Candidates must answer *four* questions in total,
and must answer *at least one* question from *each* section.

SECTION 1.

(1) (a) Let $z, w \in \mathbb{C}$. Determine whether the following are TRUE or FALSE. If they are true, give a short proof. If they are false, give a counterexample.

(i) $\overline{z+w} = \overline{z} + \overline{w}$. **[1 mark]**

(ii) $\overline{zw} = \overline{z}\,\overline{w}$. **[1 mark]**

(iii) $\overline{(z/w)} = \overline{z}/\overline{w}$. **[1 mark]**

(iv) $|z+w| = |z| + |w|$. **[1 mark]**

(v) $|z^2| = |z|^2$. **[1 mark]**

(b) Give examples of non-empty sets $X$ and $Y$ such that

(i) $X \cap Y = X \cup Y$. **[2 marks]**

(ii) $X \backslash Y = X$. **[2 marks]**

(iii) Every function $X \to Y$ is injective. **[3 marks]**

In each case, justify your answer.

(c) Let $f : X \to Y$ be a function. Prove that $f$ is injective if and only if there exists a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$, where $\mathrm{id}_X$ is the identity function on $X$. **[3 marks]**

(2) (a) Find integers $a$ and $b$ such that $51a + 36b = 3$. **[4 marks]**

(b) How many different integers divide $2^2 \times 3^2 \times 11^4$? Justify your answer. **[3 marks]**

(c) Recall that a relation $R$ on a set $X$ is called an *equivalence relation* if $R$ is reflexive (i.e. $xRx$ for all $x \in X$), $R$ is symmetric (i.e. $xRy$ implies $yRx$ for all $x, y \in X$) and $R$ is transitive (i.e. $xRy$ and $yRz$ implies $xRz$ for all $x, y, z \in X$). Describe briefly how we can define the *integers mod $n$* by using equivalence relations. Define also *addition* and *multiplication* mod $n$. **[4 marks]**

(d) Find all solutions to the congruence $x^2 \equiv 4 \bmod 6$. **[4 marks]**

SECTION 2.

(3) (a) Define what we mean by a *sequence*, and what it means for a sequence to *converge to a limit*. **[2 marks]**

(b) Determine the limits, if they exist, of the following sequences. If the limit exists, give a proof. If the limit does not exist, justify why the sequence does not converge.

(i) $a_n := \frac{1}{n}$ for all $n \in \mathbb{N}$. **[2 marks]**

(ii) $a_n := \frac{n^3-n^2}{n}$ for all $n \in \mathbb{N}$. **[2 marks]**

(iii) $a_n := \frac{1}{n} + (0.7)^n$ for all $n \in \mathbb{N}$. **[3 marks]**

(iv) $a_n := \frac{\sqrt{n^2-1}}{n}$ for all $n \in \mathbb{N}$. **[2 marks]**

(c) Give an example of

(i) a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \to 0$ as $n \to \infty$, but $\sum_{n=1}^{\infty} a_n$ is not finite. Justify your answer. **[2 marks]**

(ii) a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \to 0$ as $n \to \infty$, but $\sum_{n=1}^{\infty} a_n$ is finite. Justify your answer. **[2 marks]**

(4) (a) Define what we mean by a *series*, and what it means for a series to *converge to a limit*. **[2 marks]**

(b) Determine whether the following series converge. If they converge, find their limit and give a proof. If they do not converge, justify why.

(i) $\sum_{i=1}^{\infty} 1$. **[1 mark]**

(ii) $\sum_{i=1}^{\infty} a$, where $a \in \mathbb{R}$. **[2 marks]**

(iii) $\sum_{i=1}^{\infty} i$. **[1 mark]**

(iv) $\sum_{i=1}^{\infty} \frac{1}{i}$. **[2 marks]**

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function, and let $a \in \mathbb{R}$. Define the Taylor expansion of $f$ around $a$. **[2 marks]**

(d) Find the Taylor expansion of $\ln(1+x)$ around $a$ when

(i) $a = 0$. **[2 marks]**

(ii) $a = 1$. **[3 marks]**

## SECTION 3.

(5) (a) Let $A$ and $B$ be $n \times n$ matrices. Determine whether the following are TRUE or FALSE. If they are true, give a (short) proof. If they are false, provide a counterexample.

(i) If the trace of $A$ is zero, then $A$ is invertible.

(ii) If the determinant of $A$ is zero, then $A$ is invertible.

(iii) If $AB = BA$, then $(AB)^{-1} = A^{-1}B^{-1}$.

(iv) If $A^5 = 0$, then $A = 0$.

(v) If $A^5 = \mathbb{I}$, then $A$ is invertible.

**[5 marks]**

(b) Let $A$ be a $2 \times 2$ matrix.

(i) Define the *characteristic polynomial* of $A$, and define what we mean by the *eigenvalues* and the *eigenvectors* of $A$. **[3 marks]**

(ii) Show that the characteristic polynomial of $A$ can be written as

$$t^2 - \mathrm{Tr}(A)t + \det(A),$$

where $\mathrm{Tr}(A)$ denotes the trace of $A$, and $\det(A)$ denotes the determinant of $A$. **[3 marks]**

(c) Determine the eigenvalues and the eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 3 & 4 \end{pmatrix}.$$

**[4 marks]**

(6) (a) Let $A$ be an $n \times n$ matrix, and let $P$ be a $n \times n$ invertible matrix.

(i) Show that $A$ and $P^{-1}AP$ have the same determinant. You can use any theorems from lectures, provided that they are clearly stated. **[3 marks]**

(ii) Show that $A$ is invertible if and only if $P^{-1}AP$ is invertible. **[3 marks]**

(b) Consider the matrix

$$A := \begin{pmatrix} 0 & 3 & 6 & 9 & 33 \\ 1 & 2 & 7 & 0 & 11 \\ -2 & -2 & -10 & 7 & 2 \end{pmatrix}.$$

(i) Find the echelon form of $A$. **[3 marks]**

(ii) Find the reduced echelon form of $A$. **[3 marks]**

(iii) Hence or otherwise describe the solution set of the linear equations

$$\begin{aligned} 3x_2 + 6x_3 + 9x_4 &= 33 \\ x_1 + 2x_2 + 7x_3 &= 11 \\ -2x_1 - 2x_2 - 10x_3 + 7x_4 &= 2. \end{aligned}$$

**[3 marks]**

[End of Paper]