# Sheet 6

**6.1** Consider the adjoint representation of $\mathfrak{sl}_2$.

1. **[The order of the basis has been changed. This does not really change the question, it just permutes the matrices a bit]** Show that with respect to the basis $\{e, h, f\}$, $\mathrm{ad}_h\colon \mathfrak{sl}_2 \to \mathfrak{sl}_2$ has matrix
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
2. Find the matrices for $\mathrm{ad}_e$ and $\mathrm{ad}_f$.
3. Is this adjoint representation simple?

**6.2** (Submodules=Ideals for $L$) Consider $L$ viewed as an $L$-module. Show that the submodules of $L$ are precisely the ideals of $L$.

**6.3** (An explicit example) Consider the two-dimensional Lie algebra $L$ with basis $\{x, y\}$ and bracket $[x, y] := x$. Show that we can construct a representation of $L$ by considering $V = \mathbb{C}^2$ and defining
$$\varphi(x) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \varphi(y) := \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

**6.4** (The quotient space is a module) Suppose that $V$ is an $L$-module, with submodule $W$.

1. Show that the vector space $V/W$ becomes an $L$-module, under
$$\ell \cdot (v + W) := \ell \cdot v + W$$
for all $\ell \in L$, all $v \in V$.
2. Show that the natural map $V \to V/W$ is an $L$-module homomorphism.

**6.5** (Test for simple) If $V$ is an $L$-module and $v \in V$, consider the submodule $Lv$ generated by $v$, which by definition is the subspace of $V$ spanned by all elements of the form
$$x_1 \cdot (x_2 \cdot \ldots (x_m \cdot v)))$$
where $x_1, \ldots, x_m \in L$.

1. Show that $Lv$ is a submodule of $V$
2. Show that $V$ is simple $\iff$ $Lv = V$ for all $0 \neq v \in V$.

**6.6** (Indecomposable does not imply simple) Consider $\mathfrak{b}_2$, upper triangular $2 \times 2$ matrices. Show that the natural representation $V$ is indecomposable, but is not simple.

**6.7** (The $\mathfrak{sl}_2$ classification contains things we know!) Consider the Lie algebra $\mathfrak{sl}_2$, and the simple modules $V_n$ defined in lectures. Show that

1. $V_0$ is the trivial representation.
2. $V_1$ is the natural representation.
3. $V_2$ is the adjoint representation.