Solution to Week 10 Assignment

Q19.4. For $A := \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix}$, the characteristic equation is given by

$$\det \begin{pmatrix} 1-\lambda & -3 \\ 2 & 3-\lambda \end{pmatrix} = 0$$

i.e. $(1-\lambda)(3-\lambda)+6=0$, which re-arranges to $\lambda^2 - 4\lambda + 9 = 0$. The matrix A satisfies its own characteristic equation since

$$A^2 = \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -5 & -12 \\ 8 & 3 \end{pmatrix},$$

so

$$\begin{aligned} A^2 - 4A + 9\mathbb{I} &= \begin{pmatrix} -5 & -12 \\ 8 & 3 \end{pmatrix} - 4 \begin{pmatrix} 1 & -3 \\ 2 & 3 \end{pmatrix} + 9 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -5-4+9 & -12+12+0 \\ 8-8+0 & 3-12+9 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Q20.8. For $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the characteristic equation is given by

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

i.e. $(a-\lambda)(d-\lambda) - bc = 0$, which re-arranges to $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$. This is clearly just $\lambda^2 - (\operatorname{Tr} A)\lambda + \det A = 0$.
Since A is real, the eigenvalues of A are real if and only if the discriminant of the characteristic polynomial is non-negative, i.e. if and only if

$$(-\operatorname{Tr} A)^2 - 4\det A \geq 0,$$

i.e. if and only if $(\operatorname{Tr} A)^2 \geq 4\det A$. In terms of a, b, c and d, this condition holds if and only if $(a+d)^2 \geq 4(ad-bc)$.

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