and adopting the convention that degree one elements are denoted in lower case sans font, consider $(\mathsf{g}_1, \mathsf{g}_2, 0) \in \mathcal{C}_1$ defined by

$$\mathsf{g}_1 = \quad \begin{array}{ccccc} & \mathcal{E}_3 \xrightarrow{\begin{pmatrix} v_2 \\ -a^{r+1} \\ -1 \end{pmatrix}} \mathcal{E}_2 \xrightarrow{\begin{pmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{pmatrix}} \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \\[2em] \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\downarrow \qquad \qquad \qquad \downarrow\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \qquad \downarrow\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\[2em] \mathcal{E}_3 \longrightarrow \mathcal{E}_2 \xrightarrow{\begin{pmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{pmatrix}} \mathcal{E}_1 \xrightarrow{\begin{pmatrix} v_1 & v_2 & b^{2-r}w_1 \end{pmatrix}} \mathcal{E}_0 \end{array}$$

with g_2 being given by the same three vertical matrices, but considered on the other chart.

Similarly, define $(\mathsf{h}_1, \mathsf{h}_2, 0) \in \mathcal{C}_1$ by

$$\mathsf{h}_1 = \quad \begin{array}{ccccc} & \mathcal{E}_3 \xrightarrow{\begin{pmatrix} v_2 \\ -a^{r+1} \\ -1 \end{pmatrix}} \mathcal{E}_2 \xrightarrow{\begin{pmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{pmatrix}} \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \\[2em] \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\downarrow \qquad \qquad \qquad \downarrow\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \qquad \downarrow\begin{pmatrix} -1 & 0 & 0 \end{pmatrix} \\[2em] \mathcal{E}_3 \longrightarrow \mathcal{E}_2 \xrightarrow{\begin{pmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{pmatrix}} \mathcal{E}_1 \xrightarrow{\begin{pmatrix} v_1 & v_2 & b^{2-r}w_1 \end{pmatrix}} \mathcal{E}_0 \end{array}$$

with h_2 being given by the same three vertical matrices, but considered on the other chart.

Notation 4.2. For g and h defined above, define $\mathsf{k}_i \in \mathcal{C}_1$ by

$$\mathsf{k}_i := \begin{cases} (a^i \cdot \mathsf{g}_1, \ b^{r-2-i} \cdot \mathsf{g}_2, 0) & \text{if } 0 \leq i \leq \mathsf{r} - 2 \\ (a^{i-(\mathsf{r}+1)} \cdot \mathsf{h}_1, \ b^{(\mathsf{r}+\mathsf{s}-1)-i} \cdot \mathsf{h}_2, 0) & \text{if } \mathsf{r} + 1 \leq i \leq \mathsf{r} + \mathsf{s} - 1 \end{cases}$$

where terms such as $a^i \cdot \mathsf{g}_1$ should be interpreted as multiplying all the three vertical maps of g_1 by the coefficient a^i.

Remark 4.3. In 4.2, and in similar definitions such as 4.4 below, we say that k_i is *based on the shape* g (or h), as a reminder that its three components are polynomial multiples of the corresponding components of g (or h). This is not strictly essential, but it does help to navigate the array of constructions, and is summarised in §4.8.

Lastly consider the shape $(\mathsf{z}_1, \mathsf{z}_2, 0) \in \mathcal{C}_1$, defined as

$$\mathsf{z}_1 = \quad \begin{array}{ccccc} & \mathcal{E}_3 \xrightarrow{\begin{pmatrix} v_2 \\ -a^{r+1} \\ -1 \end{pmatrix}} \mathcal{E}_2 \xrightarrow{\begin{pmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{pmatrix}} \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \\[2em] \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\downarrow \qquad \qquad \qquad \downarrow\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \qquad \downarrow\begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \\[2em] \mathcal{E}_3 \longrightarrow \mathcal{E}_2 \xrightarrow{\begin{pmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{pmatrix}} \mathcal{E}_1 \xrightarrow{\begin{pmatrix} v_1 & v_2 & b^{2-r}w_1 \end{pmatrix}} \mathcal{E}_0 \end{array}$$

with z_2 being given by the same three vertical matrices, but considered on the other chart. This shape will appear again as a killing homotopy in §4.5 and §4.7.

4.3. **Degree Two Shapes and Generators.** Adopting the convention that degree two elements will be denoted in upper case sans font, consider the shape $\mathsf{Z} = (\mathsf{Z}_1, \mathsf{Z}_2, 0) \in \mathcal{C}_2$

defined as

$$
Z_1 =
\begin{array}{ccccccc}
& & \mathcal{E}_3 & \xrightarrow{\left(\begin{smallmatrix} v_2 \\ -a^{r+1} \\ -1 \end{smallmatrix}\right)} & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 \\
& & {\scriptstyle\left(\begin{smallmatrix}0\\-1\\0\end{smallmatrix}\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle(-1\ 0\ 0)} & & & & \\
& \mathcal{E}_3 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{(\,v_1\ v_2\ b^{2-r}w_1\,)} & \mathcal{E}_0 &
\end{array}
$$

with Z_2 being given by the same two vertical matrices, but considered on the other chart.

Notation 4.4. Set $X = (a^r \cdot Z_1,\ b^{s-1} \cdot Z_2, 0) \in \mathcal{C}_2$ and $Y = (a^{r-1} \cdot Z_1,\ b^s \cdot Z_2, 0) \in \mathcal{C}_2$.

Thus both $X, Y \in \mathcal{C}_2$ are based on the shape Z. Under the sign convention defining δ in (3.B), which for even degree is down-across$-$across-down, $\delta(X_1) = 0 = \delta(X_2)$. Further, it follows from (2.H) and (2.A) that the pair of vertical maps $(\mathcal{E}_3 \to \mathcal{E}_0)_{|\mathcal{U}_i}$ glue to give a global map, so $\mathfrak{d}(X_1, X_2) = 0$. Hence $\mathbb{D}X = 0$, and similarly $\mathbb{D}Y = 0$.
These give generators of $\mathrm{H}^2(\mathcal{C})$.

Lemma 4.5. X *and* Y *are linearly independent elements of* $\mathrm{H}^2(\mathcal{C}) \cong \mathrm{Ext}^2(\mathcal{O}_C, \mathcal{O}_C) \cong \mathbb{C}^2$.

Proof. This is similar to 4.1, with only linear independence to check. Suppose there exists $c = (c_1, c_2, c_{12}) \in \mathcal{C}_1$ such that $\mathbb{D}c = \lambda X + \mu Y$ for some $\lambda, \mu \in \mathbb{K}$. Restricting to $\mathcal{U}_1$, we have $\delta(c_1) = \lambda X_1 + \mu Y_1$, and by (3.B) and 3.1 the component $\mathcal{E}_2 \to \mathcal{E}_0$ of this element is

$$
\delta(c_1) = (v_1, v_2, a^{r+1}v_1 + v_2^{t-1}A)M + (e_1, e_2, e_3)\,\mathrm{Mat}(d_2)
$$

for some 3×3 matrix M with entries in $\mathcal{O}_{\mathcal{U}_1}$, and some $e_i \in \mathcal{O}_{\mathcal{U}_1}$, where $\mathrm{Mat}(d_2)$ is the matrix of d_2 on $\mathcal{U}_1$ from 3.1. Considering only the first component of this, and equating to that of $\lambda X_1 + \mu Y_1$, we have at once that

$$
a^{r+1}e_1 - e_3 \equiv -\lambda a^r - \mu a^{r-1} \quad \mathrm{mod}\ I_C = \langle v_1, v_2 \rangle. \tag{4.A}
$$

Considering now c_{12} on $\mathcal{U}_1$, it is also immediate that the components of $\delta c_{12} \colon \mathcal{E}_1 \to \mathcal{E}_0$ all lie in I_C, since the two summands of this map each factor through d_1. Thus the condition that $(\mathbb{D}c)_{12} = 0$ implies that

$$
c_{1|\mathcal{U}_{12}} - c_{2|\mathcal{U}_{12}} \equiv 0 \quad \mathrm{mod}\ I_C. \tag{4.B}
$$

Expressing the component $c_2 \colon \mathcal{E}_1 \to \mathcal{E}_0$ in coordinates by (e_1', e_2', e_3') for $e_i' \in \mathcal{O}_{\mathcal{U}_2}$, then modulo I_C each e_i' is a polynomial in $b \in \mathcal{O}_{\mathcal{U}_2}$. In particular, modulo I_C and working in the coordinates of $\mathcal{U}_{12} \subset \mathcal{U}_1$ with $b = a^{-1}$, (4.B) implies for $e_3'(b)$ that

$$
e_3(a) \equiv a^{r-2}e_3'(a^{-1}) \quad \mathrm{mod}\ I_C.
$$

The righthand side is a polynomial in a of degree at most $r - 2$, so (4.A) implies that $\lambda = \mu = 0$. $\qquad\square$

Now consider the shape $(G_1, G_2, 0) \in \mathcal{C}_2$, defined as

$$
G_1 =
\begin{array}{ccccccc}
& & \mathcal{E}_3 & \xrightarrow{\left(\begin{smallmatrix} v_2 \\ -a^{r+1} \\ -1 \end{smallmatrix}\right)} & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 \\
& & {\scriptstyle\left(\begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle(0\ 0\ -1)} & & & & \\
& \mathcal{E}_3 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{(\,v_1\ v_2\ b^{2-r}w_1\,)} & \mathcal{E}_0 &
\end{array}
$$

with G_2 being given by the same two vertical matrices, but considered on the other chart.

Similarly, define $(\mathsf{H}_1, \mathsf{H}_2, 0)$ where

$$
\mathsf{H}_1 =
\begin{array}{ccccc}
& \mathcal{E}_3 & \xrightarrow{\left(\begin{smallmatrix} v_2 \\ -a^{r+1} \\ -1 \end{smallmatrix}\right)} & \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \\
& \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)\downarrow & & \downarrow {\left(\begin{smallmatrix} 0 & -1 & 0 \end{smallmatrix}\right)} \\
\mathcal{E}_3 \longrightarrow & \mathcal{E}_2 \longrightarrow & \mathcal{E}_1 & \xrightarrow{\left(\begin{smallmatrix} v_1 & v_2 & b^{2-r}w_1 \end{smallmatrix}\right)} & \mathcal{E}_0
\end{array}
$$

with H_2 being given by the same two vertical matrices, but considered on the other chart. The following should be viewed as the degree two version of 4.2, with slightly different superscripts, and slightly larger intervals.

Notation 4.6. With G and H defined above, define $\mathsf{K}_i \in \mathcal{C}_2$ by

$$
\mathsf{K}_i := \begin{cases} (a^i \cdot \mathsf{G}_1, \ b^{r-1-i} \cdot \mathsf{G}_2, 0) & \text{if } 0 \leq i \leq r-1 \\ (a^{i-(r+1)} \cdot \mathsf{H}_1, \ b^{r+s-i} \cdot \mathsf{H}_2, 0) & \text{if } r+1 \leq i \leq r+s. \end{cases}
$$

4.4. **Degree Three Shapes and Generator.** Adopting the convention that degree three elements will be denoted in lower case mathfrak, consider the shape $(\mathfrak{s}_1, \mathfrak{s}_2, 0) \in \mathcal{C}_3$ defined as

$$
\mathfrak{s}_1 =
\begin{array}{ccccc}
& \mathcal{E}_3 & \xrightarrow{\left(\begin{smallmatrix} v_2 \\ -a^{r+1} \\ -1 \end{smallmatrix}\right)} & \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \\
& {-1}\downarrow & & \\
\mathcal{E}_3 \longrightarrow & \mathcal{E}_2 \longrightarrow & \mathcal{E}_1 \longrightarrow \mathcal{E}_0
\end{array}
$$

with $\mathfrak{s}_2$ being given by the same vertical matrix, but considered on the other chart.

Notation 4.7. Set $\mathfrak{s} = (a^r \cdot \mathfrak{s}_1, b^s \cdot \mathfrak{s}_2, 0) \in \mathcal{C}_3$.

It is easy to verify that $\mathbb{D}(\mathfrak{s}) = 0$, and it is clear that $\mathfrak{s}$ is a basis element for $\mathrm{H}^3(\mathcal{C}) \cong \mathbb{K}$.

4.5. **Elementary Relationships Involving $\mathbb{D}$.** This subsection establishes some elementary relationships between the degree one elements in §4.1, the degree two elements in §4.3, and the degree three elements in §4.4 under the differential $\mathbb{D}$.

Lemma 4.8. *For any $f \in \mathcal{O}_{\mathcal{U}_1}$ and $g \in \mathcal{O}_{\mathcal{U}_2}$,*

$$
\begin{aligned}
\mathbb{D}(f \cdot \mathsf{g}_1, g \cdot \mathsf{g}_2, 0) &= (\quad f \cdot \mathsf{Z}_1, \ b^{s+1}g \cdot \mathsf{Z}_2, \ (f - a^{r-2}g) \cdot \mathsf{Z}_1|_{\mathcal{U}_{12}}) \\
\mathbb{D}(f \cdot \mathsf{h}_1, g \cdot \mathsf{h}_2, 0) &= (a^{r+1}f \cdot \mathsf{Z}_1, \qquad g \cdot \mathsf{Z}_2, \ (f - a^{s-2}g) \cdot \mathsf{Z}_1|_{\mathcal{U}_{12}}) \\
\mathbb{D}(f \cdot \mathsf{z}_1, g \cdot \mathsf{z}_2, 0) &= (\quad v_2 f \cdot \mathsf{Z}_1, \quad w_2 g \cdot \mathsf{Z}_2, \ (f - a^{r+s-2}g) \cdot \mathsf{Z}_1|_{\mathcal{U}_{12}}).
\end{aligned}
$$

Proof. (1) By definition, $\mathbb{D}(f \cdot \mathsf{g}_1, g \cdot \mathsf{g}_2, 0) = (\delta(f \cdot \mathsf{g}_1), \delta(g \cdot \mathsf{g}_2), f \cdot \mathsf{g}_1|_{12} - g \cdot \mathsf{g}_2|_{12})$. We first claim that $\delta(\mathsf{g}_1) = \mathsf{Z}_1$ and $\delta(\mathsf{g}_2) = b^{s-1} \cdot \mathsf{Z}_2$. Under the sign convention in (3.B), which for degree one is down-across+across-down, the differential $\delta(\mathsf{g}_1)$ equals

$$
\begin{array}{ccccc}
& \mathcal{E}_3 \longrightarrow & \mathcal{E}_2 \longrightarrow & \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \\
\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} v_2 \\ -a^{r+1} \\ -1 \end{smallmatrix}\right) & \downarrow & & \downarrow {\left(\begin{smallmatrix} v_1 & v_2 & b^{2-r}w_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a^{r+1} & v_2 & 0 \\ v_2^{t-2}A & -v_1 & b^{2-r}w_1 \\ -1 & 0 & -v_2 \end{smallmatrix}\right)} \\
\mathcal{E}_3 \longrightarrow & \mathcal{E}_2 \longrightarrow & \mathcal{E}_1 \longrightarrow \mathcal{E}_0
\end{array}
$$

which is Z_1. Similarly $\delta(\mathsf{g}_2)$ equals

$$
\begin{array}{ccccc}
\mathcal{E}_3 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 \\
\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} w_2 \\ -1 \\ -b^{s+1} \end{smallmatrix}\right) \downarrow & & \downarrow & & \downarrow \\
\mathcal{E}_3 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0
\end{array}
$$

with the upper-right map
$$
\left(\, a^{2-s} v_1 \ w_2 \ w_1 \,\right)\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) + \left(\, 0 \ 0 \ 1 \,\right)\left(\begin{smallmatrix} 1 & w_2 & 0 \\ w_2^{t-2}B & -a^{2-s}v_1 & w_1 \\ -b^{s+1} & 0 & -w_2 \end{smallmatrix}\right)
$$

which is $b^{s+1} \cdot \mathsf{Z}_2$. To calculate the final entry $f \cdot \mathsf{g}_1|_{12} - g \cdot \mathsf{g}_2|_{12}$, we work on the $\mathcal{U}_1$ coordinates on $\mathcal{U}_{12}$. The first non-zero map in $f \cdot \mathsf{g}_1|_{12} - g \cdot \mathsf{g}_2|_{12}$ is given by

$$
\begin{array}{ccc}
\mathcal{E}_2 & \quad & \mathcal{E}_2 \xrightarrow{\left(\begin{smallmatrix} a^{r+s-1} & 0 & 0 \\ 0 & a^{s-1} & 0 \\ 0 & 0 & a^{r-1} \end{smallmatrix}\right)} \mathcal{E}_2 \\
\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & f \\ 0 & 0 & 0 \end{smallmatrix}\right)\downarrow & - & \downarrow\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & 0 & 0 \end{smallmatrix}\right) \\
\mathcal{E}_1 & & \mathcal{E}_1 \xleftarrow{\left(\begin{smallmatrix} a^{2-s} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a^{2-r} \end{smallmatrix}\right)} \mathcal{E}_1
\end{array}
\quad = \quad
\begin{array}{c}
\mathcal{E}_2 \\
(f-a^{r-2}g)\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)\downarrow \\
\mathcal{E}_1
\end{array}
$$

and the second possible non-zero map is given by

$$
\begin{array}{ccc}
\mathcal{E}_1 & \quad & \mathcal{E}_1 \xrightarrow{\left(\begin{smallmatrix} a^{s-2} & 0 & 0 \\ 0 & a^1 & 0 \\ 0 & 0 & a^{r-2} \end{smallmatrix}\right)} \mathcal{E}_1 \\
\left(\, 0 \ 0 \ f \,\right)\downarrow & - & \downarrow\left(\, 0 \ 0 \ g \,\right) \\
\mathcal{E}_0 & & \mathcal{E}_0 \xleftarrow{\ 1 \ } \mathcal{E}_0
\end{array}
\quad = \quad
\begin{array}{c}
\mathcal{E}_1 \\
(f-a^{r-2}g)\left(\, 0 \ 0 \ 1 \,\right)\downarrow \\
\mathcal{E}_0
\end{array}
$$

and so $f \cdot \mathsf{g}_1|_{12} - g \cdot \mathsf{g}_2|_{12} = (f - a^{r-2}g) \cdot \mathsf{Z}_1|_{\mathcal{U}_{12}}$ as claimed.

(2) Again by direct calculation as above, $\delta(\mathsf{h}_1) = a^{r+1} \cdot \mathsf{Z}_1$ and $\delta(\mathsf{h}_2) = \mathsf{Z}_2$. The calculation that $f \cdot \mathsf{h}_1|_{12} - g \cdot \mathsf{h}_2|_{12} = (f - a^{s-2}g) \cdot \mathsf{Z}_1|_{\mathcal{U}_{12}}$ is also similar.

(3) Here, again by direct calculation, $\delta(\mathsf{z}_1) = v_2 \cdot \mathsf{Z}_1$ and $\delta(\mathsf{z}_2) = w_2 \cdot \mathsf{Z}_2$. Now the only possible non-zero map in $f \cdot \mathsf{z}_1|_{12} - g \cdot \mathsf{z}_2|_{12}$ is the map $\mathcal{E}_2 \to \mathcal{E}_1$ given by

$$
\begin{array}{ccc}
\mathcal{E}_2 & \quad & \mathcal{E}_2 \xrightarrow{\left(\begin{smallmatrix} a^{r+s-1} & 0 & 0 \\ 0 & a^{s-1} & 0 \\ 0 & 0 & a^{r-1} \end{smallmatrix}\right)} \mathcal{E}_2 \\
\left(\begin{smallmatrix} 0 & 0 & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)\downarrow & - & \downarrow\left(\begin{smallmatrix} 0 & 0 & 0 \\ -g & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \\
\mathcal{E}_1 & & \mathcal{E}_1 \xleftarrow{\left(\begin{smallmatrix} a^{2-s} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a^{2-r} \end{smallmatrix}\right)} \mathcal{E}_1
\end{array}
$$

which is clearly $(f - a^{r+s-2}g) \cdot \mathsf{Z}_1|_{\mathcal{U}_{12}}$. $\qquad\square$

The following asserts that, although k_i is defined on two different intervals, the effect of applying $\mathbb{D}$ always looks the same.

Corollary 4.9. *For all i such that k_i is defined,*
$$
\mathbb{D}(\mathsf{k}_i) = (a^i \cdot \mathsf{Z}_1, \ b^{r+s-1-i} \cdot \mathsf{Z}_2, 0).
$$

Proof. In the first interval $\mathsf{k}_i = (a^i \cdot \mathsf{g}_1, b^{r-2-i} \cdot \mathsf{g}_2, 0)$, so the statement follows from the top line in 4.8 applied to $f = a^i$ and $g = b^{r-2-i}$, where $f - a^{r-2}g = 0$ since $a = b^{-1}$. Similarly, in the second interval $\mathsf{k}_i = (a^{i-(r+1)} \cdot \mathsf{h}_1, \ b^{(r+s-1)-i} \cdot \mathsf{h}_2, 0)$ so the statement follows from the second line in 4.8. $\qquad\square$

As a consequence of 4.9, every element in the sequence
$$
(a^0 \cdot \mathsf{Z}_1, b^{r+s-1} \cdot \mathsf{Z}_2, 0), \ldots, (a^{r-2} \cdot \mathsf{Z}_1, b^{s+1} \cdot \mathsf{Z}_2, 0), (a^{r+1} \cdot \mathsf{Z}_1, b^{s-2} \cdot \mathsf{Z}_2, 0), \ldots, (a^{r+s-1} \cdot \mathsf{Z}_1, b^0 \cdot \mathsf{Z}_2, 0)
$$

belongs to the image of $\mathbb{D}$. In contrast, by 4.5 the two 'missing' elements in this sequence, namely $\mathsf{Y} = (a^{\mathsf{r}-1} \cdot \mathsf{Z}_1, b^{\mathsf{s}} \cdot \mathsf{Z}_2, 0)$ and $\mathsf{X} = (a^{\mathsf{r}} \cdot \mathsf{Z}_1, b^{\mathsf{s}-1} \cdot \mathsf{Z}_2, 0)$, do not lie in the image of $\mathbb{D}$. The following corollary formalises this fact.

Corollary 4.10. *Given* $f(a) = \alpha_0 + \alpha_1 a + \ldots + \alpha_{\mathsf{r}+\mathsf{s}-1} a^{\mathsf{r}+\mathsf{s}-1} \in \mathbb{K}[a]$, *set* $g = b^{\mathsf{r}+\mathsf{s}-1} f(b^{-1})$, *namely* $g = \alpha_0 b^{\mathsf{r}+\mathsf{s}-1} + \alpha_1 b^{\mathsf{r}+\mathsf{s}-2} + \ldots + \alpha_{\mathsf{r}+\mathsf{s}-1}$. *Then in* $\mathcal{C}$

$$(f \cdot \mathsf{Z}_1,\, g \cdot \mathsf{Z}_2, 0) = \sum_{i=0}^{\mathsf{r}-2} \alpha_i \mathbb{D}(\mathsf{k}_i) + \alpha_{\mathsf{r}-1} \mathsf{Y} + \alpha_{\mathsf{r}} \mathsf{X} + \sum_{i=\mathsf{r}+1}^{\mathsf{r}+\mathsf{s}-1} \alpha_i \mathbb{D}(\mathsf{k}_i).$$

Proof. By construction of g, we may write

$$(f \cdot \mathsf{Z}_1,\, g \cdot \mathsf{Z}_2, 0) = \sum_{i=0}^{\mathsf{r}+\mathsf{s}-1} \alpha_i (a^i \cdot \mathsf{Z}_1,\, b^{\mathsf{r}+\mathsf{s}-1-i} \cdot \mathsf{Z}_2, 0).$$

The result then follows directly from 4.9, and the definition of X and Y. □

The next elementary relationship is an analogue of 4.8 under $\mathbb{D} \colon \mathcal{C}_2 \to \mathcal{C}_3$.

Lemma 4.11. *For any* $f \in \mathcal{O}_{\mathcal{U}_1}$ *and* $g \in \mathcal{O}_{\mathcal{U}_2}$,

$$\mathbb{D}(f \cdot \mathsf{G}_1,\, g \cdot \mathsf{G}_2,\, 0) = (\quad f \cdot \mathfrak{s}_1,\; b^{\mathsf{s}+1} g \cdot \mathfrak{s}_2,\; (f - a^{\mathsf{r}-1} g) \cdot \mathfrak{s}_1|_{\mathcal{U}_{12}})$$
$$\mathbb{D}(f \cdot \mathsf{H}_1,\, g \cdot \mathsf{H}_2,\, 0) = (a^{\mathsf{r}+1} f \cdot \mathfrak{s}_1,\qquad g \cdot \mathfrak{s}_2,\; (f - a^{\mathsf{s}-1} g) \cdot \mathfrak{s}_1|_{\mathcal{U}_{12}})$$

Proof. By definition, $\mathbb{D}(f \cdot \mathsf{G}_1,\, g \cdot \mathsf{G}_2, 0) = (\delta(f \cdot \mathsf{G}_1), \delta(g \cdot \mathsf{G}_2), f \cdot \mathsf{G}_1|_{12} - g \cdot \mathsf{G}_2|_{12})$. Under the sign convention in (3.B), which for degree two is down-across minus across-down, it is clear that $\delta(\mathsf{G}_1) = \mathfrak{s}_1$ and $\delta(\mathsf{G}_2) = b^{\mathsf{s}+1} \mathfrak{s}_2$. Now the only possible non-zero map in $f \cdot \mathsf{G}_1|_{12} - g \cdot \mathsf{G}_2|_{12}$ is the map $\mathcal{E}_2 \to \mathcal{E}_0$ given by

$$
\begin{array}{ccc}
 & \mathcal{E}_2 \xrightarrow{\left(\begin{smallmatrix} a^{\mathsf{r}+\mathsf{s}-1} & 0 & 0 \\ 0 & a^{\mathsf{s}-1} & 0 \\ 0 & 0 & a^{\mathsf{r}-1} \end{smallmatrix}\right)} \mathcal{E}_2 & \\
\mathcal{E}_2 & & \\
{\scriptstyle\left(\begin{smallmatrix} 0 & 0 & -f \end{smallmatrix}\right)}\Big\downarrow \quad - & & \Big\downarrow{\scriptstyle\left(\begin{smallmatrix} 0 & 0 & -g \end{smallmatrix}\right)} \\
\mathcal{E}_0 & \mathcal{E}_0 \xleftarrow{\quad 1 \quad} \mathcal{E}_0 &
\end{array}
$$

which is clearly $(f - a^{\mathsf{r}-1} g) \cdot \mathfrak{s}_1|_{\mathcal{U}_{12}}$. This proves the first statement. In a similar way, $\delta(\mathsf{H}_1) = a^{\mathsf{r}+1} \mathfrak{s}_1$, $\delta(\mathsf{H}_2) = \mathfrak{s}_2$ and $f \cdot \mathsf{H}_1|_{12} - g \cdot \mathsf{H}_2|_{12} = (f - a^{\mathsf{s}-1} g) \cdot \mathfrak{s}_1|_{\mathcal{U}_{12}}$. □

As in 4.9, the following asserts that although K_i is defined on two different intervals, the effect of applying $\mathbb{D}$ always looks the same.

Corollary 4.12. *For all i such that K_i is defined,*

$$\mathbb{D}(\mathsf{K}_i) = (a^i \cdot \mathfrak{s}_1,\; b^{\mathsf{r}+\mathsf{s}-i} \cdot \mathfrak{s}_2, 0).$$

Proof. In the first interval $\mathsf{K}_i = (a^i \cdot \mathsf{G}_1,\, b^{\mathsf{r}-1-i} \cdot \mathsf{G}_2, 0)$, so the statement follows from the top line in 4.11 applied to $f = a^i$ and $g = b^{\mathsf{r}-1-i}$, where $f - a^{\mathsf{r}-1} g = 0$ since $a = b^{-1}$. Similarly, in the second interval, $\mathsf{K}_i = (a^{i-(\mathsf{r}+1)} \cdot \mathsf{H}_1,\, b^{\mathsf{r}+\mathsf{s}-i} \cdot \mathsf{H}_2, 0)$ so the statement follows from the second line in 4.11. □

4.6. **Elementary Relationships Involving $\star$.** This subsection establishes some elementary relationships between the degree one elements in §4.1, and the degree two elements in §4.3, under the operation $\star$.

Lemma 4.13. *In* $\mathcal{C}$, *the following statements hold.*

$$\mathsf{x} \star \mathsf{x} = (\quad v_2^{\mathsf{t}-3} A \cdot \mathsf{Z}_1,\; b^2 w_2^{\mathsf{t}-3} B \cdot \mathsf{Z}_2, 0)$$
$$\mathsf{x} \star \mathsf{y} = \mathsf{y} \star \mathsf{x} = (\quad a v_2^{\mathsf{t}-3} A \cdot \mathsf{Z}_1,\quad b w_2^{\mathsf{t}-3} B \cdot \mathsf{Z}_2, 0)$$
$$\mathsf{y} \star \mathsf{y} = (\, a^2 v_2^{\mathsf{t}-3} A \cdot \mathsf{Z}_1,\qquad w_2^{\mathsf{t}-3} B \cdot \mathsf{Z}_2, 0)$$

Proof. Note $x \star x = (x_1 \circ x_1, x_2 \circ x_2, 0)$. Now on $\mathcal{U}_1$, the composition $x_1 \circ x_1$ is

$$
\begin{array}{ccccccc}
\mathcal{E}_3 & \xrightarrow{\left(\begin{smallmatrix} v_2 \\ -a^{r+1} \\ -1 \end{smallmatrix}\right)} & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 \\
{\scriptstyle v_2^{t-3}A\left(\begin{smallmatrix} 0 \\ -1 \\ 0 \end{smallmatrix}\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle v_2^{t-3}A\left(\,-1\ 0\ 0\,\right)} & & & & \\
\mathcal{E}_3 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{\left(\,v_1\ v_2\ b^{2-r}w_1\,\right)} & \mathcal{E}_0
\end{array}
$$

which equals $v_2^{t-3}A \cdot Z_1$. On $\mathcal{U}_2$, $x_2 \circ x_2$ it is similar, with $b^2 w_2^{t-3}B$ replacing the coefficient $v_2^{t-3}A$ in the two vertical maps above. The first statement follows.

The second and third statements follow in an identical fashion. $\qquad\square$

Lemma 4.14. *In $\mathcal{C}$, the following hold.*

$$
\begin{array}{cccc}
-g_1 \circ x_1 = G_1 & -g_1 \circ y_1 = a \cdot G_1 & -g_2 \circ x_2 = b \cdot G_2 & -g_2 \circ y_2 = G_2 \\
x_1 \circ g_1 = G_1 & y_1 \circ g_1 = a \cdot G_1 & x_2 \circ g_2 = b \cdot G_2 & y_2 \circ g_2 = G_2 \\
-h_1 \circ x_1 = H_1 & -h_1 \circ y_1 = a \cdot H_1 & -h_2 \circ x_2 = b \cdot H_2 & -h_2 \circ y_2 = H_2 \\
x_1 \circ h_1 = H_1 & y_1 \circ h_1 = a \cdot H_1 & x_2 \circ h_2 = b \cdot H_2 & y_2 \circ h_2 = H_2
\end{array}
$$

Proof. The first column holds by inspection. The second column follows from the first since $y_1 = a \cdot x_1$. The fourth is $\mathcal{U}_2$ version of the first column, and the third column follows from the fourth since $x_2 = b \cdot y_2$. $\qquad\square$

In the following, note that whenever k_i is defined, both K_i and K_{i+1} are defined.

Corollary 4.15. *For all i such that k_i is defined, in $\mathcal{C}$ there are equalities*

$$
\begin{aligned}
x \star k_i &= K_i && = -k_i \star x, \\
y \star k_i &= K_{i+1} && = -k_i \star y.
\end{aligned}
$$

Proof. In the first interval, $k_i = (a^i \cdot g_1, b^{r-2-i} \cdot g_2, 0)$, so by 4.14

$$
-k_i \star x = -(a^i \cdot g_1 \circ x_1, b^{r-2-i} \cdot g_2 \circ x_2, 0) = (a^i \cdot G_1, b^{r-1-i} \cdot G_2, 0) = K_i.
$$

Similarly, in the second interval, by 4.14

$$
-k_i \star x = -(a^{i-(r+1)} \cdot h_1 \circ x_1, b^{(r+s-1)-i} \cdot h_2 \circ x_2, 0) = (a^{i-(r+1)} \cdot H_1, b^{r+s-i} \cdot H_2, 0) = K_i.
$$

All other claims are similar. $\qquad\square$

The following deals with $\star$ between degree one and degree two inputs.

Lemma 4.16. *In $\mathcal{C}$, for all $i, j \geq 0$, there are equalities*

$$
\begin{aligned}
x \star (a^i \cdot Z_1, b^j \cdot Z_2, 0) &= -(\quad a^i \cdot \mathfrak{s}_1,\ b^{j+1} \cdot \mathfrak{s}_2, 0) = (a^i \cdot Z_1,\ b^j \cdot Z_2, 0) \star x \\
y \star (a^i \cdot Z_1, b^j \cdot Z_2, 0) &= -(a^{i+1} \cdot \mathfrak{s}_1,\quad b^j \cdot \mathfrak{s}_2, 0) = (a^i \cdot Z_1,\ b^j \cdot Z_2, 0) \star y.
\end{aligned}
$$

Proof. The first statement follow from the fact that $x_1 \circ Z_1 = -\mathfrak{s}_1 = Z_1 \circ x_1$ and $x_2 \circ Z_2 = -b \cdot \mathfrak{s}_1 = Z_2 \circ x_2$, as can be directly verified. The second statement follows similarly. $\quad\square$

4.7. **Notation for Induction.** This subsection lays down some notation useful for the induction in 5.12, and furthermore introduces some key homotopies.

Recall the polynomials A_j and B_j from §2.2.

Notation 4.17. For $i \geq 3$, set $A_{\geq i} = \sum_{j \geq i} v_2^{j-t} A_j$ and $B_{\geq i} = \sum_{j \geq i} w_2^{j-t} B_j$.

Remark 4.18. By (2.G) and the definition above, $A = A_{\geq 3}$.

Multiplying the definition by the appropriate power of v_2, it is clear that

$$
\begin{aligned}
v_2^{t-i}A_{\geq i} &= A_i + v_2^{t-i}A_{\geq i+1} \\
w_2^{t-i}B_{\geq i} &= B_i + w_2^{t-i}B_{\geq i+1}.
\end{aligned} \tag{4.C}
$$

Lemma 4.19. *For all $i \geq 3$, $A_{\geq i} = a^{r+s-t}B_{\geq i}$ and $a^{2-r-s}v_2^{t-i}A_{\geq i} = b^{i-2}w_2^{t-i}B_{\geq i}$.*

Proof. This statement $A_{\geq i} = a^{\mathsf{r}+\mathsf{s}-\mathsf{t}}B_{\geq i}$ is immediate from the last statement in 2.8. The second statement follows by the first, since $v_2 = aw_2$ by the glue (2.A). □

The following, which are based on the shape z, will be the key inductive homotopies.

Notation 4.20. For any $i \geq 3$, and for any $0 \leq k \leq i - 1$, set

$$\mathsf{e}_{i,k} := (a^k(v_2^{\mathsf{t}-i-1}A_{\geq i+1}) \cdot \mathsf{z}_1, \ b^{i-k-1}(w_2^{\mathsf{t}-i-1}B_{\geq i+1}) \cdot \mathsf{z}_2, 0) \in \mathcal{C}_1. \qquad (4.D)$$

By (4.C), all entries in (4.D) are polynomials.

Corollary 4.21. *With notation as above,*

$$\mathbb{D}(\mathsf{e}_{i,k}) = (\,a^k(v_2^{\mathsf{t}-i}A_{\geq i+1}) \cdot \mathsf{Z}_1, \ b^{i-k-1}(w_2^{\mathsf{t}-i}B_{\geq i+1}) \cdot \mathsf{Z}_2, 0).$$

Proof. Setting $f = a^k(v_2^{\mathsf{t}-i}A_{\geq i+1})$ and $g = b^{i-k-1}(w_2^{\mathsf{t}-i}B_{\geq i+1})$, which are both polynomials by (4.C) applied to $i + 1$, the result is then a direct application of the general 4.8, where the third entry is now zero since by 4.19 applied to $i + 1$ we have $a^{2-\mathsf{r}-\mathsf{s}}f = g$. □

Lemma 4.22. *For any $i \geq 3$, and for any $0 \leq k \leq i - 1$, there are equalities*

(1) $\mathsf{e}_{i,k} \star \mathsf{x} = \mathsf{e}_{i,k-1} \star \mathsf{y}$ *and* $\mathsf{x} \star \mathsf{e}_{i,k} = \mathsf{y} \star \mathsf{e}_{i,k-1}$ *provided that $k \geq 1$.*
(2) $\mathsf{x} \star \mathsf{e}_{i,k} + \mathsf{e}_{i,k} \star \mathsf{x} = -(\,a^k v_2^{\mathsf{t}-(i+1)}A_{\geq i+1} \cdot \mathsf{Z}_1, \ b^{i-k}w_2^{\mathsf{t}-(i+1)}B_{\geq i+1} \cdot \mathsf{Z}_2, 0)$

Proof. (1) Since $k \geq 1$, the element $\mathsf{e}_{i,k-1}$ is defined. The result then follows since

$$\begin{aligned}
\mathsf{x} \star \mathsf{e}_{i,k} &= (a^k v_2^{\mathsf{t}-i-1}A_{\geq i+1} \cdot \mathsf{x}_1 \circ \mathsf{z}_1, \ b^{i-k-1}w_2^{\mathsf{t}-(i+1)}B_{\geq i+1} \cdot \mathsf{x}_2 \circ \mathsf{z}_2, 0) \\
&= (a^{k-1}v_2^{\mathsf{t}-(i+1)}A_{\geq i+1} \cdot (a \cdot \mathsf{x}_1) \circ \mathsf{z}_1, \ b^{i-k}w_2^{\mathsf{t}-(i+1)}B_{\geq i+1} \cdot (b^{-1} \cdot \mathsf{x}_2) \circ \mathsf{z}_2, 0) \\
&= (a^{k-1}v_2^{\mathsf{t}-(i+1)}A_{\geq i+1} \cdot \mathsf{y}_1 \circ \mathsf{z}_1, \ b^{i-k}w_2^{\mathsf{t}-(i+1)}B_{\geq i+1} \cdot \mathsf{y}_2 \circ \mathsf{z}_2, 0) \\
&= \mathsf{y} \star \mathsf{e}_{i,k-1}.
\end{aligned}$$

The other claim is similar.
(2) By definition, $\mathsf{x} \star \mathsf{e}_{i,k} + \mathsf{e}_{i,k} \star \mathsf{x}$ equals

$$(a^k v_2^{\mathsf{t}-(i+1)}A_{\geq i+1} \cdot (\mathsf{x}_1 \circ \mathsf{z}_1 + \mathsf{z}_1 \circ \mathsf{x}_1), \ b^{i-k-1}w_2^{\mathsf{t}-(i+1)}B_{\geq i+1} \cdot (\mathsf{x}_2 \circ \mathsf{z}_2 + \mathsf{z}_2 \circ \mathsf{x}_2), 0).$$

The statement follows since $\mathsf{x}_1 \circ \mathsf{z}_1 + \mathsf{z}_1 \circ \mathsf{x}_1 = -\mathsf{Z}_1$ and $\mathsf{x}_2 \circ \mathsf{z}_2 + \mathsf{z}_2 \circ \mathsf{x}_2 = -b \cdot \mathsf{Z}_2$, as can be directly verified. □

The following products all vanish for very elementary reasons.

Lemma 4.23. *Wherever K_*, k_* and $\mathsf{e}_{*,*}$ are defined, the following products*

$\mathsf{k}_* \star \mathsf{e}_{*,*}$	$\mathsf{e}_{*,*} \star \mathsf{k}_*$	$\mathsf{k}_* \star \mathsf{k}_{*'}$	$\mathsf{e}_{*,*} \star \mathsf{e}_{*',*'}$
$\mathsf{k}_* \star \mathsf{X}$	$\mathsf{X} \star \mathsf{k}_*$	$\mathsf{k}_* \star \mathsf{Y}$	$\mathsf{Y} \star \mathsf{k}_*$
$\mathsf{e}_{*,*} \star \mathsf{X}$	$\mathsf{X} \star \mathsf{e}_{*,*}$	$\mathsf{e}_{*,*} \star \mathsf{Y}$	$\mathsf{Y} \star \mathsf{e}_{*,*}$
$\mathsf{K}_* \star \mathsf{k}_*$	$\mathsf{k}_* \star \mathsf{K}_*$	$\mathsf{K}_* \star \mathsf{e}_{*,*}$	$\mathsf{e}_{*,*} \star \mathsf{K}_*$
$\mathsf{K}_i \star \mathsf{x}$	$\mathsf{x} \star \mathsf{K}_i$	$\mathsf{K}_i \star \mathsf{y}$	$\mathsf{y} \star \mathsf{K}_i$

all equal $0_\mathcal{C}$.

Proof. Most follow at once from the shapes of these elements. The first row holds since all compositions involving z, g and h are identically zero. The second and third row holds since all compositions of Z with g, h and z are zero. For the fourth line, K_* is based on G or H, so its only nonzero maps are $\mathcal{E}_2 \to \mathcal{E}_0$, whilst k_* is based on g or h, both of which have zero map $\mathcal{E}_3 \to \mathcal{E}_2$. Thus any composition of k_* with K_* is zero. Similarly $\mathsf{e}_{*,*}$ is based on z, which again has zero map $\mathcal{E}_3 \to \mathcal{E}_2$ and the vanishing follows. The last line holds since all compositions of x and y with G and H are zero, as again can be explicitly observed. □

4.8. **Summary.** As a summary of this section, the closed elements x, y induce generators of $\mathrm{H}^1(\mathcal{C})$, and likewise X, Y induce generators of $\mathrm{H}^2(\mathcal{C})$, whilst $\mathfrak{s}$ generates $\mathrm{H}^3(\mathcal{C})$. These, together with their corresponding underlying shapes, are summarised as follows

element	x	y	X	Y	$\mathfrak{s}$
shape			Z	Z	$\mathfrak{s}$
see	§4.1		4.4	4.4	4.7

where we note that x and y are not usefully defined by shapes. In contrast, the elements

element	k_i	$\mathsf{e}_{i,k}$	K_i	$\mathbb{D}\mathsf{k}_i$	$\mathbb{D}\mathsf{e}_{i,k}$	$\mathbb{D}\mathsf{K}_i$
shape	g, h	z	G, H	Z	Z	$\mathfrak{s}$
see	4.2	4.20	4.6	4.9	4.21	4.12

will provide homotopies needed to control the Kadeishvili algorithm in §5 below.

5. THE A_∞ MINIMAL MODEL

In this section, regardless of char $\mathbb{K}$, we describe the A_∞-minimal model of the DG-algebra $\mathcal{C}$ constructed in §3.5, and in the process pin down all its higher products.

5.1. **Recap on Minimal Models.** The main result will be proved by a very careful use of the A_∞-minimal model due to Kadeishvili [K1], which we state to set notation. As in [K1], all A_∞-algebras considered in this paper are strictly unital, meaning there is an element $1 \in \mathsf{A}_0$ which is the identity with respect to multiplication m_2, and further $\mathsf{m}_n(\ldots, 1, \ldots) = 0$ for all $n \geq 3$.

Under Setup 2.4, consider the DG-algebra $\mathcal{C}$ in §3.5, and consider the finite dimensional vector space $\mathsf{A} := \mathrm{H}^*(\mathcal{C})$. Following [K1], for a homogeneous element $\mathsf{a} \in \mathsf{A}_i$ set $\hat{\mathsf{a}} = (-1)^{|\mathsf{a}|}\mathsf{a}$ where $|\mathsf{a}| = i$.

Theorem 5.1 ([K1, Theorem 1]). *There is an A_∞-structure on $\mathsf{A} = \mathrm{H}^*(\mathcal{C})$ given by*

$$\mathsf{m}_n \colon \otimes^n \mathsf{A} \to \mathsf{A} \qquad \text{for every } n \geq 1$$

with $\mathsf{m}_1 = 0$, where on homogeneous inputs $\mathsf{m}_2(\mathsf{a}_1, \mathsf{a}_2) = -\hat{\mathsf{a}}_1\mathsf{a}_2$, and furthermore an A_∞-morphism

$$\mathsf{f}_n \colon \otimes^n \mathsf{A} \to \mathcal{C} \qquad \text{for every } n \geq 1$$

for which $\mathsf{f}_1 \colon \mathsf{A} \to \mathcal{C}$ is a quasi-isomorphism.

The construction is recalled below. For notational convenience, write A_∞-maps as e.g. $\mathsf{m}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ rather than $\mathsf{m}_n(\mathsf{a}_1 \otimes \ldots \otimes \mathsf{a}_n)$.

Remark 5.2. By definition of an A_∞-algebra, the A_∞-structure maps m_n are graded of degree $2 - n$, namely

$$\mathsf{m}_n(\mathsf{A}_{i_1} \otimes \ldots \otimes \mathsf{A}_{i_n}) \subseteq \mathsf{A}_{i_1 + \ldots + i_n + 2 - n}.$$

Thus, for example, on degree one inputs $\mathsf{a}_1, \ldots, \mathsf{a}_n \in \mathsf{A}_1$, $\mathsf{m}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) \in \mathsf{A}_2$ for all $n \geq 1$. In the setup $\mathsf{A} = \mathsf{A}_0 \oplus \ldots \oplus \mathsf{A}_3$ below, it follows at once that $\mathsf{m}_n(\mathsf{b}_1, \ldots, \mathsf{b}_n) = 0$ on homogeneous inputs $\mathsf{b}_i \in \mathsf{A}_{d_i}$ if any $d_i \geq 3$, or if any two distinct i_1 and i_2 satisfy $d_{i_1} = d_{i_2} = 2$.

Notation 5.3. If $\mathsf{a} \in \mathcal{C}$ with $\mathbb{D}(\mathsf{a}) = 0$, write $[\mathsf{a}]$ for a viewed in cohomology $\mathrm{H}^*(\mathcal{C}) = \mathsf{A}$. To implement Kadeishvili's construction we must, once and for all, choose (closed) elements $\{b_i\}$ of $\mathcal{C}$ which descend to a basis $\{[b_i]\}$ of A. This defines an injective map of vector spaces

$$\iota \colon \mathsf{A} \longrightarrow \mathcal{C}$$

sending $[b_i] \mapsto b_i$, which is a quasi-isomorphism of complexes of vector spaces (where A has trivial boundary maps). We make this choice now, using the generators of 4.8:

$$\iota \colon \qquad [1_\mathcal{C}] \mapsto 1_\mathcal{C}, \qquad \begin{matrix} [\mathsf{x}] \mapsto \mathsf{x} \\ [\mathsf{y}] \mapsto \mathsf{y} \end{matrix} , \qquad \begin{matrix} [\mathsf{X}] \mapsto \mathsf{X} \\ [\mathsf{Y}] \mapsto \mathsf{Y} \end{matrix} , \qquad [\mathfrak{s}] \mapsto \mathfrak{s},$$

where recall $1_\mathcal{C} = (1, 1, 0)$. For inductive purposes, set $\mathsf{f}_1 = \iota$.

The task is to construct all the higher products $\mathsf{m}_n\colon \mathsf{A}^{\otimes n}\to \mathsf{A}$. The construction works inductively, constructing m_n and f_n in tandem, employing the following auxiliary sequence U_n. Removing the $k=0$, $j=n$ term from [K1, (2)], following [K1][1] for any $n\geq 2$ define $\mathsf{U}_n\colon \mathsf{A}^{\otimes n}\to \mathcal{C}$ in terms of already defined $\mathsf{f}_{<n}$ and $\mathsf{m}_{<n}$ by

$$
\begin{aligned}
\mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) := \sum_{\ell=1}^{n-1} (-1)^{|\mathsf{f}_\ell(\mathsf{a}_1,\ldots,\mathsf{a}_\ell)|+1}\, \mathsf{f}_\ell(\mathsf{a}_1,\ldots,\mathsf{a}_\ell)\star \mathsf{f}_{n-\ell}(\mathsf{a}_{\ell+1},\ldots,\mathsf{a}_n) \\
-\sum_{k=0}^{n-2}\sum_{j=2}^{\mathfrak{m}_{n,k}} (-1)^k\, \mathsf{f}_{n-j+1}(\hat{\mathsf{a}}_1,\ldots,\hat{\mathsf{a}}_k,\mathsf{m}_j(\mathsf{a}_{k+1},\ldots,\mathsf{a}_{k+j}),\mathsf{a}_{k+j+1},\ldots,\mathsf{a}_n)
\end{aligned}
\tag{5.A}
$$

where $\mathfrak{m}_{n,k} = \min\{n-k, n-1\}$. The condition [K1, (2′)] that the f_n determine an A_∞-morphism is

$$
(\iota\mathsf{m}_n - \mathsf{U}_n)(\mathsf{a}_1,\ldots,\mathsf{a}_n) = \mathbb{D}\mathsf{f}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n).
\tag{5.B}
$$

As explained in more detail below, passing to cohomology by applying $[-]$, equation (5.B) defines m_n, which in turn defines $\iota\mathsf{m}_n$, which in turn (up to choice) determines f_n.

Notation 5.4. For closed $\mathsf{c}_i \in \mathcal{C}$, to ease notation throughout we will write $\mathsf{m}_n(\mathsf{c}_1,\ldots,\mathsf{c}_n)$ as shorthand for $\mathsf{m}_n([\mathsf{c}_1],\ldots,[\mathsf{c}_n])$, and similarly for f_n and U_n. In other words, since the inputs to m_n, f_n and U_n must be cohomology classes, our convention is that should inputs not be this, make them so. This is a mild abuse of notation, but it is unambiguous.

Concretely, the above translates into the following. For $\bullet, \blacktriangle \in \{\mathsf{x},\mathsf{y},\mathsf{X},\mathsf{Y},\mathfrak{s}\}$, by the above abuse of notation

$$
\mathsf{U}_2(\bullet,\blacktriangle) := (-1)^{|\iota(\bullet)|+1}\iota(\bullet)\star \iota(\blacktriangle) = (-1)^{|\bullet|+1}(\bullet\star\blacktriangle).
$$

The right hand side is (closed and) determined by f_1, and thus so are both $\mathsf{U}_2(\bullet,\blacktriangle)$ and $[\mathsf{U}_2(\bullet,\blacktriangle)]$. Using (5.B) defines $\mathsf{m}_2(\bullet,\blacktriangle)$, as expressing this known $[\mathsf{U}_2(\bullet,\blacktriangle)]$ in terms of the chosen basis $[b_i]$, there exists scalars α_i such that

$$
\mathsf{m}_2(\bullet,\blacktriangle) \overset{(5.B)}{=} [\mathsf{U}_2(\bullet,\blacktriangle)] = \sum \alpha_i[b_i].
$$

Applying ι it follows that $\iota\mathsf{m}_2(\bullet,\blacktriangle) = \sum \alpha_i b_i$, and so on the chain level (5.B) is simply

$$
\sum \alpha_i b_i - (-1)^{|\bullet|+1}(\bullet\star\blacktriangle) = \mathbb{D}\mathsf{f}_2(\bullet,\blacktriangle).
\tag{5.C}
$$

The left hand side is all determined, so determines some choice $\mathsf{f}_2(\bullet,\blacktriangle)\colon \mathsf{A}^{\otimes 2}\to \mathcal{C}$. It should be noted that f_2 is not unique, but some choice of $\mathsf{f}_2(\bullet,\blacktriangle)$ satisfying (5.C) can always be made. In the course of the proof below we will use this freedom explicitly, see 5.7 and (0) in the proof of 5.12.

We next move to determine m_3 and f_3. Now by definition

$$
\begin{aligned}
\mathsf{U}_3(\bullet,\blacktriangle,\blacksquare) := \quad & (-1)^{|\bullet|+1}\bullet\star \mathsf{f}_2(\blacktriangle,\blacksquare) + (-1)^{|\bullet|+|\blacktriangle|}\mathsf{f}_2(\bullet,\blacktriangle)\star\blacksquare \\
& + (-1)^{|\bullet|}\mathsf{f}_2(\bullet,\mathsf{m}_2(\blacktriangle,\blacksquare)) - \mathsf{f}_2(\mathsf{m}_2(\bullet,\blacktriangle),\blacksquare),
\end{aligned}
\tag{5.D}
$$

which is fully determined by the previous data of m_2 and f_2. Expressing this known cohomology class $[\mathsf{U}_3(\bullet,\blacktriangle,\blacksquare)]$ in terms of the basis $[b_i]$, there exists scalars β_i such that

$$
\mathsf{m}_3(\bullet,\blacktriangle,\blacksquare) \overset{(5.B)}{=} [\mathsf{U}_3(\bullet,\blacktriangle,\blacksquare)] = \sum \beta_i[b_i].
$$

Applying ι it follows that $\iota\mathsf{m}_3(\bullet,\blacktriangle,\blacksquare) = \sum \beta_i b_i$, and so on the chain level (5.B) is simply

$$
\sum \beta_i b_i - \mathsf{U}_3(\bullet,\blacktriangle,\blacksquare) = \mathbb{D}\mathsf{f}_3(\bullet,\blacktriangle,\blacksquare).
\tag{5.E}
$$

Again, since the left hand side is all determined, this gives some non-unique choice of $\mathsf{f}_3(\bullet,\blacktriangle,\blacksquare)\colon \mathsf{A}^{\otimes 3}\to \mathcal{C}$.

The calculation continues in this manner. The chain-level U_n is determined by smaller m_i and f_i. This then determines $[\mathsf{U}_n]$, which can be written in terms of the basis $[b_i]$. In turn, applying $[-]$ to (5.B), the expression for $[\mathsf{U}_n]$ determines m_n and thus $\iota\mathsf{m}_n$, which in turn by (5.B) gives some non-unique choice of f_n.

---

[1][K1] writes instead $+\sum(-1)^{k+1}$, which is equivalent, and also writes the second sum $\sum_{j=2}^{n-1}$, with the convention that terms are zero when they do not make sense.

5.2. **Degree One m_2 Inputs.** This subsection calculates m_2 on all degree one inputs, namely combinations of x and y. The point is that this is described by a very particular linear combination of X and Y, up to some homotopies of the form $\mathbb{D}(\mathsf{k}_i)$, plus in all cases some higher order term of the form $\mathbb{D}(\mathsf{e})$. This latter term will, in later subsections, allow all higher products m_n to be computed inductively.

Proposition 5.5. *In $\mathcal{C}$, the following statements hold*

$$\mathsf{x} \star \mathsf{x} = \lambda_{30}\mathsf{X} + \lambda_{21}\mathsf{Y} + \mathbb{D}(\lambda_{12}\mathsf{k}_{r-2} + \lambda_{03}\mathsf{k}_{r-3} + \mathsf{e}_{3,0})$$
$$\mathsf{x} \star \mathsf{y} = \mathsf{y} \star \mathsf{x} = \lambda_{21}\mathsf{X} + \lambda_{12}\mathsf{Y} + \mathbb{D}(\lambda_{30}\mathsf{k}_{r+1} + \lambda_{03}\mathsf{k}_{r-2} + \mathsf{e}_{3,1})$$
$$\mathsf{y} \star \mathsf{y} = \lambda_{12}\mathsf{X} + \lambda_{03}\mathsf{Y} + \mathbb{D}(\lambda_{30}\mathsf{k}_{r+2} + \lambda_{21}\mathsf{k}_{r+1} + \mathsf{e}_{3,2}).$$

Proof. By 4.18, $A = A_{\geq 3}$. Applying (4.C) to $i = 3$ gives $v_2^{t-3} A_{\geq 3} = A_3 + v_2^{t-3} A_{\geq 4}$, and so combining gives $v_2^{t-3} A = A_3 + v_2^{t-3} A_{\geq 4}$. Similarly $w_2^{t-3} B = B_3 + w_2^{t-3} B_{\geq 4}$.

Thus, decomposing 4.13 we may write

$$\mathsf{x} \star \mathsf{x} = \big(\ A_3 \cdot \mathsf{Z}_1,\ b^2 B_3 \cdot \mathsf{Z}_2,\ 0\big) + \big(\ v_2^{t-3} A_{\geq 4} \cdot \mathsf{Z}_1,\ b^2 w_2^{t-3} B_{\geq 4} \cdot \mathsf{Z}_2,\ 0\big)$$
$$\mathsf{x} \star \mathsf{y} = \mathsf{y} \star \mathsf{x} = \big(\ aA_3 \cdot \mathsf{Z}_1,\ bB_3 \cdot \mathsf{Z}_2,\ 0\big) + \big(\ av_2^{t-3} A_{\geq 4} \cdot \mathsf{Z}_1,\ bw_2^{t-3} B_{\geq 4} \cdot \mathsf{Z}_2,\ 0\big)$$
$$\mathsf{y} \star \mathsf{y} = \big(a^2 A_3 \cdot \mathsf{Z}_1,\ B_3 \cdot \mathsf{Z}_2,\ 0\big) + \big(a^2 v_2^{t-3} A_{\geq 4} \cdot \mathsf{Z}_1,\ w_2^{t-3} B_{\geq 4} \cdot \mathsf{Z}_2,\ 0\big).$$

By 4.21, the final terms can be rewritten, to give

$$\mathsf{x} \star \mathsf{x} = \big(\ A_3 \cdot \mathsf{Z}_1,\ b^2 B_3 \cdot \mathsf{Z}_2,\ 0\big) + \mathbb{D}(\mathsf{e}_{3,0})$$
$$\mathsf{x} \star \mathsf{y} = \mathsf{y} \star \mathsf{x} = \big(\ aA_3 \cdot \mathsf{Z}_1,\ bB_3 \cdot \mathsf{Z}_2,\ 0\big) + \mathbb{D}(\mathsf{e}_{3,1})$$
$$\mathsf{y} \star \mathsf{y} = \big(a^2 A_3 \cdot \mathsf{Z}_1,\ B_3 \cdot \mathsf{Z}_2,\ 0\big) + \mathbb{D}(\mathsf{e}_{3,2}).$$

Now writing 2.7 backwards

$$A_3 = \lambda_{03} a^{r-3} + \lambda_{12} a^{r-2} + \lambda_{21} a^{r-1} + \lambda_{30} a^r$$
$$aA_3 = \lambda_{03} a^{r-2} + \lambda_{12} a^{r-1} + \lambda_{21} a^r \quad + \lambda_{30} a^{r+1}$$
$$a^2 A_3 = \lambda_{03} a^{r-1} + \lambda_{12} a^r \quad + \lambda_{21} a^{r+1} + \lambda_{30} a^{r+2},$$

using 4.10 we may then write

$$\big(\ A_3 \cdot \mathsf{Z}_1,\ b^2 B_3 \cdot \mathsf{Z}_2,\ 0\big) = \lambda_{03}\mathbb{D}(\mathsf{k}_{r-3}) + \lambda_{12}\mathbb{D}(\mathsf{k}_{r-2}) + \lambda_{21}\mathsf{Y} \quad\quad + \lambda_{30}\mathsf{X}$$
$$\big(\ aA_3 \cdot \mathsf{Z}_1,\ bB_3 \cdot \mathsf{Z}_2,\ 0\big) = \lambda_{03}\mathbb{D}(\mathsf{k}_{r-2}) + \lambda_{12}\mathsf{Y} \quad\quad + \lambda_{21}\mathsf{X} \quad\quad + \lambda_{30}\mathbb{D}(\mathsf{k}_{r+1})$$
$$\big(a^2 A_3 \cdot \mathsf{Z}_1,\ B_3 \cdot \mathsf{Z}_2,\ 0\big) = \lambda_{03}\mathsf{Y} \quad\quad + \lambda_{12}\mathsf{X} \quad\quad + \lambda_{21}\mathbb{D}(\mathsf{k}_{r+1}) + \lambda_{30}\mathbb{D}(\mathsf{k}_{r+2}),$$

and the statement follows. $\qquad\square$

In particular, in the notation of §5.1, since on degree one inputs $\mathsf{U}_2(\bullet, \blacktriangle) = \bullet \star \blacktriangle$, applying $[-]$ then ι to 5.5 shows that

$$\iota\mathsf{m}_2(\mathsf{x}, \mathsf{x}) = \lambda_{30}\mathsf{X} + \lambda_{21}\mathsf{Y}$$
$$\iota\mathsf{m}_2(\mathsf{x}, \mathsf{y}) = \iota\mathsf{m}_2(\mathsf{y}, \mathsf{x}) = \lambda_{21}\mathsf{X} + \lambda_{12}\mathsf{Y} \tag{5.F}$$
$$\iota\mathsf{m}_2(\mathsf{y}, \mathsf{y}) = \lambda_{12}\mathsf{X} + \lambda_{03}\mathsf{Y}.$$

Substituting (5.F) and 5.5 directly into the left side of (5.C), it follows that we may choose

$$\mathsf{f}_2(\mathsf{x}, \mathsf{x}) = -(\lambda_{12}\mathsf{k}_{r-2} + \lambda_{03}\mathsf{k}_{r-3} + \mathsf{e}_{3,0})$$
$$\mathsf{f}_2(\mathsf{x}, \mathsf{y}) = \mathsf{f}_2(\mathsf{y}, \mathsf{x}) = -(\lambda_{30}\mathsf{k}_{r+1} + \lambda_{03}\mathsf{k}_{r-2} + \mathsf{e}_{3,1}) \tag{5.G}$$
$$\mathsf{f}_2(\mathsf{y}, \mathsf{y}) = -(\lambda_{30}\mathsf{k}_{r+2} + \lambda_{21}\mathsf{k}_{r+1} + \mathsf{e}_{3,2}).$$

5.3. **All m_2 Products.** By Remark 5.2, if one of the inputs for m_2 has degree three, or if both have degree two, then the product is necessarily zero. Thus the only remaining m_2 to consider are those having one input of degree one, and one input of degree two.

Lemma 5.6. *In $\mathcal{C}$, the following statements hold.*

$$\mathsf{x} \star \mathsf{X} = -\mathfrak{s} = \mathsf{X} \star \mathsf{x} \qquad\qquad \mathsf{x} \star \mathsf{Y} = -\mathbb{D}(\mathsf{K}_{r-1}) = \mathsf{Y} \star \mathsf{x}$$
$$\mathsf{y} \star \mathsf{Y} = -\mathfrak{s} = \mathsf{Y} \star \mathsf{y} \qquad\qquad \mathsf{y} \star \mathsf{X} = -\mathbb{D}(\mathsf{K}_{r+1}) = \mathsf{X} \star \mathsf{y}$$

Proof. Since $\mathsf{Y} = (a^{\mathsf{r}-1} \cdot \mathsf{Z}_1, b^{\mathsf{s}} \cdot \mathsf{Z}_2, 0)$ and $\mathsf{X} = (a^{\mathsf{r}} \cdot \mathsf{Z}_1, b^{\mathsf{s}-1} \cdot \mathsf{Z}_2, 0)$, it follows directly from 4.16 that

$$\mathsf{x} \star \mathsf{X} = -(a^{\mathsf{r}} \cdot \mathfrak{s}_1, \ b^{\mathsf{s}} \cdot \mathfrak{s}_2, 0) = \mathsf{X} \star \mathsf{x} \qquad \mathsf{x} \star \mathsf{Y} = -(a^{\mathsf{r}-1} \cdot \mathfrak{s}_1, \ b^{\mathsf{s}+1} \cdot \mathfrak{s}_2, 0) = \mathsf{Y} \star \mathsf{x}$$
$$\mathsf{y} \star \mathsf{Y} = -(a^{\mathsf{r}} \cdot \mathfrak{s}_1, \ b^{\mathsf{s}} \cdot \mathfrak{s}_2, 0) = \mathsf{Y} \star \mathsf{y} \qquad \mathsf{y} \star \mathsf{X} = -(a^{\mathsf{r}+1} \cdot \mathfrak{s}_1, \ b^{\mathsf{s}-1} \cdot \mathfrak{s}_2, 0) = \mathsf{X} \star \mathsf{y}.$$

Now the statement follows since $\mathfrak{s} = (a^{\mathsf{r}} \cdot \mathfrak{s}_1, b^{\mathsf{s}} \cdot \mathfrak{s}_2, 0)$, and 4.12. □

Now $\mathsf{m}_2(\bullet, \blacktriangle) = (-1)^{|\bullet|+1}[\bullet \star \blacktriangle]$, so applying $[-]$ then ι to 5.6 gives

$$\begin{aligned}
\iota\mathsf{m}_2(\mathsf{x}, \mathsf{X}) &= -\mathfrak{s} & \iota\mathsf{m}_2(\mathsf{x}, \mathsf{Y}) &= 0_{\mathcal{C}} \\
\iota\mathsf{m}_2(\mathsf{X}, \mathsf{x}) &= \ \ \mathfrak{s} & \iota\mathsf{m}_2(\mathsf{Y}, \mathsf{x}) &= 0_{\mathcal{C}} \\
\iota\mathsf{m}_2(\mathsf{y}, \mathsf{Y}) &= -\mathfrak{s} & \iota\mathsf{m}_2(\mathsf{y}, \mathsf{X}) &= 0_{\mathcal{C}} \\
\iota\mathsf{m}_2(\mathsf{Y}, \mathsf{y}) &= \ \ \mathfrak{s} & \iota\mathsf{m}_2(\mathsf{X}, \mathsf{y}) &= 0_{\mathcal{C}}.
\end{aligned} \qquad (5.\mathrm{H})$$

Substituting (5.H) and 5.6 directly into the left hand side of (5.C), it follows that

$$\begin{aligned}
0_{\mathcal{C}} &= \mathbb{D}\mathsf{f}_2(\mathsf{x}, \mathsf{X}) & \mathbb{D}(\mathsf{K}_{\mathsf{r}-1}) &= \mathbb{D}\mathsf{f}_2(\mathsf{x}, \mathsf{Y}) \\
0_{\mathcal{C}} &= \mathbb{D}\mathsf{f}_2(\mathsf{X}, \mathsf{x}) & -\mathbb{D}(\mathsf{K}_{\mathsf{r}-1}) &= \mathbb{D}\mathsf{f}_2(\mathsf{Y}, \mathsf{x}) \\
0_{\mathcal{C}} &= \mathbb{D}\mathsf{f}_2(\mathsf{y}, \mathsf{Y}) & \mathbb{D}(\mathsf{K}_{\mathsf{r}+1}) &= \mathbb{D}\mathsf{f}_2(\mathsf{y}, \mathsf{X}) \\
0_{\mathcal{C}} &= \mathbb{D}\mathsf{f}_2(\mathsf{Y}, \mathsf{y}) & -\mathbb{D}(\mathsf{K}_{\mathsf{r}+1}) &= \mathbb{D}\mathsf{f}_2(\mathsf{X}, \mathsf{y}),
\end{aligned}$$

thus we may choose

$$\begin{aligned}
\mathsf{f}_2(\mathsf{x}, \mathsf{X}) = 0_{\mathcal{C}} \quad \mathsf{f}_2(\mathsf{y}, \mathsf{Y}) = 0_{\mathcal{C}} \quad \mathsf{f}_2(\mathsf{x}, \mathsf{Y}) = \mathsf{K}_{\mathsf{r}-1} \quad \mathsf{f}_2(\mathsf{y}, \mathsf{X}) = \mathsf{K}_{\mathsf{r}+1} \\
\mathsf{f}_2(\mathsf{X}, \mathsf{x}) = 0_{\mathcal{C}} \quad \mathsf{f}_2(\mathsf{Y}, \mathsf{y}) = 0_{\mathcal{C}} \quad \mathsf{f}_2(\mathsf{Y}, \mathsf{x}) = -\mathsf{K}_{\mathsf{r}-1} \quad \mathsf{f}_2(\mathsf{X}, \mathsf{y}) = -\mathsf{K}_{\mathsf{r}+1}.
\end{aligned} \qquad (5.\mathrm{I})$$

5.4. Simple Maps and Consequences of Degree. As the collection of $\mathsf{a}_1 \otimes \ldots \otimes \mathsf{a}_n$ with each $\mathsf{a}_i \in \{\mathsf{x}, \mathsf{y}, \mathsf{X}, \mathsf{Y}, \mathfrak{s}\}$ is linearly independent in the tensor algebra $\mathrm{T}_{\mathbb{K}}(\mathsf{A})$, the map f_n may be defined by making a choice for each $\mathsf{f}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n)$ independently, compatible with (5.B). The following is clear, and is well-known.

Lemma 5.7. *Suppose that f_i is defined for $1 \le i \le n-1$, and further that the defining equation (5.A) gives $\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}$ where $\mathsf{a}_1, \ldots, \mathsf{a}_n \in \{\mathsf{x}, \mathsf{y}, \mathsf{X}, \mathsf{Y}, \mathfrak{s}\}$. Then $\mathsf{m}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = 0_{\mathsf{A}}$ and we may choose the map f_n so that $\mathsf{f}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}$.*

Proof. Since ι is injective, (5.B) is satisfied only if $\mathsf{m}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = [\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n)] = 0_{\mathsf{A}}$. Since $\mathbb{D}(0_{\mathcal{C}}) = 0_{\mathcal{C}} = (\iota\mathsf{m}_n - \mathsf{U}_n)(\mathsf{a}_1, \ldots, \mathsf{a}_n)$, the choice $\mathsf{f}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) := 0_{\mathcal{C}}$ is compatible with (5.B). □

Convention 5.8. If $\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}$, we choose f_n so that $\mathsf{f}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}$.

Usually controlling U_n and f_n is hard, but properties of both will be significantly simplified in our setting by the following.

Definition 5.9. We say that a collection of maps $\{\mathsf{f}_\ell \colon \mathsf{A}^{\otimes \ell} \to \mathcal{C}\}_{\ell=1}^k$ is *simple* if for all $1 \le \ell \le k$, and for all inputs $\mathsf{a}_i \in \{\mathsf{x}, \mathsf{y}, \mathsf{X}, \mathsf{Y}, \mathfrak{s}\}$,

$$\mathsf{f}_\ell(\mathsf{a}_1, \ldots, \mathsf{a}_\ell) = (\mathsf{S}, \mathsf{T}, 0) \in \mathcal{C}_{d_1 + \cdots + d_\ell + 1 - \ell}$$

for some chains S, T that depend on ℓ and the a_i, where $d_i = |\mathsf{a}_i|$.

It will turn out, as part of our inductive process, that we may choose the f_n to be simple to any degree. As such, later we will require the following two results.

Lemma 5.10. *Fix $n \ge 2$. Suppose that the set of maps $\{\mathsf{f}_1, \ldots, \mathsf{f}_{n-1}\}$ is simple and consider inputs $\mathsf{a}_i \in \{\mathsf{x}, \mathsf{y}, \mathsf{X}, \mathsf{Y}, \mathfrak{s}\}$ for $1 \le i \le n$ satisfying $\sum(|\mathsf{a}_i| - 1) \ge 2$. Then the following statements hold.*

(1) *For all ℓ such that $1 \le \ell \le n-1$,*

$$\mathsf{f}_\ell(\mathsf{a}_1, \ldots, \mathsf{a}_\ell) \star \mathsf{f}_{n-\ell}(\mathsf{a}_{\ell+1}, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}.$$

(2) $\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}.$

Proof. (1) Set $d_i = |\mathsf{a}_i|$. Each map f_n has degree $1 - n$, so that

$$\mathsf{f}_\ell(\mathsf{a}_1, \ldots, \mathsf{a}_\ell) \in \mathcal{C}_{d_1 + \cdots + d_\ell + 1 - \ell} \quad \text{and} \quad \mathsf{f}_{n-\ell}(\mathsf{a}_{\ell+1}, \ldots, \mathsf{a}_n) \in \mathcal{C}_{d_{\ell+1} + \cdots + d_n + 1 - (n-\ell)}.$$

By assumption, both f_ℓ and $\mathsf{f}_{n-\ell}$ are simple, and so multiplying these two chains together by $\star$ still gives an element of the form $(\mathsf{S}, \mathsf{T}, 0) \in \mathcal{C}_d$, where

$$d := (d_1 + \ldots + d_\ell + 1 - \ell) + (d_{\ell+1} + \cdots + d_n + 1 - (n-\ell)) = 2 - n + \sum d_i.$$

If $d \geq 4$, then trivially $\mathsf{S} = \mathsf{T} = 0$ since by definition $\mathcal{C}_4 = 0 \oplus 0 \oplus \check{\mathrm{C}}^1(\mathcal{U}, \mathcal{H}om^3(\mathcal{E}, \mathcal{E}))$ and $\mathcal{C}_{\geq 5} = 0$. Since $d \geq 4$ is equivalent to $\sum(d_i - 1) \geq 2$, the claim follows.

(2) By (1), the top right sum in (5.A) is zero. Now again using the assumption that the f_i are all simple, any term of the bottom line of (5.A) is of the form

$$\mathsf{f}_{n-j+1}(\hat{\mathsf{a}}_1, \ldots, \hat{\mathsf{a}}_k, \mathsf{m}_j(\mathsf{a}_{k+1}, \ldots, \mathsf{a}_{k+j}), \mathsf{a}_{k+j+1}, \ldots, \mathsf{a}_n) = (\mathsf{S}, \mathsf{T}, 0) \in \mathcal{C}_e$$

where $e = \sum d_i + 2 - n$. Since by hypothesis $\sum d_i \geq n + 2$, necessarily $e \geq 4$ and thus again $\mathsf{S} = \mathsf{T} = 0$. $\square$

Whilst 5.10 deals with $\sum(|\mathsf{a}_i| - 1) \geq 2$, the case $\sum(|\mathsf{a}_i| - 1) = 1$ is mildly more tricky, and requires more assumptions.

Proposition 5.11. *For $n \geq 3$, suppose that the set of maps $\{\mathsf{f}_1, \ldots, \mathsf{f}_{n-1}\}$ are simple, and have been chosen to adhere to Convention 5.8.*

(1) *Consider inputs $\mathsf{a}_i \in \{\mathsf{x}, \mathsf{y}, \mathsf{X}, \mathsf{Y}\}$ with $1 \leq i \leq n$, where precisely one of the a_i has degree two. Then*

$$\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = \sum_{\ell=1}^{n-1} (-1)^{|\mathsf{f}_\ell(\mathsf{a}_1, \ldots, \mathsf{a}_\ell)| + 1} \mathsf{f}_\ell(\mathsf{a}_1, \ldots, \mathsf{a}_\ell) \star \mathsf{f}_{n-\ell}(\mathsf{a}_{\ell+1}, \ldots, \mathsf{a}_n). \qquad (5.\mathrm{J})$$

(2) *Suppose in addition that*
   (a) *for any choices of $\mathsf{b}_* \in \{\mathsf{x}, \mathsf{y}\}$ for $2 \leq \ell \leq n - 1$ each $\mathsf{f}_\ell(\mathsf{b}_1, \ldots, \mathsf{b}_\ell) \in \mathcal{C}_1$ is a $\mathbb{K}$-linear combination of k_i terms and $\mathsf{e}_{j,j'}$ terms, and*
   (b) *for $\mathsf{b} \in \{\mathsf{x}, \mathsf{y}\}$ and $\mathsf{B} \in \{\mathsf{X}, \mathsf{Y}\}$ both $\mathsf{f}_2(\mathsf{b}, \mathsf{B})$ and $\mathsf{f}_2(\mathsf{B}, \mathsf{b})$ are polynomial multiples of some K_i.*
   Then $\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}$.

Proof. (1) The bottom line of (5.A) is a linear combination of terms of the form $\mathsf{f}_\ell(\mathsf{b}_1, \ldots, \mathsf{b}_\ell)$ where $\mathsf{b}_i \in \{\mathsf{x}, \mathsf{y}, \mathsf{X}, \mathsf{Y}\}$, $2 \leq \ell \leq n - 1$, and

$$|\mathsf{b}_1| + \ldots + |\mathsf{b}_\ell| = \ell + 2.$$

By 5.10(2) applied to ℓ, $\mathsf{U}_\ell(\mathsf{b}_1, \ldots, \mathsf{b}_\ell) = 0_{\mathcal{C}}$. By the assumption that Convention 5.8 holds, $\mathsf{f}_\ell(\mathsf{b}_1, \ldots, \mathsf{b}_\ell) = 0_{\mathcal{C}}$. Thus the bottom line of (5.A) is zero, proving the claim.

(2) We work by induction on $n \geq 3$. For the initial case $n = 3$,

$$\mathsf{U}_3(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3) = \pm \mathsf{a}_1 \star \mathsf{f}_2(\mathsf{a}_2, \mathsf{a}_3) \pm \mathsf{f}_2(\mathsf{a}_1, \mathsf{a}_2) \star \mathsf{a}_3.$$

Term by term, if the inputs to f_2 have degree one, then the singleton is X or Y and so the term vanishes by assumption (a) and 4.23. On the other hand, if one of the arguments to f_2 has degree two, then the singleton is x or y, and so the term vanishes by assumption (b) and 4.23. Thus $\mathsf{U}_3(\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3) = 0_{\mathcal{C}}$.

If $n > 3$, then each individual term in (5.J) vanishes as follows:

- $\mathsf{a}_1 \star \mathsf{f}_{n-1}(\mathsf{a}_2, \ldots, \mathsf{a}_n)$: if $|\mathsf{a}_1| = 1$ then by induction $\mathsf{U}_{n-1}(\mathsf{a}_2, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}$, so by the assumption that Convention 5.8 has been adhered to, $\mathsf{f}_{n-1}(\mathsf{a}_2, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}$, and thus the term vanishes. Otherwise $\mathsf{a}_1 = \mathsf{X}$ or Y, and so the term vanishes by (a) and 4.23.
- $\mathsf{f}_2(\mathsf{a}_1, \mathsf{a}_2) \star \mathsf{f}_{n-2}(\mathsf{a}_3, \ldots, \mathsf{a}_n)$: if $n = 4$, then by (a) and (b) this is a product of a multiple of K_* with k_i and $\mathsf{e}_{j,j'}$ terms, so vanishes by 4.23. For $n > 4$, if $|\mathsf{a}_1| = |\mathsf{a}_2| = 1$ then the second factor $\mathsf{f}_{n-2}(\mathsf{a}_3, \ldots, \mathsf{a}_n) = 0_{\mathcal{C}}$ again by induction and Convention 5.8. Otherwise, one of $\mathsf{a}_1, \mathsf{a}_2$ has degree two, so the term vanishes by (b), (5.I) and 4.23.
- $\mathsf{f}_\ell(\mathsf{a}_1, \ldots, \mathsf{a}_\ell) \star \mathsf{f}_{n-\ell}(\mathsf{a}_{\ell+1}, \ldots, \mathsf{a}_n)$ with $\ell, n - \ell \geq 3$: one of the two factors vanishes again by induction and Convention 5.8.

Similarly, the terms $f_{n-2}(a_1, \ldots, a_{n-2}) \star f_2(a_{n-1}, a_n)$ and $f_{n-1}(a_1, \ldots, a_{n-1}) \star a_n$ vanish by symmetry. Thus $U_n(a_1, \ldots, a_n) = 0_{\mathcal{C}}$, as required. $\qquad\square$

5.5. **All Higher m_n Products.** The following is the main result of this paper. Note that the sum in (1) generalises the right hand side of the displayed equations in 5.5.

Theorem 5.12. *For any $n \geq 2$ and any decomposition $n = j + k$ with $j, k \geq 0$, the following statements hold.*

(1) *For all sequences $a_1, \ldots, a_n \in \{x, y\}$, where there are j occurrences of x and k occurrences of y, in $\mathcal{C}$ there is an equality*

$$U_n(a_1, \ldots, a_n) = \lambda_{j+1,k} X + \lambda_{j,k+1} Y - \mathbb{D}f_n(a_1, \ldots, a_n)$$

where

$$f_n(a_1, \ldots, a_n) = -\left(\sum_{i=0}^{j-1} \lambda_{i,n+1-i}\, k_{r+i-(j+1)} + \sum_{i=j+2}^{n+1} \lambda_{i,n+1-i}\, k_{r+i-(j+1)} + e_{n+1,k} \right).$$

In particular, U_n is not affected by the order of the sequence of degree one inputs.

(2) *Suppose that $n \geq 3$ and $a_1, \ldots, a_n \in \{x, y, X, Y, \mathfrak{s}\}$, with $\sum |a_i| \geq n + 1$. Then $U_n(a_1, \ldots, a_n) = 0_{\mathcal{C}}$.*

The case $n = 2$ has already been considered: since $U_2(a_1, a_2) = (-1)^{|a_1|+1} a_1 \star a_2$, on $a_i \in \{x, y, X, Y, \mathfrak{s}\}$ the possible nonzero values of U_2 are 4.13 and 5.6.

Proof. Fix some $n \geq 3$. By induction we can *choose* f_i for $2 \leq i < n$ such that the following three conditions hold.

(1′) The condition (1) holds for all $i < n$. Indeed, the case $n = 2$ for (1) was established in 5.5, after recalling that on degree one inputs $U_2(\bullet, \blacktriangle) = \bullet \star \blacktriangle$.

(S) The collection $\{f_i\}_{i=1}^{n-1}$ is simple, since both f_1 and f_2 are simple by definition, (5.G) and (5.I).

(0) Whenever $U_i(a_1, \ldots, a_i) = 0_{\mathcal{C}}$ for some $i < n$ and $a_1, \ldots, a_i \in \{x, y, X, Y, \mathfrak{s}\}$, then $f_i(a_1, \ldots, a_i) = 0_{\mathcal{C}}$. That is, we adhere to Convention 5.8.

With these choices, given degree one inputs $b_1, \ldots, b_{n-1} \in \{x, y\}$ where there are j occurrences of x and $\mathfrak{k}$ occurrences of y, the following statements hold.

(a) After passing to cohomology, writing in terms of the basis, then applying ι, as explained in §5.1 (and underneath 5.5) $m_{n-1}(b_1, \ldots, b_{n-1}) = \lambda_{j+1,\mathfrak{k}} X + \lambda_{j,\mathfrak{k}+1} Y$.

(b) There are equalities

$$f_{n-1}(b_1, \ldots, b_{n-1}) = -\left(\sum_{i=0}^{j-1} \lambda_{i,n-i}\, k_{r+i-(j+1)} + \sum_{i=j+2}^{n} \lambda_{i,n-i}\, k_{r+i-(j+1)} + e_{n,\mathfrak{k}} \right)$$

$$= -\left(\sum_{i=\mathfrak{k}+2}^{n} \lambda_{n-i,i}\, k_{r+\mathfrak{k}-i} + \sum_{i=0}^{\mathfrak{k}-1} \lambda_{n-i,i}\, k_{r+\mathfrak{k}-i} + e_{n,\mathfrak{k}} \right)$$

where the second line is just the reindexing $i \mapsto n - i$, using $n - 1 = j + \mathfrak{k}$.

(c) For any $2 \leq \ell \leq n - 2$,

$$f_\ell(b_1, \ldots, b_\ell) \star f_{n-\ell}(b_{\ell+1}, \ldots, b_n) = 0_{\mathcal{C}}.$$

Indeed, each f_ℓ evaluated on degree one inputs is a linear combination of k_* and $e_{*,*}$, and these multiply to zero by 4.23.

Now consider $U_n(a_1, \ldots, a_n)$ for $a_1, \ldots, a_n \in \{x, y\}$, where there are j occurrences of x and k occurrences of y. We consider the two rows of (5.A) separately. In the first row, for each $2 \leq \ell \leq n - 2$, the term $f_\ell(a_1, \ldots, a_\ell) \star f_{n-\ell}(a_{\ell+1}, \ldots, a_n)$ vanishes by (c). In the second row of (5.A), whenever j is such that $2 \leq j \leq n - 2$ the term

$$f_{n-j+1}(\hat{a}_1, \ldots, \hat{a}_k, m_j(a_{k+1}, \ldots, a_{k+j}), a_{k+j+1}, \ldots, a_n)$$

has $n - j \geq 2$ arguments of degree one and a single argument $m_j(a_{k+1}, \ldots, a_{k+j})$ of degree two. This vanishes by 5.11(2) and (0) above, since the hypotheses there hold by (b) and

(5.I). These vanishing statements leave only

$$\begin{aligned} \mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) =\ & \mathsf{a}_1 \star \mathsf{f}_{n-1}(\mathsf{a}_2,\ldots,\mathsf{a}_n) + \mathsf{f}_{n-1}(\mathsf{a}_1,\ldots,\mathsf{a}_{n-1}) \star \mathsf{a}_n \\ & - \mathsf{f}_2(\mathsf{a}_1, \mathsf{m}_{n-1}(\mathsf{a}_2,\ldots,\mathsf{a}_n)) - \mathsf{f}_2(\mathsf{m}_{n-1}(\mathsf{a}_1,\ldots,\mathsf{a}_{n-1}), \mathsf{a}_n). \end{aligned} \tag{5.K}$$

The heart of the proof is now to verify the claim that

$$\mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) = (\, a^k v_2^{\mathsf{t}-(n+1)} A_{\geq n+1} \cdot \mathsf{Z}_1, \ b^j w_2^{\mathsf{t}-(n+1)} B_{\geq n+1} \cdot \mathsf{Z}_2, \ 0)$$

holds. The proof of this claim splits into two cases.

Case 1: $\mathsf{a}_1 = \mathsf{a}_n$. In this case, the number of occurrences ℓ of y in $\mathsf{a}_1,\ldots,\mathsf{a}_{n-1}$ equals the number of occurrences of y in $\mathsf{a}_2,\ldots,\mathsf{a}_n$. Similarly for the number of x's. The inductive hypothesis (a) implies that $\mathsf{m}_{n-1}(\mathsf{a}_2,\ldots,\mathsf{a}_n) = \mathsf{m}_{n-1}(\mathsf{a}_1,\ldots,\mathsf{a}_{n-1})$, and this clearly has degree two. But by (5.I) f_2 is antisymmetric on any mix of degree one and degree two inputs, thus the bottom two terms in (5.K) cancel to give

$$\mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) = \mathsf{a}_1 \star \mathsf{f}_{n-1}(\mathsf{a}_2,\ldots,\mathsf{a}_n) + \mathsf{f}_{n-1}(\mathsf{a}_1,\ldots,\mathsf{a}_{n-1}) \star \mathsf{a}_n. \tag{5.L}$$

Now again since the number of occurrences ℓ of y in $\mathsf{a}_1,\ldots,\mathsf{a}_{n-1}$ equals the number of occurrences of y in $\mathsf{a}_2,\ldots,\mathsf{a}_n$, and similarly for the x's, the inductive hypothesis (b) implies that $\mathsf{f}_{n-1}(\mathsf{a}_2,\ldots,\mathsf{a}_n) = \mathsf{f}_{n-1}(\mathsf{a}_1,\ldots,\mathsf{a}_{n-1})$, and furthermore both terms equal expressions of the form

$$\begin{cases} -\sum \lambda_* \mathsf{k}_* - \mathsf{e}_{n,k} & \text{if } \mathsf{a}_1 = \mathsf{a}_n = \mathsf{x} \text{ (so } \ell = k) \\ -\sum \lambda_* \mathsf{k}_* - \mathsf{e}_{n,k-1} & \text{if } \mathsf{a}_1 = \mathsf{a}_n = \mathsf{y} \text{ (so } \ell = k-1). \end{cases}$$

Substituting this into (5.L), and using the fact that every degree one input anti-commutes with k_i by 4.15, the above (5.L) simplifies to

$$\begin{aligned} \mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) &= \begin{cases} -(\mathsf{x} \star \mathsf{e}_{n,k} \ \ \ + \mathsf{e}_{n,k} \star \mathsf{x}) & \text{if } \mathsf{a}_1 = \mathsf{a}_n = \mathsf{x} \\ -(\mathsf{y} \star \mathsf{e}_{n,k-1} + \mathsf{e}_{n,k-1} \star \mathsf{y}) & \text{if } \mathsf{a}_1 = \mathsf{a}_n = \mathsf{y} \end{cases} \\ &= -(\mathsf{x} \star \mathsf{e}_{n,k} \ \ \ + \mathsf{e}_{n,k} \star \mathsf{x}) && \text{(by 4.22(1))} \\ &= (\, a^k v_2^{\mathsf{t}-(n+1)} A_{\geq n+1} \cdot \mathsf{Z}_1, \ b^j w_2^{\mathsf{t}-(n+1)} B_{\geq n+1} \cdot \mathsf{Z}_2, \ 0) && \text{(by 4.22(2))} \end{aligned}$$

since $n = j + k$, which verifies the claim in Case 1.

Case 2: $\mathsf{a}_1 \neq \mathsf{a}_n$. Write k_1 for the number of occurrences of y in $\mathsf{a}_1,\ldots,\mathsf{a}_{n-1}$, and k_2 for the number of occurrences of y in $\mathsf{a}_2,\ldots,\mathsf{a}_n$. Thus, by inductive hypothesis (a),

$$\begin{aligned} \mathsf{m}_{n-1}(\mathsf{a}_1,\ldots,\mathsf{a}_{n-1}) &= \lambda_{n-k_1,k_1} \mathsf{X} + \lambda_{n-k_1-1,k_1+1} \mathsf{Y} \\ \mathsf{m}_{n-1}(\mathsf{a}_2,\ldots,\mathsf{a}_n) &= \lambda_{n-k_2,k_2} \mathsf{X} + \lambda_{n-k_2-1,k_2+1} \mathsf{Y}, \end{aligned}$$

and so the general formula (5.K) reads

$$\begin{aligned} \mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) =\ & \mathsf{a}_1 \star \mathsf{f}_{n-1}(\mathsf{a}_2,\ldots,\mathsf{a}_n) && + \mathsf{f}_{n-1}(\mathsf{a}_1,\ldots,\mathsf{a}_{n-1}) \star \mathsf{a}_n \\ & - \mathsf{f}_2(\mathsf{a}_1, \lambda_{n-k_2,k_2} \mathsf{X} + \lambda_{n-k_2-1,k_2+1} \mathsf{Y}) - \mathsf{f}_2(\lambda_{n-k_1,k_1} \mathsf{X} + \lambda_{n-k_1-1,k_1+1} \mathsf{Y}, \mathsf{a}_n). \end{aligned}$$

Substituting in the inductive hypothesis (b) for the f_{n-1} terms,

$$\begin{aligned} \mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) =\ & -\mathsf{a}_1 \star (\sum \lambda_* \mathsf{k}_* - \mathsf{e}_{n,k_2}) && - (\sum \lambda_* \mathsf{k}_* - \mathsf{e}_{n,k_1}) \star \mathsf{a}_n \\ & - \mathsf{f}_2(\mathsf{a}_1, \lambda_{n-k_2,k_2} \mathsf{X} + \lambda_{n-k_2-1,k_2+1} \mathsf{Y}) - \mathsf{f}_2(\lambda_{n-k_1,k_1} \mathsf{X} + \lambda_{n-k_1-1,k_1+1} \mathsf{Y}, \mathsf{a}_n). \end{aligned}$$

Since a_n anti-commutes with any k_*, and up to sign on the bottom right we can swap the order of the inputs by (5.I), it follows that $\mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) - (\mathsf{a}_1 \star \mathsf{e}_{n,k_2} + \mathsf{e}_{n,k_1} \star \mathsf{a}_n)$ equals

$$\begin{aligned} & -\mathsf{a}_1 \star (\sum \lambda_* \mathsf{k}_*) && + \mathsf{a}_n \star (\sum \lambda_* \mathsf{k}_*) \\ & -\mathsf{f}_2(\mathsf{a}_1, \lambda_{n-k_2,k_2} \mathsf{X} + \lambda_{n-k_2-1,k_2+1} \mathsf{Y}) && + \mathsf{f}_2(\mathsf{a}_n, \lambda_{n-k_1,k_1} \mathsf{X} + \lambda_{n-k_1-1,k_1+1} \mathsf{Y}). \end{aligned} \tag{5.M}$$

We now claim that (5.M) is zero. To ease notation, the proof splits into two subcases.

Subcase 2(a) $\mathsf{a}_1 = \mathsf{x}$, so $\mathsf{a}_n = \mathsf{y}$. The precise indices on the top leftmost term in (5.M) are

$$-\mathsf{a}_1 \star (\sum \lambda_* \mathsf{k}_*) = -\mathsf{x} \star \left(\sum_{i=k_2+2}^{n} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_2-i} + \sum_{i=0}^{k_2-1} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_2-i} \right).$$

By 4.15 $\mathsf{x} \star \mathsf{k}_i = \mathsf{K}_i$, thus $-\mathsf{a}_1 \star (\sum \lambda_* \mathsf{k}_*)$ is the sum $-\sum \lambda_{n-i,i} \mathsf{K}_{\mathsf{r}+k_2-i}$ containing all $\lambda_{n-i,i}$ terms except $i = k_2$ and $i = k_2 + 1$. Now since $\mathsf{a}_1 = \mathsf{x}$, by (5.I), it follows that

$$-\mathsf{f}_2(\mathsf{a}_1, \lambda_{n-k_2,k_2}\mathsf{X} + \lambda_{n-k_2-1,k_2+1}\mathsf{Y}) = -\lambda_{n-k_2-1,k_2+1}\mathsf{K}_{\mathsf{r}-1}.$$

Thus the two leftmost terms in (5.M) equal $-\sum \lambda_{n-i,i} \mathsf{K}_{\mathsf{r}+k_2-i}$, where the sum contains all $\lambda_{n-i,i}$ terms except $i = k_2$.

On the other hand, the precise indices on the top rightmost term in (5.M) are

$$\mathsf{a}_n \star \left(\sum \lambda_* \mathsf{k}_*\right) = \mathsf{y} \star \left(\sum_{i=k_1+2}^{n} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_1-i} + \sum_{i=0}^{k_1-1} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_1-i} \right).$$

Since $\mathsf{a}_1 = \mathsf{x}$, it follows that $k_1 = k_2 - 1$, and thus

$$\mathsf{a}_n \star \left(\sum \lambda_* \mathsf{k}_*\right) = \mathsf{y} \star \left(\sum_{i=k_2+1}^{n} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_2-1-i} + \sum_{i=0}^{k_2-2} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_2-1-i} \right).$$

By 4.15 $\mathsf{y} \star \mathsf{K}_i = \mathsf{K}_{i+1}$, thus $\mathsf{a}_n \star (\sum \lambda_* \mathsf{k}_*)$ is the sum $\sum \lambda_{n-i,i}\mathsf{K}_{\mathsf{r}+k_2-i}$ containing all $\lambda_{n-i,i}$ terms except $i = k_2$ and $i = k_2 - 1$. Again by (5.I) it follows that

$$\mathsf{f}_2(\mathsf{a}_n, \lambda_{n-k_1,k_1}\mathsf{X} + \lambda_{n-k_1-1,k_1+1}\mathsf{Y}) = \lambda_{n-k_1,k_1}\mathsf{K}_{\mathsf{r}+1} = \lambda_{n-k_2+1,k_2-1}\mathsf{K}_{\mathsf{r}+1}.$$

Thus the two rightmost terms in (5.M) equal $+\sum \lambda_{n-i,i}\mathsf{K}_{\mathsf{r}+k_2-i}$ where the sum contains all $\lambda_{n-i,i}$ terms except $i = k_2$.

Combining, the leftmost terms in (5.M) cancel the rightmost terms, and thus (5.M) is zero, as claimed.

Subcase 2(b) $\mathsf{a}_1 = \mathsf{y}$, so $\mathsf{a}_n = \mathsf{x}$ and now $k_2 = k_1 - 1$. In a similar manner to the above,

$$-\mathsf{a}_1 \star \left(\sum \lambda_* \mathsf{k}_*\right) = -\mathsf{y} \star \left(\sum_{i=k_2+2}^{n} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_2-i} + \sum_{i=0}^{k_2-1} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_2-i} \right)$$

$$= -\mathsf{y} \star \left(\sum_{i=k_1+1}^{n} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_1-1-i} + \sum_{i=0}^{k_1-2} \lambda_{n-i,i}\, \mathsf{k}_{\mathsf{r}+k_1-1-i} \right)$$

$$= \sum_{i=k_1+1}^{n} \lambda_{n-i,i}\, \mathsf{K}_{\mathsf{r}+k_1-i} + \sum_{i=0}^{k_1-2} \lambda_{n-i,i}\, \mathsf{K}_{\mathsf{r}+k_1-i},$$

and so the two leftmost terms of (5.M) now equal $-\sum \lambda_{n-i,i}\mathsf{K}_{\mathsf{r}+k_1-i}$ where the sum contains all $\lambda_{n-i,i}$ terms except $i = k_1$. Similarly the rightmost terms of (5.M) give the same sum, with the sign swapped. Thus again the leftmost and rightmost terms cancel, so (5.M) is zero.

It follows in all cases that (5.M) is zero, and so consequently

$$\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = -(\mathsf{a}_1 \star \mathsf{e}_{n,k_2} + \mathsf{e}_{n,k_1} \star \mathsf{y}).$$

Now if $\mathsf{a}_1 = \mathsf{x}$, necessarily $\mathsf{a}_n = \mathsf{y}$ and so there $k_1 = k - 1$ and $k_2 = k$. Similarly, if $\mathsf{a}_1 = \mathsf{y}$, then $k_1 = k$ and $k_2 = k - 1$. Consequently, it follows that

$$\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = \begin{cases} -(\mathsf{x} \star \mathsf{e}_{n,k} + \mathsf{e}_{n,k-1} \star \mathsf{y}) & \text{if } \mathsf{a}_1 = \mathsf{x} \\ -(\mathsf{y} \star \mathsf{e}_{n,k-1} + \mathsf{e}_{n,k} \star \mathsf{x}) & \text{if } \mathsf{a}_1 = \mathsf{y} \end{cases}$$

$$= -(\mathsf{x} \star \mathsf{e}_{n,k} + \mathsf{e}_{n,k} \star \mathsf{x}) \qquad\qquad\qquad\quad \text{(by 4.22(1))}$$

$$= (\,a^k v_2^{\mathsf{t}-(n+1)} A_{\geq n+1} \cdot \mathsf{Z}_1,\; b^j w_2^{\mathsf{t}-(n+1)} B_{\geq n+1} \cdot \mathsf{Z}_2, 0), \qquad \text{(by 4.22(2))}$$

which verifies the claim in Case 2.

Thus always $\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n) = (\,a^k v_2^{\mathsf{t}-(n+1)} A_{\geq n+1} \cdot \mathsf{Z}_1,\; b^j w_2^{\mathsf{t}-(n+1)} B_{\geq n+1} \cdot \mathsf{Z}_2, 0)$, and from here the proof is elementary. Decomposing the right hand side using (4.C),

$\mathsf{U}_n(\mathsf{a}_1, \ldots, \mathsf{a}_n)$

$$= (\,a^k A_{n+1} \cdot \mathsf{Z}_1,\; b^j B_{n+1} \cdot \mathsf{Z}_2, 0) + (\,a^k v_2^{\mathsf{t}-(n+1)} A_{\geq n+2} \cdot \mathsf{Z}_1,\; b^j w_2^{\mathsf{t}-(n+1)} B_{\geq n+2} \cdot \mathsf{Z}_2, 0)$$

$$= (\,a^k A_{n+1} \cdot \mathsf{Z}_1,\; b^j B_{n+1} \cdot \mathsf{Z}_2, 0) + \mathbb{D}(\mathsf{e}_{n+1,k}). \qquad\qquad\qquad \text{(by 4.21)}$$

Now since by definition (reading 2.7 backwards)

$$a^k A_{n+1} = \lambda_{0,n+1} a^{r-(n+1)+k} + \ldots + \lambda_{j,k+1}\, a^{r-1} + \lambda_{j+1,k}\, a^r + \ldots + \lambda_{n+1,0}\, a^{r+k},$$

applying 4.10 it follows that

$$(a^k A_{n+1} \cdot \mathsf{Z}_1,\ b^j B_{n+1} \cdot \mathsf{Z}_2,\ 0) = \lambda_{j+1,k}\, \mathsf{X} + \lambda_{j,k+1}\mathsf{Y}$$

$$+ \mathbb{D}\left(\sum_{i=0}^{j-1} \lambda_{i,n+1-i}\, \mathsf{k}_{r+i-(j+1)} + \sum_{i=j+2}^{n+1} \lambda_{i,n+1-i}\, \mathsf{k}_{r+i-(j+1)} \right).$$

Consequently we can choose f_n on degree one inputs to satisfy the condition in (1).

For (2), consider $\mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n)$ with $\mathsf{a}_i \in \mathsf{A}_{d_i}$. If $\sum d_i \geq n+2$, then $\mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) = 0_{\mathcal{C}}$ by 5.10(2). If $\sum d_i = n+1$, that is one of the inputs has degree two and all the others have degree one, then $\mathsf{U}_n(\mathsf{a}_1,\ldots,\mathsf{a}_n) = 0_{\mathcal{C}}$ by 5.11(2): as before, the hypotheses there hold by induction hypothesis (b), (0) and (5.I).

This establishes (1) and (2) for n. By 5.7, we can choose the map f_n to be zero whenever at least one input is not degree one. Since further on degree one inputs f_n is a combination of k and e, it follows that f_n is simple. This verifies (S) and (0) replacing n by $n+1$. Thus (1'), (S) and (0) all hold replacing n by $n+1$, allowing the induction to proceed. $\square$

5.6. **Summary.** The previous subsections combine to verify the following, which is the main result in the introduction. To be consistent with the notation there (where x and X are the classes in cohomology), this subsection introduces the one further abuse of notation that x,X now also denote their classes in cohomology $[\mathsf{x}],[\mathsf{X}]$, etc. Since no more calculations with chains are needed, this introduces no ambiguities.

Thus consider the graded vector space $\mathsf{A} = \bigoplus_{i \in \mathbb{Z}} \mathsf{A}_i$, where

$$\mathsf{A}_i = \begin{cases} \mathbb{K} & \text{if } i = 0,3 \\ \mathbb{K}^2 & \text{if } i = 1,2 \\ 0 & \text{else}, \end{cases}$$

and write 1 for the basis of A_0, x,y for the basis of A_1, and X,Y for the basis of A_2, and $\mathfrak{s}$ for the basis of A_3. Recall that all A_∞-algebras in this paper are strictly unital.

Corollary 5.13. *The following defines an* A_∞*-structure on* A*, and furthermore the resulting* A_∞*-algebra is quasi-isomorphic to the* DG*-algebra* $\mathcal{C}$.

(1) *For any* $n \geq 2$ *and any decomposition* $n = j + k$ *with* $j,k \geq 0$,

$$\mathsf{m}_n(\underbrace{\mathsf{x},\ldots,\mathsf{x}}_{j}, \underbrace{\mathsf{y},\ldots,\mathsf{y}}_{k}) = \lambda_{j+1,k}\, \mathsf{X} + \lambda_{j,k+1}\mathsf{Y},$$

where the λ*'s are the coefficients from the glue in* (2.A).
(2) *More generally,* m_n *with* $n \geq 2$ *applied only to degree one inputs (so, combinations of* x *and* y*) does not depend on the order of those degree one inputs, and thus is determined by* (1) *above.*
(3) *The only other non-zero products are*

$$-\mathsf{m}_2(\mathsf{x},\mathsf{X}) = \mathfrak{s} = \mathsf{m}_2(\mathsf{X},\mathsf{x}) \qquad - \mathsf{m}_2(\mathsf{y},\mathsf{Y}) = \mathfrak{s} = \mathsf{m}_2(\mathsf{Y},\mathsf{y}).$$

Proof. This is now a direct consequence of 5.12 and (5.H), given those followed the construction for Kadeishvili's Theorem 5.1 outlined in §5.1. $\square$

6. Corollaries

This section works over $\mathbb{K} = \mathbb{C}$, or any algebraically closed field of characteristic zero (which we still denote $\mathbb{C}$), and gives the deformation theory and mirror symmetry consequences of the previous sections. Throughout, given scalars $\lambda_{jk} \in \mathbb{C}$ subject to 2.4, consider the variety $\mathcal{X}$ defined by (2.A), with associated $\mathbb{P}^1 \cong \mathrm{C} \subset \mathcal{X}$.

6.1. **NC Deformation Theory Summary.** Maintaining the setup above, the sheaf $\mathcal{O}_C \in \operatorname{coh}\mathcal{X}$ gives rise to the noncommutative deformation functor

$$\operatorname{Def}_{\mathcal{O}_C} \colon \mathfrak{art}_1 \to \mathrm{Set}$$

defined in e.g. the survey [K3, 2.1] or [B]. As is well known e.g. [K3, §2], this functor admits a prorepresentable hull, written Λ_{def}, called the NC deformation algebra.

As is also well-known (see e.g. [T, p615], or [K3, §5]), Λ_{def} can be explicitly presented using the A_∞-structure of any DGA quasi-isomorphic to $\operatorname{End}_X^{\mathrm{DG}}(\mathcal{I})$ as in §3.5, where $\mathcal{I}$ is an injective resolution of $\mathcal{O}_C$. Indeed, the A_∞-morphisms give a morphism

$$\mathsf{m} := \sum_{i \geq 2} \mathsf{m}_i \colon \bigoplus_{i \geq 2} \operatorname{Ext}_{\mathcal{X}}^1(\mathcal{O}_C, \mathcal{O}_C)^{\otimes i} \to \operatorname{Ext}_{\mathcal{X}}^2(\mathcal{O}_C, \mathcal{O}_C)$$

which dualises to give a presentation

$$\Lambda_{\mathrm{def}} \cong \frac{\mathbb{C}\langle\!\langle \operatorname{Ext}_{\mathcal{X}}^1(\mathcal{O}_C, \mathcal{O}_C)^* \rangle\!\rangle}{\operatorname{Image}(\mathsf{m}^*)} = \frac{\mathbb{C}\langle\!\langle x, y \rangle\!\rangle}{\operatorname{Image}(\mathsf{m}^*)}.$$

By 3.5, we can compute the above instead using the A_∞-algebra structure on the more manageable DGA $\mathcal{C}$, and thus below we will freely use 5.13 to describe Λ_{def}.

6.2. **Superpotentials and Necklaces.** Following the conventions in e.g. [DWZ, BW, D], consider the $\mathbb{C}$-linear map $\partial_x \colon \mathbb{C}\langle\!\langle x, y \rangle\!\rangle \to \mathbb{C}\langle\!\langle x, y \rangle\!\rangle$ which simply 'strikes off' the leftmost x of each monomial. Thus, on monomials

$$\partial_x(m) = \begin{cases} n & \text{if } m = xn \\ 0 & \text{otherwise,} \end{cases}$$

with ∂_y being defined similarly. In contrast, the cyclic derivative is the $\mathbb{C}$-linear map $\delta_x \colon \mathbb{C}\langle\!\langle x, y \rangle\!\rangle \to \mathbb{C}\langle\!\langle x, y \rangle\!\rangle$ which on monomials sends

$$x_{i_1} \ldots x_{i_t} \mapsto \sum_{j=1}^{t} \partial_x(x_{i_j} x_{i_{j+1}} \ldots x_{i_t} \cdot x_{i_1} \ldots x_{i_{j-1}}),$$

with δ_y being defined similarly. For $f \in \mathbb{C}\langle\!\langle \mathsf{x} \rangle\!\rangle$, the Jacobi algebra is defined to be

$$\mathcal{J}\mathrm{ac}(f) := \frac{\mathbb{C}\langle\!\langle x, y \rangle\!\rangle}{(\!(\delta_x f, \delta_y f)\!)}$$

where $(\!(\delta_x f, \delta_y f)\!)$ is the closure of the two-sided ideal $(\delta_x f, \delta_y f)$.

Recall from the introduction that the free necklace polynomial is defined to be

$$\mathcal{N}_{j,k}(x, y) := \frac{1}{j+k} \sum_{m \in \mathcal{O}\mathrm{rb}_{j,k}} |m| \cdot p_m.$$

Remark 6.1. It is immediate from the definition that, if we instead work up to cyclic rotation, we may replace $|m| p_m$ with the m distinct representatives of the orbit. In this way, up to cyclic permutation, $\mathcal{N}_{j,k}$ is clearly then equal to all terms with j occurrences of x in the free algebra expansion of $\frac{1}{j+k}(x+y)^{j+k}$.

Notation 6.2. Write $\mathcal{M}\mathrm{ono}_{j,k}$ for the sum of all monomials in x and y, inside the free algebra, where there are j occurrences of x, and k occurrences of y.

For calibration, $\mathcal{M}\mathrm{ono}_{2,2} = xxyy + xyyx + yyxx + yxxy + xyxy + yxyx$. The following is elementary, where $\mathbb{K} = \mathbb{C}$ is used to allow the denominators in the proof.

Lemma 6.3. $\delta_x(\mathcal{N}_{j+1,k}) = \mathcal{M}\mathrm{ono}_{j,k}$ and $\delta_y(\mathcal{N}_{j,k+1}) = \mathcal{M}\mathrm{ono}_{j,k}$.

Proof. This is follows at once from the definitions, since

$$\delta_x(\mathcal{N}_{j+1,k}) = \sum_{m \in \mathcal{O}\mathrm{rb}_{j+1,k}} \frac{|m|}{j+1+k} \cdot \delta_x(p_m) = \mathcal{M}\mathrm{ono}_{j,k}. \qquad \square$$

6.3. **Deformation Theory Corollaries.** The following is now immediate, and is one of the main results.

Corollary 6.4. *The* NC *deformation algebra* Λ_{def} *of* $\mathcal{O}_{\mathrm{C}} \in \mathrm{coh}\, \mathfrak{X}$ *is described by*

$$\Lambda_{\mathrm{def}} \cong \mathcal{J}\mathrm{ac}(\mathsf{W}) = \frac{\mathbb{C}\langle\!\langle x, y \rangle\!\rangle}{(\!(\delta_x \mathsf{W}, \delta_y \mathsf{W})\!)}$$

where $\mathsf{W} = \sum \lambda_{jk}\, \mathcal{N}_{j,k} \in \mathbb{C}\langle\!\langle x, y \rangle\!\rangle$ *is the sum of free necklace polynomials, and the* λ_{jk} *are the data in the glue* (1.A).

Proof. As in [K3, below 4.3], and summarised in §6.1 above, the NC deformation algebra Λ_{def} can be presented as the completed tensor algebra over A_1^*, subject to the relations induced by the map

$$\mathsf{A}_2^* \to \hat{\mathrm{T}}_{\mathbb{C}}(\mathsf{A}_1^*) \tag{6.A}$$

which is dual to $\sum_{i=2}^{\infty} \mathsf{m}_i$. Now, in 5.13 the term $\lambda_{j,k}\mathsf{X}$ appears precisely in those m_n when there are $j-1$ occurrences of x and k occurrences of y. Similarly the term $\lambda_{j,k}\mathsf{Y}$ appears precisely in those m_n when there are j occurrences of x and $k-1$ occurrences of y. Writing $x = \mathsf{x}^*$ and $y = \mathsf{y}^*$, then under (6.A) it follows that

$$\mathsf{X}^* \mapsto \sum_{j+k\geq 3} \lambda_{j,k}\, \mathcal{M}\mathrm{ono}_{j-1,k}(x, y),$$

$$\mathsf{Y}^* \mapsto \sum_{j+k\geq 3} \lambda_{j,k}\, \mathcal{M}\mathrm{ono}_{j,k-1}(x, y).$$

The first relation is thus $\sum_{j+k\geq 3} \lambda_{j,k}\, \mathcal{M}\mathrm{ono}_{j-1,k}$, which equals $\delta_x \mathsf{W}$ by 6.3. In a similar way, the second relation equals $\delta_y \mathsf{W}$. $\square$

The above then immediately recovers Katz–Namba [K2, N].

Corollary 6.5. *Classical commutative deformations of* $\mathcal{O}_{\mathrm{C}} \in \mathrm{coh}\, \mathfrak{X}$ *are prorepresented by*

$$\Lambda_{\mathrm{def}}^{\mathrm{ab}} \cong \mathcal{J}\mathrm{ac}(\mathsf{W})^{\mathrm{ab}} = \frac{\mathbb{C}[\![x, y]\!]}{(\delta_x \mathsf{V}, \delta_y \mathsf{V})}$$

where $\mathsf{V} = \sum \lambda_{j,k}\, \mathcal{N}_{j,k}^{\mathrm{ab}} = \sum \frac{\lambda_{j,k}}{j+k}\binom{j+k}{k} x^j y^k$.

Proof. This follows from 6.4, since as is standard the versal space for commutative deformations is the abelianisation of the noncommutative version (e.g. [T, (13)]). $\square$

6.4. **Mirror Models.** The next application of 5.13 is categorical. Given a quiver with superpotential (Q, W), Ginzburg [G] associates a 3-CY category $\mathcal{D}_{\mathsf{W}}$. It is a basic question to find geometric models for such categories, on both the A- and B-sides of mirror symmetry.

Corollary 6.6. *Let* $\mathsf{W} \in \mathbb{C}\langle x, y \rangle$ *and consider the associated* 3-CY *category* $\mathcal{D}_{\mathsf{W}}$. *If there exist scalars* λ_{jk} *for which* $\mathsf{W} = \sum \lambda_{jk}\, \mathcal{N}_{j,k}$, *then there exists a smooth* 3-*fold* $\mathfrak{X}$ *and rational curve* $\mathrm{C} \subset \mathfrak{X}$ *such that*

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathfrak{X}) \supset \langle \mathcal{O}_{\mathrm{C}} \rangle \cong \mathcal{D}_{\mathsf{W}}.$$

Proof. On one hand, the category $\langle \mathcal{O}_{\mathrm{C}} \rangle$ can be described using the A_{∞}-structure on the DG-endomorphism ring of $\mathcal{O}_{\mathrm{C}}$, which is computed in 5.13 above. On the other hand, by definition $\mathcal{D}_{\mathsf{W}} = \mathcal{D}_{\mathrm{fd}}(\Gamma)$ where Γ is the Ginzburg DGA associated to the quiver with potential. This category has a canonical heart, generated by the simple modules indexed over the vertices of the quiver, which in our case (the two-loop quiver) consists of a unique simple S. Thus $\mathcal{D}_{\mathsf{W}}$ is described using the A_{∞}-structure on the DG-endomorphism ring of S, which by [K4, §A.15] is entirely encoded by the pairing and the superpotential on degree one inputs. But this is precisely the A_{∞}-structure described in 5.13 above, and so the result follows. $\square$

6.5. **Relationship to Physics.** The majority of the physics literature for rational curves (e.g. [K2, AK, CM]) considers superpotentials viewed inside the commutative power series ring $\mathbb{C}[\![x, y]\!]$. However, [F, §3.3.1] observed that ordering ambiguities arise when D5 branes are considered, and proposed the matrix rule [F, (3.18)] over 10 years before noncommutative deformation theory entered rational curves [DW]. This subsection explains why Ferrari's predicted rule is consistent with 6.4, and in the process establishes 6.4 as the mathematically precise formulation of these physical predictions.

In 2003 Ferrari [F, (3.18)] predicted that in any commutative potential, $x^j y^k$ should get replaced by a noncommutative term in variables X and Y, under the rule

$$x^j y^k \longrightarrow \frac{j!\, k!}{(j+k)!} \oint_{C_0} \frac{dz}{2\pi\mathrm{i}}\, z^{-k-1}\mathrm{tr}(X + Yz)^{j+k}.$$

Taking the coefficient to the other side, the prediction can be rewritten

$$\binom{j+k}{k} x^j y^k \longrightarrow \oint_{C_0} \frac{dz}{2\pi\mathrm{i}}\, z^{-k-1}\mathrm{tr}(X + Yz)^{j+k}$$

and so the commutative potential $\mathsf{V} = \sum \frac{\lambda_{jk}}{j+k}\binom{j+k}{k} x^j y^k$ of 6.5 (known to [K2, N]) should be replaced with an element in the free algebra, via the rule

$$\mathsf{V} \longrightarrow \mathbb{W} = \sum_{j,k} \frac{\lambda_{jk}}{j+k} \oint_{C_0} \frac{dz}{2\pi\mathrm{i}}\, z^{-k-1}\mathrm{tr}(X + Yz)^{j+k}. \tag{6.B}$$

There is a mild ambiguity about whether the right hand side should be viewed in the free algebra F in variables X and Y, or in its quotient $F/[F, F]$. Regardless, each contour integral in the right hand side (6.B) can be viewed as an element of F, where the integral is determined by simply computing the residue. As the coefficient of the $1/z$ term is precisely the sum of all terms containing k occurrences of Y in the expansion of $(X + Y)^{j+k}$, and this equals $(j + k)\, \mathcal{N}_{j,k}$, it follows that[2]

$$\frac{\lambda_{jk}}{j+k} \oint_{C_0} \frac{dz}{2\pi\mathrm{i}}\, z^{-j-1}\mathrm{tr}(X + Yz)^{j+k} = \frac{\lambda_{jk}}{j+k} \cdot (j+k)\, \mathcal{N}_{j,k}(X, Y)$$

and so $\mathbb{W} = \sum \lambda_{jk}\, \mathcal{N}_{j,k}$, which is the noncommutative potential in 6.4.

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[2]To calibrate the trace, use [F, (3.19), (3.20)] applied to $V_Y = 0 = V$ and $V_X(X) = X^3$. There, $E = z^2 w^3$ and so the additional glueing term is $\delta_w E = 3z^2 w^2$. Thus in the notation of this paper, $\lambda_{30} = 3$ whilst all other $\lambda_{jk} = 0$. Ferrari [F, (3.20)] predicts $\mathbb{W} = V_X(X) = X^3$, which equals $3\frac{1}{3}X^3 = \lambda_{30}\, \mathcal{N}_{3,0}$.

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