ACCELERATED ALGEBRA AND CALCULUS

Assignment 2 – Solution

(a) **Claim:** Two non-zero complex numbers $re^{i\theta}$ and $se^{i\phi}$ are equal if and only if $r = s$ and $\phi = \theta + 2k\pi$ for some $k \in \mathbb{Z}$.

Proof. The question really means that $re^{i\theta}$ and $se^{i\phi}$ are in **polar form**, so that we have $r, s > 0$.

We now prove the two parts of the statement separately.

$\boxed{\Rightarrow}$ Given that $r = s$ and $\phi = \theta + 2k\pi$ for some $k \in \mathbb{Z}$, we have

$$\begin{aligned} se^{i\phi} &= re^{i(\theta+2k\pi)} \\ &= r\left(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right) \\ &= r\left(\cos\theta + i\sin\theta\right) \\ &= re^{i\theta}. \end{aligned}$$

$\boxed{\Leftarrow}$ Now if $re^{i\theta} = se^{i\phi}$, we can take absolute values to find

$$\begin{aligned} |re^{i\theta}| &= |se^{i\phi}| \\ |r| &= |s|, \end{aligned}$$

but since $r, s > 0$ this means $r = s$. So $re^{i\theta} = se^{i\phi}$ implies

$$\begin{aligned} e^{i\theta} &= e^{i\phi} \\ \cos\theta + i\sin\theta &= \cos\phi + i\sin\phi. \end{aligned}$$

Comparing real and imaginary parts, this means

$$\cos\theta = \cos\phi \quad \text{and} \quad \sin\theta = \sin\phi,$$

and, due to the periods of the sine and cosine functions, this means we must have $\phi = \theta + 2k\pi$.

□

(b) **Claim:** For $z, w \in \mathbb{C}$ we have $e^z = e^w$ if and only if $w = z + 2k\pi i$ for some $k \in \mathbb{Z}$.

Proof. Suppose $z = a + ib$ and $w = x + iy$. Using the definition of the complex exponential,

$$e^z = e^w \iff e^a e^{ib} = e^x e^{iy}.$$

Putting $r = e^a$, $\theta = b$, $s = e^x$ and $\phi = y$ in the result proved in (a), we get that this is equivalent to

$$e^a = e^x \quad \text{and} \quad y = b + 2k\pi \text{ for some } k \in \mathbb{Z}.$$

Since $e^a = e^x$ if and only if $a = x$, this is equivalent to

$$w = x + iy = a + i(b + 2k\pi) = z + 2k\pi i.$$

□

(c) To show that

$$\exp : \{z \in \mathbb{C} \mid \mathcal{Im}\, z \in (\pi, \pi]\} \to \{z \in \mathbb{C} \mid z \neq 0\}$$

is a bijection, we show that it is **injective** and **surjective**.

- To see that exp is injective, suppose we have $\exp z = \exp w$; we aim to show that $z = w$. Now, by (b), we have have $w = z + 2k\pi i$ for some $k \in \mathbb{Z}$. Since z and w are in the domain of exp, we know $\mathcal{Im}\, z, \mathcal{Im}\, w \in (-\pi, \pi]$, so the only way to have
$$\mathcal{Im}\, w = \mathcal{Im}\, z + 2k\pi i$$
is to have $k = 0$, i.e. $z = w$. So exp is injective.
- To see that it is surjective, let $re^{i\theta}$ be an aribtrary element of the codomain. We suppose it is written in polar form with $\theta \in (-\pi, \pi]$. Then
$$\exp(\ln r + i\theta) = e^{\ln r} e^{i\theta} = re^{i\theta};$$
thus $\ln r + i\theta$ is an element of the domain that is mapped to re^{θ}.

(d) Suppose $z = re^{i\theta} \neq 0$, with $\theta \in (-\pi, \pi]$. Then

$$z = re^{i\theta} = e^{\ln r} e^{i\theta} = e^{\ln r + i\theta},$$

so $\operatorname{Log} z = \ln r + i\theta$.

(e) We say that $c \in \mathbb{C}$ is a **logarithm of** z if $e^c = z$.

Claim: If c is a logarithm of z then $c = \operatorname{Log} z + 2k\pi i$ for some $k \in \mathbb{Z}$.

Proof. Given that $e^c = z$, note that $z = e^{\operatorname{Log} z}$, so $e^c = e^{\operatorname{Log} z}$. Applying ($b$), we have

$$c = \operatorname{Log} z + 2k\pi i$$

for some $k \in \mathbb{Z}$. □

(f) $\operatorname{Log}(i) = \operatorname{Log}(e^{i\frac{\pi}{2}})$, so applying ($d$),

$$\operatorname{Log}(i) = \ln 1 + i\tfrac{\pi}{2} = i\tfrac{\pi}{2}.$$

Using the result of (e), the set of all logarithms of i is

$$\left\{ i\tfrac{\pi}{2} + 2k\pi i \,|\, k \in \mathbb{Z} \right\}.$$

(g) From the definition, the principal value of i^i is

$$e^{i \operatorname{Log} i} = e^{i \times i \frac{\pi}{2}} = e^{-\frac{\pi}{2}}.$$

The set of all values of i^i is

$$\begin{aligned} &\left\{ e^{ic} \,|\, c \text{ is a logarithm of } i \right\} \\ =& \left\{ e^{ic} \,|\, c = (2k\pi + \tfrac{\pi}{2})i, k \in \mathbb{Z} \right\} \\ =& \left\{ e^{-(2k\pi + \frac{\pi}{2})} \,|\, k \in \mathbb{Z} \right\}. \end{aligned}$$

(h) The set of values of $z^{1/2}$ is

$$\begin{aligned} &\left\{ e^{\frac{1}{2}c} \,|\, c \text{ is a logarithm of } z \right\} \\ =& \left\{ e^{\frac{1}{2}(\operatorname{Log} z + 2k\pi i)} \,|\, k \in \mathbb{Z} \right\} \\ =& \left\{ e^{\frac{1}{2} \operatorname{Log} z} e^{k\pi i} \,|\, k \in \mathbb{Z} \right\}, \end{aligned}$$

and since $e^{k\pi i} = \pm 1$, this is simply

$$\left\{ e^{\frac{1}{2} \operatorname{Log} z}, -e^{\frac{1}{2} \operatorname{Log} z} \right\}.$$

Similarly, the values of z^2 are

$$\begin{aligned} &\left\{ e^{2(\operatorname{Log} z + 2k\pi i)} \,|\, k \in \mathbb{Z} \right\} \\ =& \left\{ e^{\frac{1}{2} \operatorname{Log} z} e^{4k\pi i} \,|\, k \in \mathbb{Z} \right\} \\ =& \left\{ e^{\frac{1}{2} \operatorname{Log} z} \right\}. \end{aligned}$$