Sheet 4

4.1 (Our first weight decomposition) Consider $L = \mathfrak{d}_n$, diagonal $n \times n$ matrices, which is a subalgebra of $\mathfrak{gl}(V)$ for $V = \mathbb{C}^n$. Define $\lambda_i \colon L \to \mathbb{C}$ by

$$\begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ & & & \vdots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix} \mapsto \alpha_i$$

Show that $V_{\lambda_i} = \mathrm{Span}\{e_i\}$ and that

$$V = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_n}.$$

4.2 Consider $L = \mathfrak{b}_n$, upper triangular $n \times n$ matrices. Show that e_1 is an eigenvector for L, find the corresponding weight and determine its weight space.

4.3 (Basics before the Invariance Lemma)

1. (Easy!) If $f, g \colon V \to V$ satisfy $f \circ g = g \circ f$, show that $g \colon \ker f \to \ker f$, i.e. $\ker f$ is g-invariant.
2. Show that Lemma 5.4 in the book really does generalise this statement.

4.4 Check the induction step in the proof of the Invariance Lemma.

4.5 (Part 1 generalises the $i\ell = \ell i + [i, \ell]$ trick)

1. Show that for any linear maps $f, g \colon V \to V$, and for any $m \geq 1$
$$f \circ g^m = g^m \circ f + \sum_{k=1}^{m} \binom{m}{k} g^{m-k} \circ f_k$$
where $f_1 = [f, g]$ and $f_k = [f_{k-1}, g]$ for all $k \geq 2$.
2. Deduce that
$$\mathrm{ad}\, g^m = \sum_{k=1}^{m} (-1)^k \binom{m}{k} G^{m-k} \circ (\mathrm{ad}\, g)^k$$
where $G^{m-k} \colon \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ is defined by sending $X \mapsto g^{m-k} X$.