Solution(s) to Week 2 Assignment

(a)(i) If $A \subseteq F$, then $|A| \leq |F|$.

Solution 1: Since $A \subseteq F$, every element in A is an element in F. This means that if there are n elements in A, there are at least n elements in F. Hence $|A| \leq |F|$.

Solution 2: Since $A \subseteq F$, every element in A is an element in F. It may be true that F has other elements in addition to the ones that are in A, so $|A| \leq |F|$.

(other variations of the above two solutions are also possible)

(a)(ii) $|A| \leq |A \cup B|$.

Solution: Since $A \subseteq A \cup B$, this follows immediately from (a)(i) (with $F := A \cup B$).

(a)(iii) $|A \cup B| \leq |A| + |B|$.

Solution: If $A \cap B = \emptyset$ then the list of elements of $A \cup B$ consists of the complete list of the elements of A adjoined with the complete list of the elements of B. Hence in this case $|A \cup B| = |A| + |B|$. On the other hand, if $A \cap B \neq \emptyset$ (i.e. A and B have elements in common) then elements in $A \cap B$ are listed only once in $A \cup B$. However in $|A| + |B|$, the elements in $A \cap B$ are counted twice, hence $|A \cup B| < |A| + |B|$.

(again, many other solutions are possible)

(a)(iv) $|A \cup B| \geq \max(|A|, |B|)$.

Solution: Since $A \subseteq A \cup B$, by (a)(i) $|A \cup B| \geq |A|$. But also $B \subseteq A \cup B$, so again by (a)(i) $|A \cup B| \geq |B|$. This shows that $|A \cup B|$ is greater than or equal to both $|A|$ and $|B|$, hence it follows that $|A \cup B| \geq \max(|A|, |B|)$.

(a)(v) $|A \cap B| \leq \min(|A|, |B|)$.

Solution: Since $A \cap B \subseteq A$, by (a)(i) $|A \cap B| \leq |A|$. But also $A \cap B \subseteq B$, so again by (a)(i) $|A \cap B| \leq |B|$. This shows that $|A \cap B|$ is less than or equal to both $|A|$ and $|B|$, hence it follows that $|A \cap B| \leq \min(|A|, |B|)$.

(b)(ii) $|A| = |A \cup B| \iff B \subseteq A$.

Solution 1: ($\Rightarrow$) Suppose that $|A| = |A \cup B|$. Then since $A \subseteq A \cup B$, it follows that $A = A \cup B$. In particular, there cannot be a member of B which does not belong to A, so $B \subseteq A$.
($\Leftarrow$) Suppose that $B \subseteq A$, then certainly $A = A \cup B$. In particular their sizes are the same, i.e. $|A| = |A \cup B|$.

Solution 2: (proves both directions at the same time) Since $A \subseteq A \cup B$, $|A| = |A \cup B| \iff A = A \cup B$. But this holds if and only if $B \subseteq A$.

(b)(iii) $|A \cup B| = |A| + |B| \iff A \cap B = \emptyset$.

Solution 1: The proof of (c) shows that $|A \cup B| = |A| + |B| - |A \cap B|$. Hence

$$|A \cup B| = |A| + |B| \iff |A \cap B| = 0 \iff A \cap B = \emptyset.$$

Solution 2: (direct, not using (c)) ($\Leftarrow$) Suppose that $A \cap B = \emptyset$, then the list of elements of $A \cup B$ consists of the complete list of the elements of A adjoined with the complete list of elements of B. Hence $|A \cup B| = |A| + |B|$.
($\Rightarrow$) Suppose that $|A \cup B| = |A| + |B|$. If $A \cap B \neq \emptyset$ we can find an element $a \in A \cap B$. But this element is listed only once in $A \cup B$, whereas it is counted twice in $|A| + |B|$. This implies that $|A \cup B| < |A| + |B|$, which is a contradiction. Hence $A \cap B = \emptyset$.

(b)(iv) $|A \cup B| = \max(|A|, |B|) \iff A \subseteq B$ or $B \subseteq A$.
Solution: From the definition of maximum, we have that

$$|A \cup B| = \max(|A|, |B|) \iff |A \cup B| = |A| \text{ or } |A \cup B| = |B|. \qquad (1)$$

But by (b)(ii) we have

$$|A \cup B| = |A| \iff B \subseteq A \qquad (2)$$

and

$$|A \cup B| = |B| \iff A \subseteq B. \qquad (3)$$

Combining (1), (2) and (3) gives

$$|A \cup B| = \max(|A|, |B|) \iff A \subseteq B \text{ or } B \subseteq A.$$

Remark: You might find it easier to first argue the ($\Rightarrow$) direction (i.e. suppose that $|A \cup B| = \max(|A|, |B|)$ and argue that either $A \subseteq B$ or $B \subseteq A$), then second argue the ($\Leftarrow$) direction (i.e. suppose that $A \subseteq B$ or $B \subseteq A$, then argue that $|A \cup B| = \max(|A|, |B|)$)

(b)(v) $|A \cap B| = \min(|A|, |B|) \iff A \subseteq B$ or $B \subseteq A$.
Solution: From the definition of minimum, we have that

$$|A \cap B| = \min(|A|, |B|) \iff |A \cap B| = |A| \text{ or } |A \cap B| = |B|. \qquad (4)$$

Since $A \cap B \subseteq A$,

$$|A \cap B| = |A| \iff A = A \cap B \iff A \subseteq B. \qquad (5)$$

Similarly since $A \cap B \subseteq B$,

$$|A \cap B| = |B| \iff B = A \cap B \iff B \subseteq A. \qquad (6)$$

Combining (4), (5) and (6) gives

$$|A \cap B| = \min(|A|, |B|) \iff A \subseteq B \text{ or } B \subseteq A.$$

Remark: Again, you might find it easier to first argue the ($\Rightarrow$) direction (i.e. suppose that $|A \cap B| = \min(|A|, |B|)$ and argue that either $A \subseteq B$ or $B \subseteq A$), then second argue the ($\Leftarrow$) direction (i.e. suppose that $A \subseteq B$ or $B \subseteq A$, then argue that $|A \cap B| = \min(|A|, |B|)$).

(c) $|A \cup B| = |A| + |B| - |A \cap B|$.

(d) Proof: (similar to (a)(iii)) If $A \cap B = \emptyset$ then the list of elements of $A \cup B$ consists of the complete list of the elements of A adjoined with the complete list of elements of B. Hence in this case $|A \cup B| = |A| + |B| = |A| + |B| - |A \cap B|$. On the other hand, if $A \cap B \neq \emptyset$ (i.e. A and B have elements in common) then elements in $A \cap B$ are listed only once in $A \cup B$. However in $|A| + |B|$, the elements in $A \cap B$ are counted twice, hence $|A \cup B| = |A| + |B| - |A \cap B|$.

(There are many other possible proofs of this)

(e) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

Even although the question did not ask for a proof, here is one:

$$
\begin{aligned}
|A \cup B \cup C| &= |(A \cup B) \cup C| \\
&\overset{(c)}{=} |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| \\
&\overset{(c)}{=} |A \cup B| + |C| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \\
&= |A \cup B| + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&\overset{(c)}{=} (|A| + |B| - |A \cap B|) + |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| .
\end{aligned}
$$

2.4. (i) Consider the function $f : [0,1] \to [0,1]$ defined by $f(x) := 1$. Since $f(0) = f(1)$ with $0 \neq 1$, f is not injective. Further, since $0 \in [0,1]$ and there does not exist $a \in [0,1]$ such that $f(a) = 0$, it follows that f is not surjective.

(ii) Consider the function $g : [0,1] \to [0,1]$ defined by $g(x) := 4(x - \frac{1}{2})^2$. Since $g(0) = g(1)$ with $0 \neq 1$, g is not injective. However, for all $y \in [0,1]$ there exists $x \in [0,1]$ such that $g(x) = y$ (to see this just draw the graph of the function; I cannot do this easily on the computer I am using), hence it follows that g is surjective.

(iii) Consider the function $h : [0,1] \to [0,1]$ defined by $h(x) := \frac{x}{2}$. Suppose that $x_1, x_2 \in [0,1]$ with $h(x_1) = h(x_2)$. Then $\frac{x_1}{2} = \frac{x_2}{2}$ and so $x_1 = x_2$. This shows that h is injective. However, $1 \in [0,1]$ and there does not exist $x \in [0,1]$ such that $h(x) = 1$ (since if $\frac{x}{2} = 1$ then $x = 2$, which does not belong to the domain), hence it follows that h is not surjective.

(iv) Consider the function $j : [0,1] \to [0,1]$ defined by $j(x) := 1 - x$. Suppose that $x_1, x_2 \in [0,1]$ with $j(x_1) = j(x_2)$. Then $1 - x_1 = 1 - x_2$ and so $x_1 = x_2$. This shows that j is injective. Also, if $y \in [0,1]$ then take $x := 1 - y \in [0,1]$. Since $j(x) = 1 - (1 - y) = y$, it follows that j is surjective.

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