Sheet 7

7.1 Let L be a Lie algebra.

1. Show that the Killing form $\kappa(-,-)\colon L \times L \to \mathbb{C}$ defined by
$$\kappa(x,y) := \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$$
is a symmetric bilinear form.
2. Prove that the Killing form is associative, i.e.
$$\kappa([a,b],c) = \kappa(a,[b,c])$$
for all $a,b,c \in L$.

7.2 Using Q6.1, calculate the Killing form $\kappa_{\mathfrak{sl}_2}(a,b)$ for $a,b \in \{e,f,h\}$.

7.3 Show that $\kappa_{\mathfrak{gl}_n}(x,y) = 2n\,\mathrm{Tr}(xy) - 2\,\mathrm{Tr}(x)\,\mathrm{Tr}(y)$ for all $x,y \in \mathfrak{gl}_n$ by using the following strategy:

1. Recall the matrices e_{ij}, and the relations in Q2.6. Using these, write a formula for $\mathrm{ad}_{e_{ij}} \circ \mathrm{ad}_{e_{k\ell}}(e_{gh})$.
2. Deduce that
$$\kappa_{\mathfrak{gl}_n}(e_{ij}, e_{k\ell}) = 2n\,\mathrm{Tr}(e_{ij}e_{k\ell}) - 2\,\mathrm{Tr}(e_{ij})\,\mathrm{Tr}(e_{k\ell}).$$
3. By linearity, deduce the result.

7.4 Compute directly (using a basis of your choice!) the Killing form for $\mathfrak{b}_2$.

7.5 Show that the Killing form on L is identically zero if L is a nilpotent Lie algebra.