Sheet 1

Recall that all vector spaces in this course are over $\mathbb{C}$.

1.1 (Basic manipulations) If L is a Lie algebra, and $x, y \in L$, show that:

1. $[x, 0] = 0 = [0, x]$.
2. If $[x, y] \neq 0$, then x and y are linearly independent.

1.2 (Common subalgebras) Recall that $\mathfrak{gl}_n$ is just the set of all $n \times n$ matrices.

1. Verify that $\mathfrak{gl}_n$ is a Lie algebra with bracket $[x, y] = xy - yx$.
2. Show that $\mathfrak{sl}_n := \{x \in \mathfrak{gl}_n \mid \mathrm{Tr}(x) = 0\}$ is a subalgebra of $\mathfrak{gl}_n$.
3. Show that upper triangular matrices
$$\mathfrak{b}_n := \{x \in \mathfrak{gl}_n \mid x_{ij} = 0 \text{ for } i > j\}$$
is a subalgebra of $\mathfrak{gl}_n$.
4. Show that strictly upper triangular matrices
$$\mathfrak{n}_n := \{x \in \mathfrak{gl}_n \mid x_{ij} = 0 \text{ for } i \geq j\}$$
is a subalgebra of $\mathfrak{gl}_n$.
5. Show that diagonal matrices
$$\mathfrak{d}_n := \{x \in \mathfrak{gl}_n \mid x_{ij} = 0 \text{ for } i \neq j\}$$
is a subalgebra of $\mathfrak{gl}_n$.
6. Pick an $n \times n$ matrix S, and define
$$\mathfrak{gl}_S := \{x \in \mathfrak{gl}_n \mid x^T S = -Sx\}.$$
Show that $\mathfrak{gl}_S$ is a subalgebra of $\mathfrak{gl}_n$. As a special case,
$$\mathfrak{so}_n := \{x \in \mathfrak{gl}_n \mid x^T = -x\}$$
is a subalgebra of $\mathfrak{gl}_n$.

1.3 (A useful fact for later) Show that as a vector space, $\mathfrak{sl}_n$ has a basis
$$\{e_{ij} \mid i \neq j\} \cup \{e_{ii} - e_{i+1\,i+1} \mid 1 \leq i < n\}.$$

1.4 (A counterexample) Following [EW, p4], verify that for $n \geq 2$, $\mathfrak{b}_n$ is a subalgebra of $\mathfrak{gl}_n$, but is not an ideal

1.5 (Centres)

1. If L is a Lie algebra, verify that its centre $Z(L)$ is an ideal.
2. (A bit fiddly) Show that $Z(\mathfrak{gl}_n) = \{\lambda\,\mathrm{Id} \mid \lambda \in \mathbb{C}\}$.
3. Deduce that $Z(\mathfrak{sl}_n) = \{0\}$.

1.6 (Similar to groups) If $\varphi\colon L \to M$ is a homomorphism of Lie algebras, show that

1. The kernel $\ker\varphi$ is an ideal of L
2. The image $\mathrm{Im}\,\varphi$ is a subalgebra of M.

1.7 (Classifying abelian Lie algebras is easy) Suppose L and M are abelian Lie algebras. Show that $L \cong M$ if and only if L and M have the same dimension.

1.8 (Associativity Question, optional) If L is a Lie algebra, by definition $[-,-]$ is an operation. If we temporarily denote the operation $a \cdot b := [a, b]$, then in all previous mathematical structures you have studied, brackets do not matter, namely

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for all $a, b, c \in L$. However, in Lie algebras they do! Prove the following:

$$[a, [b, c]] = [[a, b], c] \text{ for all } a, b, c \in L \iff [x, y] \in Z(L) \text{ for all } x, y \in L$$