Candidates must answer *four* questions in total,
and must answer *at least one* question from *each* section.

SECTION 1.

(1) (a) Prove that $\sqrt{2} \notin \mathbb{Q}$. **[3 marks]**

(b) Consider the function

$$\begin{array}{rcl} \phi : \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{R} \\ (a, b) & \mapsto & a + b\sqrt{2}. \end{array}$$

Prove that ϕ is an injection. **[3 marks]**

(c) Find the number of injective functions from the set $\{1, 2\}$ to the set $\{1, 2, 3\}$. **[3 marks]**

(d) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *strictly decreasing* if $f(x) < f(y)$ whenever $x < y$.

(i) Prove that if f is strictly decreasing, then f is injective. **[2 marks]**

(ii) If f is strictly decreasing, is f surjective? Justify your answer. **[2 marks]**

(e) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *decreasing* if $f(x) \leq f(y)$ whenever $x \leq y$. Is a decreasing function always injective? Justify your answer. **[2 marks]**

(2) (a) Determine whether the following are TRUE or FALSE. If they are true, give a short proof. If they are false, give a counterexample.

(i) If X is a set containing n elements, then its power set $\mathcal{P}(X)$ contains $n!$ elements. **[2 marks]**

(ii) If X and Y are sets, then $X \cap Y$ is always a subset of $X \cup Y$. **[2 marks]**

(iii) Let $f : X \to Y$ be a function. Then f is surjective if and only if there exists a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$, where id_Y denotes the identity function on Y. **[4 marks]**

(b) Describe the solution set of $x \in \mathbb{R}$ such that:

(i) $|x| > x$. **[2 marks]**

(ii) $|x + 1| + |x - 1| < 1$. **[2 marks]**

(c) Give an example of a surjective function $\mathbb{N} \to \mathbb{Z}$. Justify your answer. **[3 marks]**

SECTION 2.

(3) (a) Define what we mean by a *series*, and what it means for a series to *converge to a limit.* **[2 marks]**

(b) Determine whether the following statements are TRUE or FALSE. If they are true give a (short) proof, if they are false give a counterexample.

[Please turn over]

(i) The series $\sum_{k=1}^{\infty} \frac{1}{k}$ does not have a limit. **[2 marks]**

(ii) If $\sum_{n=1}^{\infty} a_n$ has a limit, then $a_n \to 0$ as $n \to \infty$. **[2 marks]**

(iii) If $a_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ has a limit. **[2 marks]**

(c) For $x \in \mathbb{R}$, consider the series $\sum_{n=1}^{\infty}(n+1)x^n$.

(i) Denote $s_k := \sum_{n=1}^{k}(n+1)x^n$. Show that

$$(1-x)s_k = \frac{1-x^{k+1}}{1-x} - (k+1)x^{k+1}.$$

[2 marks]

(ii) Hence, or otherwise, show that when $|x| < 1$, $\sum_{n=1}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}$. **[3 marks]**

(d) Determine the rational number which has decimal expansion 0.12121212.... **[2 marks]**

(4) (a) Let $n \in \mathbb{N}$ with $n \geq 2$.

(i) Show by induction that the product of $n-1$ terms

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

[3 marks]

(ii) Hence or otherwise determine the limit of the sequence

$$a_n := \prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right).$$

You may use any result from lectures, provided that it is clearly stated. **[3 marks]**

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Define the Maclaurin series of f. **[1 mark]**

(c) Determine the Maclaurin series for each of the following functions:

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x$. **[3 marks]**

(ii) $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = e^{5x} + 1$. **[2 marks]**

(d) For the function f defined in (c)(i), determine the values of x for which the Maclaurin series converges to $f(x)$. In your answer, you may use any theorems or results from lectures, provided that they are clearly stated. **[3 marks]**

SECTION 3.

(5) (a) Give an example of:

(i) A 3×3 matrix B such that $B^3 = B$.
(ii) A 3×3 matrix B such that $B^3 \neq B$.
(iii) A 3×3 matrix B such that $B^{\mathrm{T}} = B$.
(iv) A 3×3 matrix B with determinant 4 and trace 3.

[5 marks]

(b) Suppose that A is an $n \times n$ real matrix. Define what we mean by the *eigenvalues* of A, and the *eigenvectors* of A. **[2 marks]**

(c) Determine whether the following are TRUE or FALSE. If they are true give a (short) proof. If they are false, provide a counterexample.

(i) Every real matrix has a real eigenvalue.
(ii) Every 2×2 matrix has two distinct eigenvalues.
(iii) 0 is an eigenvalue of every matrix.

[5 marks]

(d) Find the eigenvalues and the eigenvectors of the following matrix:

$$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}.$$

[3 marks]

(6) (a) (i) By Gaussian elimination or otherwise, find the inverses of the following matrices:

$$A := \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \qquad B := \begin{pmatrix} 0 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

[6 marks]

(ii) Hence determine the inverse of the matrix

$$C := \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

[4 marks]

(b) Find the solution set of the following linear equations:

$$\begin{aligned} x - y + z &= 12 \\ 3x - 2y + 2z &= 1 \\ x + 5y + z &= 3. \end{aligned}$$

[5 marks]

[End of Paper]