Sheet 2

2.1 (A very good exercise! A bit fiddly, but worth it) Recall $\mathfrak{sl}_2$ has basis

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

1. Prove that $\mathfrak{sl}_2$ has no non-trivial ideals (later in the course, this means that $\mathfrak{sl}_2$ is *simple*).
2. Deduce that $\mathfrak{sl}_2' = \mathfrak{sl}_2$.

2.2 Prove directly (without appealing to general results for ideals) that $\mathfrak{gl}_2 \cong \mathbb{C} \oplus \mathfrak{sl}_2$.

2.3 (Ideals and direct sums) Let L_1, L_2 be Lie algebras, and $I_1 \subseteq L_1$ and $I_2 \subseteq L_2$ be ideals.

1. Show that the subspace
$$I_1 \oplus I_2 \subseteq L_1 \oplus L_2$$
is an ideal of $L_1 \oplus L_2$.
2. Do all ideals of $L_1 \oplus L_2$ arise in this way? (*hint:* consider $\mathbb{C} = L_1 = L_2$).
3. Show that $Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$ and $(L_1 \oplus L_2)' = L_1' \oplus L_2'$.

2.4 (Will be used later) **[Question has been corrected]** Let L_1, L_2 be Lie algebras, and inside $L_1 \oplus L_2$ consider

$$L_1 \cong \{(x, 0) \mid x \in L_1\}$$
$$L_2 \cong \{(0, y) \mid y \in L_2\},$$

which are both ideals of $L_1 \oplus L_2$. For the rest of the question, suppose that L_1 and L_2 do not have any non-trivial ideals, and are not abelian (i.e. are *simple*).

1. If J is a non-zero ideal of $L_1 \oplus L_2$ such that $J \cap L_1 = \{0\}$ and $J \cap L_2 = \{0\}$, show that $J = \{0\}$.
2. Deduce that $L_1 \oplus L_2$ has only two non-trivial ideals.

2.5 Let $x \in \mathfrak{gl}_n$ be a diagonal matrix with entries $\lambda_1, \dots, \lambda_n$. Show that $\mathrm{ad}_x \colon \mathfrak{gl}_n \to \mathfrak{gl}_n$ is also diagonalisable, with eigenvalues $\lambda_i - \lambda_j$ for $1 \le i, j \le n$.

2.6 Recall from [EW,p3] the matrices e_{ij}, which satisfy the relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}.$$

Using these relations, show that $\mathfrak{gl}_n' = \mathfrak{sl}_n$.