Revision lecture: Year 3 Group Theory 2012. (Thursday 19th April at 11am)

ie for every pair, one from G, one from X, specify a unique element in X

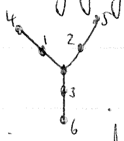
Group actions: Let G be a group, X a non-empty set. An action of G on X is a map $G\times X\longrightarrow X$ (written $(g,x)\mapsto g\cdot x$) such that (i) $g\cdot(h\cdot x)=(gh)\cdot x$ $\forall g,h\in G$ $\forall x\in X$ (ii) $e\cdot x=x$ $\forall x\in X$.

If $x\in X$, the orbit $\mathrm{orb}_G(x)=\{g\cdot x \mid g\in G\}$ "everything you can get by hitting x with elements of G".

the stabilizer $\mathrm{Stab}_G(x)=\{g\in G \mid g\cdot x=x\}$.

Key result: orbit-stabilizer theorem: if G is finite group acting on a set X, and $x\in X$, then $|\mathrm{orb}_G(x)|\cdot|\mathrm{Stab}_G(x)|=|G|$.

Uses: (1) practical; eg working out number of elements in a group (§4.5.10) §6.6.2, Pólya counting
(2) theoretical: information on conjugacy (class equation §5.3.1), Sylow's theorem, ...

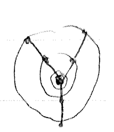
Example. (Q4, 2008) Consider the graph 

(i) Symmetry group of graph is isomorphic to S_3: any symmetry must fix the middle vertex, and is fully determined by a permutation of vertices 1, 2 and 3.

(ii) Calculate orb(1) and Stab(1). The orbit containing 1 is (by definition) all vertices you can reach by applying symmetries to the vertex 1. Since symmetries preserve the valency of vertices, $\mathrm{orb}_G(1)\subseteq\{1,2,3\}$.

Now let g = rotation by $2\pi/3$  then
$e\cdot 1=1$ (so $1\in\mathrm{orb}_G(1)$)
$g\cdot 1=3$ (so $3\in\mathrm{orb}_G(1)$)
$g^2\cdot 1=2$ (so $2\in\mathrm{orb}_G(1)$)

Hence $\{1,2,3\}\subseteq\mathrm{orb}_G(1)$, so $\mathrm{orb}_G(1)=\{1,2,3\}$.

[aside: similarly $\{4,5,6\}$ is an orbit. Picture this with circles  $X=\{\text{middle}\}\cup\{1,2,3\}\cup\{4,5,6\}$ is disjoint union of orbits]

Now $\mathrm{Stab}_G(1)=\{g\in G \mid g\cdot 1=1\}$. For 1 to be fixed, by (i) the symmetry must be $\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$ or $\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$, ie e or (23). Hence $\mathrm{Stab}_G(1)=\{e,(23)\}$.

[Note: check $|\mathrm{orb}_G(1)|\cdot|\mathrm{Stab}_G(1)|=3\times 2=6$, which equals $|S_3|=|G|$. Good!]

Conjugacy (special case of group actions)

G acts on itself via the rule $h\cdot g:=hgh^{-1}$ $\forall h,g\in G$. This is called the conjugacy action.

The orbits of this action are called the conjugacy classes: hence the conjugacy class containing a specified $g\in G$ is $\mathrm{orb}_G(g)=\{h\cdot g \mid h\in G\}=\{hgh^{-1} \mid h\in G\}$.

[To work out the conjugacy classes of a given group can be difficult (!). For symmetric groups there is an easy answer (§6.5.2), and dihedral groups can be done directly (see Q5.8).]

Note: with the above action, $\mathrm{Stab}_G(g)=\{h\in G \mid h\cdot g=g\}=\{h\in G \mid hgh^{-1}=g\}=\{h\in G \mid hg=gh\}:=C(g)$, called the centralizer of g.

Orbit-stabilizer $\Rightarrow$ size of conjugacy class $\times$ size of centralizer $=|G|$.

Note: we don't really see / use the full power of conjugacy classes in this course, so they might seem a bit bizarre.

One use: if H is a subgroup of G, then H is a normal subgp $\overset{5.2.1}{\Leftrightarrow}$ H is a union of conjugacy classes

[here, if you happen to know the conjugacy classes, eg for S_n or D_n, then working out which subgroups are normal is easy]

Isomorphic Groups To show $G \cong H$, possible techniques:

① (direct) write down a map $G \xrightarrow{\phi} H$. Show its a group homomorphism (ie $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ $\forall g_1, g_2 \in G$) which is bijective.

② (direct) write down a map $H \xrightarrow{\theta} G$. Show its a group homomorphism which is bijective.

[Note: writing down a map $H \to G$ might be easier than writing down a map $G \to H$!]

③ If you want to show $G/K \cong H$, use first isomorphism theorem:

First, write down map $G \xrightarrow{\phi} H$. Show that (i) ϕ is gp homomorphism (ii) ϕ is surjective (iii) $\text{Ker}\,\phi = K$, then by 1st iso theorem $G/\text{Ker}\,\phi \cong \text{im}\,\phi \underset{(iii)}{\overset{(ii)}{\Rightarrow}} G/K \cong H$.

④ use non-explicit techniques, for example

(1) all cyclic gps of order n are isomorphic. Hence if G and H are both cyclic of order n, $G \cong H$.

(2) all non-abelian gps of order $2p$ (where p is odd prime) are isomorphic to D_p (§7.2.2)

(3) If H, K subgps of G with $H \cap K = \{e\}$ and $hk = kh$ $\forall h \in H$ $\forall k \in K$, then $HK \cong H \times K$.

⌞ was used in §3.2.5 to show that $D_6 \cong D_3 \times \mathbb{Z}_2$. (see also §3.2.6 parts 2 & 3)

To show $G \ncong H$ use following logic: if $G \cong H$, then G and H would have the same __. Since they don't have the same __, $G \ncong H$.

eg 1. If $G \cong H$, then $|G| = |H|$. Hence $|G| \neq |H| \Rightarrow G \ncong H$. (shows eg $D_5 \ncong D_4$)

2. If $G \cong H$, then G and H have the same number of elements of order 4. Hence if these numbers differ, $G \ncong H$ (eg $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$)

3. ⋮ lots of possibilities

eg: If G has no subgp of order 6 but H does, then $G \ncong H$ (shows $A_4 \ncong D_6$ see Q6.11)

If G has five elements of order 2 and H only has two elements of order 2, then $G \ncong H$ (shows $D_4 \ncong Q$ in Q7.6).